INVERSE PROBLEMS FOR EVOLUTION EQUATIONS WITH FRACTIONAL INTEGRALS AT BOUNDARY-VALUE CONDITIONS

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Abstract. The following problem is considered: to find a solution and a term of a first-order differential equation in a Banach space if the initial-value condition and an excessive condition containing the fractional Riemann–Liouville integral are given. We show that the solvability of the considered problem depends on the distributions of zeroes of the Mittag-Leffler function.

1. Problem Posing

Let E be a Banach space and A be a linear closed operator such that its domain $D(A)$ is a dense subset of E, while its resolvent set is not empty. Consider a problem of finding a function $u(t) \in C^1((0,1], E)$ belonging to $D(A)$ for $t \in (0,1]$ and the parameter $p \in E$ if the following relations are given:

$$u'(t) = Au(t) + t^{k-1}p, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

$$\lim_{t\to 1} I^{\beta}u(t) = u_1, \tag{1.3}$$

where $k > 0$, $I^{\beta}u(t) = \dfrac{1}{\Gamma(\beta)}\displaystyle\int\limits_0^t (t-s)^{\beta-1}u(s)\,ds$ is the left-sided fractional Riemann–Liouville integral of order $\beta > 0$, and $\Gamma(\cdot)$ is the gamma-function.

Following the traditional terminology, we call problem (1.1)–(1.3) an *inverse problem* to distinguish it from the Cauchy problem (1.1), (1.2) with a known element $p \in E$, which is called a *direct problem*.

The considered problem can be treated as follows: to reconstruct the nonstationary term $t^{k-1}p$ of Eq. (1.1) by means of condition (1.3), which is an additional nonlocal boundary-value condition containing the fractional Riemann–Liouville integral.

Unlike inverse problems for the equation

$$u'(t) = Au(t) + \varphi(t)p \tag{1.4}$$

with a continuous function $\varphi(t)$, Eq. (1.1) contains a term singular for $0 < k < 1$. It contains a power function $\varphi(t) = t^{k-1}$. Therefore, we have to use an operation of fractional integrodifferentiation to find a solution of problem (1.1), (1.2) (see Theorem 2.1). This allows us to set additional condition (1.3) by means of a fractional integral. This integral can be treated as a Cesàro mean over the segment

[0, 1]. Then we prove that it is better (from the solvability point of view) to set a Cesàro mean over the segment [0, 1] than to set a final value $u(1)$.

To find a review of publications about inverse problems for Eq. (1.4) with various assumptions regarding the operator A, see [4, 5, 12, 14, 15].

In the present paper, we consider the case where the operator A is a generator of a k-times integrated semigroup (IS) $T_k(t)$, where k is the positive parameter from Eq. (1.1).

The notion of an integrated semigroup allows us to weaken the assumptions for the resolving operator of the Cauchy problem for an abstract first-order differential equation (see [1, 8, 10]). Therefore, the assumptions for the resolvent of the operator A are weakened as well. This requires an additional smoothness of the initial data.

Definition. Let $k > 0$. A one-parametric family of linear bounded operators $T_k(t), t \geq 0$, is called a *k times IS* if the following conditions are satisfied:

(1) $\Gamma(k)T_k(t)T_k(s) = \int\limits_s^{s+t} (t+s-r)^{k-1}T_k(r)\, dr - \int\limits_0^t (t+s-r)^{k-1}T_k(r)\, dr, \quad t, s \geq 0.$

(2) $T_k(0) = 0$.

(3) For any $x \in E$, the function $T_k(t)x$ is continuous with respect to $t \geq 0$.

(4) There exist constants $M_0 > 0$ and $\omega \in \mathbb{R}$ such that

$$\|T_k(t)\| \leq M_0 \exp(\omega t), \quad t \geq 0. \tag{1.5}$$

(5) To define the generator A of the IS $T_k(t)$, we define its domain $D(A)$ as the set of elements $x \in E$ such that there exists an element $y \in E$ $(y = y(x))$ satisfying the relation

$$T_k(t) - \frac{t^k}{\Gamma(k+1)}x = \int\limits_0^t T_k(s)y\, ds, \quad t \geq 0; \tag{1.6}$$

then we assign $Ax = y$.

Examples of integrated semigroups are given below (see [8]).

Example 1.1. Let a linear operator $B\colon D(B) \to E$ be a generator of a cosine operator-function $C(t)$. Let

$$C_1(t) = \int\limits_0^t C(s)\, ds,$$

$$T_1(t) = \begin{pmatrix} C_1(t) & \int\limits_0^t C_1(s)\, ds \\ C(t) - I & C_1(t) \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}.$$

Then the family $T_1(t)$ is a one time IS with generator A in the space $E \times E$.

This example reflects the following well-known fact: in general, the unform well-posedness of the Cauchy problem for the second-order equation

$$v''(t) = Bv(t)$$

is not equivalent to the uniform well-posedness in the space $E \times E$ of the Cauchy problem for the first-order equation

$$u'(t) = Au(t)$$

though the former problem is reduced to the latter one.

In the sequel, $\{k\}$ and $[k]$ denote the fractional and the integer part of k respectively.

Example 1.2. Any elliptic differential operator

$$A = \sum_{|\alpha|\le m} a_\alpha D^\alpha,\ \alpha = (\alpha_1,\dots,\alpha_n),\ D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},\ p(x) = \sum_{|\alpha|\le m} a_\alpha i^{|\alpha|} x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

generates an $[n/2+2]$-times IS in the spaces $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, and $L_p(\mathbb{R}^n)$, $1\le p\le\infty$, provided that the order of that elliptic operator exceeds $n/2$ and $\sup\limits_{x\in R^n} \operatorname{Re} p(x) < \infty$.

2. Inverse Problems with Nonstationary Inhomogeneous Terms

To investigate inverse problem (1.1)–(1.3), we need to express the solution of problem (1.1), (1.2) via u_0 and p, which are known. Introduce the denotation $D^k = \left(\frac{d}{dt}\right)^k$ if $k\in\mathbb{N}$ and $D^k = \left(\frac{d}{dt}\right)^{[k]+1} I^{1-\{k\}}$ if $k>0$, $k\notin\mathbb{N}$. In the sequel, we will use the expression $2-[1-k]$ equal to $k+1$ for $\{k\}=0$ and to $[k]+2$ for $\{k\}\ne 0$.

Theorem 2.1. *Let $k>0$, A be a generator of a k-times IS $T_k(t)$, $p\in D(A)$, and $u_0\in D\left(A^{2-[1-k]}\right)$. Then the function*

$$u(t) = \Gamma(k)T_k(t)p + D^kT_k(t)u_0 \tag{2.1}$$

is a unique solution of problem (1.1), (1.2) *and*

$$\begin{gathered}\|u(t)\| \le M\exp(\omega t)\left(t^k\|p\| + t^{[k]+1}\|A^{[k]+1}u_0\|\right) + \sum_{j=0}^{[k]}\frac{t^j}{j!}\|A^ju_0\|,\quad k\notin\mathbb{N},\\ \|u(t)\| \le Mt^k\exp(\omega t)\left(\|p\| + \|A^ku_0\|\right) + \sum_{j=0}^{k}\frac{t^j}{j!}\|A^ju_0\|,\quad k\in\mathbb{N}.\end{gathered} \tag{2.2}$$

Proof. Substituting $t^kv(t)$ for $u(t)$, we reduce problem (1.1), (1.2) to the problem

$$v'(t) + \frac{k}{t}v(t) = Av(t) + \frac{p}{t},$$

$$\lim_{t\to 0} t^kv(t) = u_0;$$

in [6], it is proved that it is uniquely solvable and a representation of its solution as well as its estimate for $k\in\mathbb{N}$ are found.

To prove the theorem for $k\notin\mathbb{N}$, we compute $D^kT_k(t)u_0$ first. Taking into account relation (1.6), we see that

$$\begin{aligned} D^kT_k(t)u_0 - u_0 &= \left(\frac{d}{dt}\right)^{[k]+1}\frac{1}{\Gamma(1-\{k\})}\int_0^t (t-\tau)^{-\{k\}}\int_0^\tau T_k(s)Au_0\,ds\,d\tau\\ &= \left(\frac{d}{dt}\right)^{[k]+1}\frac{1}{\Gamma(2-\{k\})}\int_0^t (t-s)^{1-\{k\}}T_k(s)Au_0\,ds\\ &= \left(\frac{d}{dt}\right)^{[k]} I^{1-\{k\}}T_k(t)Au_0 = D^{k-1}T_k(t)Au_0.\end{aligned} \tag{2.3}$$

If $k>1$, then, similarly to (2.3), we obtain the relation

$$D^{k-1}T_k(t)Au_0 - tAu_0 = D^{k-2}T_k(t)A^2u_0,$$

etc. This yields the relation

$$D^k T_k(t)u_0 = \sum_{j=0}^{[k]} \frac{t^j}{j!} A^j u_0 + I^{1-\{k\}} T_k(t) A^{[k]+1} u_0. \tag{2.4}$$

Now we check whether the function $u(t)$ defined by (2.1) is a solution of problem (1.1), (1.2). From (2.1), (1.6), and (2.3), we have

$$u'(t) = t^{k-1}p + \Gamma(k)T_k(t)Ap + D^k T_k(t)Au_0 = Au(t) + t^{k-1}p;$$

hence, the function $u(t)$ satisfies Eq. (1.1).

Relations (1.6) and (2.4) imply the validity of initial-value condition (1.2) since

$$\lim_{t\to 0} u(t) = \Gamma(k) \lim_{t\to 0} T_k(t)p + \lim_{t\to 0} D^k T_k(t)u_0 = u_0.$$

Estimate (2.2) follows from (2.1), (1.5), (1.6), and (2.4). Indeed,

$$\|u(t)\| \le M_0 \Gamma(k) t^k \exp(\omega t)\|p\| + \sum_{j=0}^{[k]} \frac{t^j}{j!}\|A^j u_0\|$$

$$+ \frac{M_0}{\Gamma(1-\{k\})} \int_0^t (t-\tau)^{-\{k\}} \tau^k \exp(\omega\tau)\, d\tau \, \|A^{[k]+1}u_0\|$$

$$\le M \exp(\omega t)\left(t^k\|p\| + t^{[k]+1}\|A^{[k]+1}u_0\|\right) + \sum_{j=0}^{[k]} \frac{t^j}{j!}\|A^j u_0\|.$$

Finally, to prove the uniqueness of the solution of the considered problem, we note that, substituting $w(t) + \Gamma(k)T_k(t)p$ for $u(t)$, we reduce (1.1), (1.2) to the problem

$$w'(t) = Aw(t), \qquad w(0) = u_0,$$

which is uniquely solvable due to [8, Theorem 1.2]. □

Remark 2.1. If an operator A generates a γ-times IS $T_\gamma(t)$, $0 < \gamma < k$, then the smoothness of the initial element in the condition of Theorem 2.1 can weaken: $u_0 \in D\left(A^{2-[1-\gamma]}\right)$.

Getting back to inverse problem (1.1)–(1.3), we note, using Theorem 2.1, that problem (1.1), (1.2) is reduced to the following one: to find a function $u(t)$ and an element $p \in D(A)$ such that

$$u(t) = \Gamma(k)T_k(t)p + D^k T_k(t)u_0, \qquad u_0 \in D(A^{2-[1-k]}). \tag{2.5}$$

From (2.5) and (1.3), we obtain the following equation for finding the unknown element p:

$$\lim_{t\to 1} I^\beta T_k(t)p = \frac{1}{\Gamma(k)}\left(u_1 - \lim_{t\to 1} I^\beta D^k T_k(t)u_0\right), \quad u_1 \in D(A).$$

In the operator form, it reads

$$Bp = q, \tag{2.6}$$

where

$$q = \frac{1}{\Gamma(k)}\left(u_1 - \lim_{t\to 1} I^\beta D^k T_k(t)u_0\right), \quad q \in D(A), \tag{2.7}$$

$$Bp = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} T_k(s)p\, ds, \quad B : D(A) \to D(A). \tag{2.8}$$

Thus, the unique solvability question for problem (1.1)–(1.3) is reduced to the question whether the bounded operator B defined by (2.8) has an inverse operator defined on a subset of the Banach space E. To answer the latter question, we obtain a (more convenient) representation of the operator B

with the aid of the resolvent $R(\lambda) = (\lambda I - A)^{-1}$, restricting the domain of the operator B to the set $D\left(A^{[k]+1}\right)$, which is dense in E.

Theorem 2.2. *Let $k, \beta > 0$, and A be a generator of a k-times IS $T_k(t)$. Then the following representation is valid for any $p \in D\left(A^{[k]+1}\right)$:*

$$Bp = \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)}{(\lambda - z)^{[k]}} R(z)(\lambda I - A)^{[k]} p \, dz, \tag{2.9}$$

where $E_{\alpha,\mu}(z) = \sum\limits_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \mu)}$ is the Mittag-Leffler function, $\lambda \in \rho(A)$, $\rho(A)$ is the resolvent set of the operator A, and $\operatorname{Re}\lambda > \sigma > \omega$.

Proof. It is proved in [9] that if the operator A generates an IS $T_k(t)$, then it has a resolvent $R(\lambda)$ in the half-plane Re $\lambda > \omega$, the estimate

$$\left\| \frac{d^n}{d\lambda^n} \left(\frac{R(\lambda)}{\lambda^k} \right) \right\| \leq \frac{Mn!}{(\text{Re } \lambda - \omega)^{n+1}}, \quad n = 0, 1, 2, \ldots, \tag{2.10}$$

is valid for the said resolvent, and

$$R(\lambda) = \lambda^k \int\limits_0^{\infty} \exp(-\lambda t) T_k(t) \, dt. \tag{2.11}$$

First, let $p \in D\left(A^l\right)$, where $l = [k] + 2$. Then $p = R^l(\lambda) p_0$, $p_0 \in E$; using the Hilbert identity, from (2.8) and (2.11), obtain that

$$\begin{aligned} Bp &= \frac{1}{\Gamma(\beta)} \int\limits_0^1 (1-s)^{\beta-1} \, ds \, \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs)}{z^k} R(z) R^l(\lambda) p_0 \, dz \\ &= \frac{1}{\Gamma(\beta)} \int\limits_0^1 (1-s)^{\beta-1} \, ds \, \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs)}{z^k} \left(\frac{R(z)p_0}{(\lambda - z)^l} - \frac{R^l(\lambda)p_0}{\lambda - z} - \ldots - \frac{R(\lambda)p_0}{(\lambda-z)^l} \right) dz \\ &= \frac{1}{\Gamma(\beta)} \int\limits_0^1 (1-s)^{\beta-1} \, ds \, \frac{1}{2\pi i} I^k \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs) R(z) p_0}{(\lambda-z)^l} \, dz \end{aligned} \tag{2.12}$$

(the integrals of all functions of the kind $\frac{\exp(zs) R^j(\lambda) p_0}{(\lambda - z)^{l+1-j}}$, $j = 1, 2, \ldots, l$, over the line Re $z = \sigma$ vanish due to the Jordan lemma).

Applying in (2.12) the semigroup property of the fractional integration, we have

$$Bp = \frac{1}{\Gamma(k+\beta)} \int\limits_0^1 (1-s)^{k+\beta-1} \, ds \, \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs) R(z) p_0}{(\lambda - z)^l} \, dz. \tag{2.13}$$

The latter integral converges absolutely. Therefore, changing the order of integrating and using the relation

$$z^{\mu} E_{1,\mu+1}(\lambda z) = \frac{1}{\Gamma(\mu)} \int\limits_0^z e^{\lambda z} (z - s)^{\mu - 1} \, ds, \quad \mu > 0, \tag{2.14}$$

(see [3, relation 1.17]), we deduce the following representation from (2.13):

$$Bp = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)R(z)p_0}{(\lambda - z)^l}\, dz$$

$$= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)R(z)\left((\lambda - z)I + (zI - A)\right)(\lambda I - A)^{l-1}p}{(\lambda - z)^l}\, dz \tag{2.15}$$

$$= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)}{(\lambda - z)^{l-1}} R(z)(\lambda I - A)^{l-1}p\, dz, \quad p \in D(A^l).$$

If $p_1 = (\lambda I - A)^{l-1}p$, then $p_1 \in D(A)$ and $p = R^{l-1}(\lambda)p_1$. Then relation (2.15) takes the following form:

$$BR^{l-1}(\lambda)p_1 = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)}{(\lambda - z)^{l-1}} R(z)p_1\, dz, \quad p_1 \in D(A). \tag{2.16}$$

The left-hand side and the right-hand side of relation (2.16) are bounded operators coinciding with each other in $D(A)$. Since $D(A)$ is dense in E, it follows that (2.16) holds for all $p_1 \in E$. Then $p = R^{l-1}(\lambda)p_1 \in D\left(A^{l-1}\right) = D\left(A^{[k]+1}\right)$. The following representation is valid for the specified p:

$$Bp = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)}{(\lambda - z)^{l-1}} R(z)(\lambda I - A)^{l-1}p\, dz$$

$$= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)}{(\lambda - z)^{l-1}} R(z)\left((\lambda - z)I + (zI - A)\right)(\lambda I - A)^{l-2}p\, dz$$

$$= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,k+\beta+1}(z)}{(\lambda - z)^{[k]}} R(z)(\lambda I - A)^{[k]}p\, dz.$$

□

Before coming to the investigation of the solvability of inverse problem (1.1)–(1.3) in the general case, we provide a criterion of its solvability in the case where the operator A is bounded.

Theorem 2.3. *Let k and β be positive real numbers and A be a bounded operator. Then problem* (1.1)–(1.3) *has a unique solution for any $u_0, u_1 \in E$ if and only if*

$$E_{1,k+\beta+1}(z) \neq 0, \quad z \in \sigma(A). \tag{2.17}$$

Proof. Let U be an open set of the complex plane such that $\sigma(A) \subset U$ and the boundary of U denoted by Ξ consists of a finite set of rectifiable curves oriented towards the positive direction. Then the operator e^{sA} is represented by the relation

$$e^{sA} = \frac{1}{2\pi i}\int\limits_{\Xi} e^{sz}R(z)\, dz$$

(see [2, pp. 608, 609]), while the operator B from (2.8) can be represented as follows:

$$Bp = \frac{1}{\Gamma(k+\beta)}\int\limits_0^1 (1-s)^{k+\beta-1}\frac{1}{2\pi i}\int\limits_{\Xi} e^{sz}R(z)p\, dz\, ds. \tag{2.18}$$

Changing the order of integration in (2.18) and taking (2.14) into account, we obtain the following representation:

$$Bp = \frac{1}{2\pi i\, \Gamma(k+\beta)} \int\limits_{\Xi} \int\limits_0^1 (1-s)^{k+\beta-1} e^{sz}\, ds R(z)p\, dz = \frac{1}{2\pi i} \int\limits_{\Xi} E_{1,k+\beta+1}(z) R(z)p.$$

This means that the operator B is an analytic function of the operator A, i.e., $B = E_{1,k+\beta+1}(A)$. By virtue of the operator spectrum mapping theorem (see [2, pp. 608, 609]), we have $\sigma(B) = E_{1,k+\beta+1}(\sigma(A))$. Thus, the origin does not belong to the spectrum of the operator B if and only if the function $E_{1,k+\beta+1}(z)$ does not vanish on the spectrum of the operator A. □

It follows from Theorem 2.3 that the location of zeroes of the function $E_{1,k+\beta+1}(z)$ determines the unique solvability of problem (1.1)–(1.3) with a bounded operator A. The following example (see [7, p. 485]) shows that, in general, this criterion is not valid even for generators of C_0-semigroups.

Example 2.1. Consider the Banach space l_2 of number sequences

$$\{u_m\} : \quad \sum_{m=-\infty}^{\infty} |u_m|^2 < +\infty$$

and define a linear unbounded operator $A\{u_m\} = \{im\, u_m\}$, $m \in \mathbb{Z}$, on the set

$$D(A) = \left\{ \{u_m\} \in l_2 : \quad \sum_{m=-\infty}^{\infty} |m\, u_m|^2 < +\infty \right\}.$$

For $m \in \mathbb{Z}$, introduce the following notation: $U(t) = \{u_m(t)\}$, $U_0 = \{(u_0)_m\}$, $U_1 = \{(u_1)_m\}$, and $P = \{p_m\}$. For $k + \beta = 1$, consider the problem

$$U'(t) = AU(t) + t^{k-1}P, \quad t \in (0, 1], \tag{2.19}$$

$$U(0) = U_0, \quad \lim_{t\to 1} I^{\beta} U(t) = U_1. \tag{2.20}$$

The spectrum $\sigma(A)$ of the operator A consists of numbers $\{im\}$, $m \in \mathbb{Z}$, while the zeroes μ_n of the function

$$E_{1,k+\beta+1}(z) = E_{1,2}(z) = \frac{\exp(z) - 1}{z}$$

are computed explicitly: $\mu_n = 2n\pi i$, $n \in \mathbb{Z} \setminus \{0\}$.

Since π is irrational, it follows that the relation $im = 2n\pi i$ is impossible for the considered values of m and n. Hence, all zeroes $\mu_n = 2n\pi i$ are regular points of the considered operator A.

Problem (2.19), (2.20) is uniquely solvable if and only if the operator B defined by (2.8) is invertible and the inverse operator is defined on $D(A)$. Here, we have

$$BP = \frac{1}{\Gamma(\beta)} \int\limits_0^1 (1-s)^{\beta-1} T_k(s) P\, ds = \left\{ \frac{1}{\Gamma(\beta)} \int\limits_0^1 (1-s)^{\beta-1} E_{1,k+1}(ims)\, ds\; p_m \right\}$$

$$= \{E_{1,k+\beta+1}(im)\, p_m\} = \{E_{1,2}(im)\, p_m\} = \left\{ \frac{\exp(im) - 1}{im}\, p_m \right\}, \quad m \in \mathbb{Z}.$$

If the operator B^{-1} is defined on $D(A)$, then $|\exp(im) - 1| \geq \delta > 0$, $m \in \mathbb{Z} \setminus \{0\}$, but it is impossible because the set $\{\exp(im)\}$, $m \in \mathbb{Z} \setminus \{0\}$, is dense in the unit circle (see [7, p. 485]).

It follows from the latter example that additional restrictions for the operator A and the values u_0 and u_1 are needed in order to obtain sufficient conditions of the unique solvability for problem (1.1)–(1.3).

Zeroes of the Mittag-Leffler function are important for the further results as well, so we provide the needed results about their location from [13]. It is proved in [13, Theorem 1] that if $k + \beta > 0$, then the zeroes can be enumerated to satisfy the following condition: all zeroes μ_n, $n \in \mathbb{Z} \setminus \{0\}$, of the function $E_{1,k+\beta+1}(z)$ such that their absolute values are large enough are simple and the following asymptotics is valid as $n \to \pm\infty$:

$$\mu_n = 2\pi n i + (k+\beta-1)\left(\ln 2\pi|n| + \frac{\pi i}{2}\operatorname{sign} n\right) - \ln\Gamma(k+\beta) + O\left(\frac{\ln|n|}{|n|}\right), \quad n \to \pm\infty. \tag{2.21}$$

Moreover, it follows from [13, Theorem 3] that all zeroes μ_n are located in the half-plane $\operatorname{Re} z < k+\beta-1$ for $0 < k+\beta < 1$, at the imaginary axis for $k+\beta = 1$, and in the half-plane $\operatorname{Re} z > k+\beta-1$ for $k+\beta > 1$. Further, we find a necessary uniqueness condition for the solution of inverse problem (1.1)–(1.3).

Theorem 2.4. *Let k and β be positive real numbers. Let A be a linear closed operator in E. Suppose that $(u(t), p)$ is a solution of inverse problem* (1.1)–(1.3)*. This solution is unique if and only if any zero of the entire function $E_{1,k+\beta+1}(z)$ is different from any eigenvalue of the operator A.*

Proof. Suppose, to the contrary, that μ_n belongs to the countable set of zeroes of the function $E_{1,k+\beta+1}(z)$ and μ_n is an eigenvalue of the operator A corresponding to an eigenvector $h_n \neq 0$.

Introduce the function $w(t) = \psi(t)h_n$ and select a scalar function $\psi(t)$ such that the function $w(t)$ satisfies Eq. (1.1) with $p = h_n$ and condition (1.2) with $u_0 = 0$. It is easy to see that the function $\psi(t)$ should satisfy the following Cauchy problem:

$$\psi'(t) = \mu_n \psi(t) + t^{k-1}, \tag{2.22}$$

$$\lim_{t\to 0} \psi(t) = 0. \tag{2.23}$$

By virtue of Theorem 2.1, problem (2.22), (2.23) has a unique solution. Taking into account (2.14), one can represent it as follows: $\psi(t) = \Gamma(k) I^k \exp(\mu_n t) = \Gamma(k) t^k E_{1,k+1}(\mu_n t)$.

Since μ_n is a zero of the function $E_{1,k+\beta+1}(z)$, we have

$$\lim_{t\to 1} I^\beta w(t) = \Gamma(k) \lim_{t\to 1} I^\beta \left(t^k E_{1,k+1}(\mu_n t)\right) = \Gamma(k) E_{1,k+\beta+1}(\mu_n) = 0.$$

Thus, the function $w(t) = \psi(t)h_n$ satisfies Eq. (1.1) with $p = h_n$ and conditions (1.2) and (1.3) for $u_0 = u_1 = 0$. This contradicts the assumption about the uniqueness of the solution because the pair $(u(t) + w(t), p + h_n)$ satisfies problem (1.1)–(1.3) as well. □

Now we find sufficient conditions of the unique solvability of problem (1.1)–(1.3). We have to assume (see Theorem 2.4) that any zero μ_n of the function $E_{1,k+\beta+1}(z)$ is different from any eigenvalue of the operator A. Also, to ensure solvability, we assume that they belong to the resolvent set $\rho(A)$. If $k+\beta > 1$, then this condition is imposed only on a finite number of zeros located to the left of the line $\{\operatorname{Re} z = \sigma > \omega\}$ because the other zeroes belong to $\rho(A)$ due to (2.21).

Theorem 2.5. *Let k and β be positive numbers, $k+\beta \le 1$, $\sigma > \omega$, A be a generator of a k-times IS $T_k(t)$, $u_0 \in D\left(A^{5+[k]-[1-k]}\right)$, and $u_1 \in D\left(A^{[k]+4}\right)$. If any zero μ_n, $n \in \mathbb{Z}\setminus\{0\}$, of the function $E_{1,k+\beta+1}(z)$ such that $\operatorname{Re}\mu_n < \sigma$ belongs to $\rho(A)$ and there exists a positive d such that*

$$\sup_{Re\ \mu_n < \sigma} \left\| \frac{R(\mu_n)}{\mu_n^k} \right\| \le d, \tag{2.24}$$

then problem (1.1)–(1.3) *has a unique solution.*

Proof. As we have noted above, the proof of the unique solvability of problem (1.1)–(1.3) (or operator equation (2.6)) is reduced to the proof of the invertibility of the bounded operator defined by (2.8) (respectively, (2.9)). Since $D(A)$ is invariant with respect to $T_k(t)$ (see [8, Proposition 1.2]) for $u_0 \in D\left(A^{5+[k]-[1-k]}\right)$ and $u_1 \in D\left(A^{[k]+4}\right)$, it follows that the right-hand side q of Eq. (2.6) belongs to $D\left(A^{[k]+4}\right)$. Let us prove that condition (2.24) implies that the operator B has an inverse operator $B^{-1} : D\left(A^{[k]+4}\right) \to D(A)$.

Since any zero μ_n of the function $E_{1,k+\beta+1}(z)$ such that $\operatorname{Re}\mu_n < \sigma$ belongs to $\rho(A)$, it follows that there exists its circular neighborhood Ω_n belonging to $\rho(A)$ as well. Let Γ be a contour in the complex plane containing of the line $\{\operatorname{Re} z = \sigma > \omega\}$ and the boundaries γ_n of the circular neighborhoods Ω_n, i.e., $\Gamma = \{\operatorname{Re} z = \sigma\} \cup \gamma_n$. The spectrum $\sigma(A)$ is located inside the domain such that the contour Γ is its boundary.

Let $\lambda \in \rho(A)$ and $\operatorname{Re}\lambda > \sigma > \omega$. Consider the bounded operator

$$\Upsilon q = \frac{-1}{2\pi i} \int\limits_{\Gamma} \frac{R(z)q\, dz}{E_{1,k+\beta+1}(z)(\lambda - z)^{[k]+3}}, \quad \Upsilon : E \to E. \tag{2.25}$$

Note that the latter integral absolutely converges by virtue of the choice of the contour Γ, estimates (2.10) and (2.24), asymptotic relation (2.21), and the following asymptotic behavior of the Mittag-Leffler function as $|z| \to \infty$ (see [3, p. 134]):

$$E_{1,\mu}(z) = z^{1-\mu}e^{z} - \sum_{j=1}^{n} \frac{1}{\Gamma(\mu - j)\, z^{j}} + O\left(|z|^{-n-1}\right), \quad |\arg z| \le \nu\pi, \quad \nu \in \left(\frac{1}{2}, 1\right), \tag{2.26}$$

$$E_{1,\mu}(z) = - \sum_{j=1}^{n} \frac{1}{\Gamma(\mu - j)\, z^{j}} + O\left(|z|^{-n-1}\right), \quad \nu\pi \le |\arg z| \le \pi. \tag{2.27}$$

Indeed,

$$\frac{1}{2\pi i} \int\limits_{\cup\gamma_n} \frac{R(z)q\, dz}{E_{1,k+\beta+1}(z)(\lambda - z)^{[k]+3}} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{R\left(\mu_n\right)q}{E'_{1,k+\beta+1}\left(\mu_n\right)\left(\lambda - \mu_n\right)^{[k]+3}}. \tag{2.28}$$

It is known from [3, (1.5), p. 118]) that

$$E'_{1,k+\beta+1}\left(\mu_n\right) = \mu_n^{-1}\left(E_{1,k+\beta}\left(\mu_n\right) - (k+\beta)E_{1,k+\beta+1}\left(\mu_n\right)\right).$$

Hence, taking into account asymptotic relations (2.26) and (2.21), we obtain

$$E'_{1,k+\beta+1}\left(\mu_n\right) = \frac{1}{\mu_n}\left(\frac{\mu_n^{1-k-\beta}(2\pi|n|)^{k+\beta-1}e^{i\operatorname{Im}\mu_n}}{\Gamma(k+\beta)} - \frac{1}{\Gamma(k+\beta-1)\mu_n}\right.$$

$$\left. - \frac{(k+\beta)\mu_n^{-k-\beta}(2\pi|n|)^{k+\beta-1}e^{i\operatorname{Im}\mu_n}}{\Gamma(k+\beta)} - \frac{k+\beta}{\Gamma(k+\beta)\mu_n} + O\left(\frac{1}{|\mu_n|^2}\right)\right).$$

Thus, we have

$$\left|E'_{1,k+\beta+1}\left(\mu_n\right)\right| = \frac{1}{|\mu_n|}\left(\frac{1}{\Gamma(k+\beta)} + O\left(\frac{1}{|\mu_n|}\right)\right). \tag{2.29}$$

By virtue of relation (2.29), condition (2.24), and asymptotic expansion (2.21), series (2.28) absolutely converges. Therefore, the integral over $\cup\gamma_n$ absolutely converges as well.

The convergence of integral (2.25) over the line $\{\operatorname{Re} z = \sigma\}$ follows from (2.24) and (2.27).

Let $q \in D\left(A^{[k]+1}\right)$ and $\sigma < \sigma_1 < \operatorname{Re}\lambda$. Then, substituting (2.9) in (2.25) and applying the Hilbert identity, we obtain

$$\begin{aligned}\Upsilon Bq &= \frac{-1}{2\pi i} \int\limits_{\Gamma} \frac{R(z)\, dz}{E_{1,k+\beta+1}(z)(\lambda - z)^{[k]+3}}\, \frac{1}{2\pi i} \int\limits_{\sigma_1-\infty}^{\sigma_1+\infty} \frac{E_{1,k+\beta+1}(\xi)R(\xi)(\lambda I - A)^{[k]}q\, d\xi}{(\lambda - \xi)^{[k]}} \\ &= \frac{-1}{(2\pi i)^2} \int\limits_{\Gamma} \int\limits_{\sigma_1-\infty}^{\sigma_1+\infty} \frac{E_{1,k+\beta+1}(\xi)}{E_{1,k+\beta+1}(z)(\lambda - z)^{[k]+3}(\lambda - \xi)^{[k]}}\, \frac{R(z) - R(\xi)}{\xi - z}(\lambda I - A)^{[k]}q\, d\xi dz.\end{aligned} \tag{2.30}$$

The integral in (2.30) absolutely converges; therefore, changing the order of integrating, we have

$$\Upsilon Bq = \frac{-1}{(2\pi i)^2}\int\limits_{\Gamma}\frac{R(z)(\lambda I - A)^{[k]}q\,dz}{E_{1,k+\beta+1}(z)(\lambda - z)^{[k]+3}}\int\limits_{\sigma_1-\infty}^{\sigma_1+\infty}\frac{E_{1,k+\beta+1}(\xi)\,d\xi}{(\lambda-\xi)^{[k]}(\xi - z)}$$
$$-\frac{1}{(2\pi i)^2}\int\limits_{\sigma_1-\infty}^{\sigma_1+\infty}\frac{E_{1,k+\beta+1}(\xi)R(\xi)(\lambda I - A)^{[k]}q\,d\xi}{(\lambda-\xi)^{[k]}}\int\limits_{\Gamma}\frac{dz}{E_{1,k+\beta+1}(z)(\lambda - z)^{[k]+3}(\xi - z)}. \tag{2.31}$$

The internal integral of the second term of (2.31) is equal to zero by virtue of the choice of the contour Γ and the Jordan lemma. To compute the integrals of the first term, we use the integral Cauchy theorem. Thus, the following relation holds for any $q \in D\left(A^{[k]+1}\right)$:

$$\Upsilon Bq = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{R(z)(\lambda I - A)^{[k]}q\,dz}{(z-\lambda)^{2[k]+3}} = \frac{R^{(2[k]+2)}(\lambda)(\lambda I - A)^{[k]}q}{(2[k]+2)!}$$
$$= R^{2[k]+3}(\lambda)(\lambda I - A)^{[k]}q = R^{[k]+3}(\lambda)q.$$

The commuting operators Υ, B, and $R(\lambda)$ are bounded and the domain $D\left(A^{[k]+1}\right)$ is dense in E. Therefore, the relation $\Upsilon Bq = R^{[k]+3}(\lambda)q$ holds for $q \in E$ and $\Upsilon B : E \to D\left(A^{[k]+3}\right)$ as well. This means that the operator $B^{-1}q = (\lambda I - A)^{[k]+3}\Upsilon q$ is inverse for B if $q \in D\left(A^{[k]+3}\right)$. Indeed, we have

$$BB^{-1}q = B(\lambda I - A)^{[k]+3}\Upsilon q = R^{[k]+3}(\lambda)(\lambda I - A)^{[k]+3}q = q, \qquad q \in D(A^{[k]+3}),$$
$$B^{-1}Bq = (\lambda I - A)^{[k]+3}\Upsilon Bq = q, \qquad q \in E.$$

Regarding solutions of problem (1.1)–(1.3), any p belonging to $D(A)$ is represented as $p = (\lambda I - A)^{[k]+3}\Upsilon q$, where the element $q \in D\left(A^{[k]+4}\right)$ is defined by relation (2.7), the operator Υ is defined by relation (2.25), $\lambda \in \rho(A)$, $\operatorname{Re}\lambda > \sigma > \omega$, and the function $u(t)$ can be found via relation (2.5). ☐

If $k + \beta > 1$, then condition (2.24) is satisfied because the number of zeroes such that $\operatorname{Re}\mu_n < \sigma$ is finite. Therefore, formulating the next theorem, we may omit condition (2.24). Taking this into account, we prove the next theorem in the same way as Theorem 2.5.

Theorem 2.6. *Let $k > 0$, $\beta > 0$, $k + \beta > 1$, $\sigma > \omega$, A be a generator of a k-times IS $T_k(t)$, $u_0 \in D\left(A^{5+[k]-[1-k]}\right)$, and $u_1 \in D\left(A^{[k]+4}\right)$. If any zero μ_n, $n = 1, 2, \dots, n_0$, of the function $E_{1,k+\beta+1}(z)$ such that $\operatorname{Re}\mu_n < \sigma$ belongs to $\rho(A)$, then problem (1.1)–(1.3) has a unique solution.*

Remark 2.2. Suppose that the operator A is a generator of a semigroup $T(t)$ strongly differentiable for $t > 0$. Then its resolvent set contains the half-plane $\{\lambda : \operatorname{Re}\lambda \geq \omega\}$ and a set of the kind $\{\lambda : \operatorname{Re}\lambda \geq a - b\ln|\operatorname{Im}\lambda|\}$ (see [11]). Then the assertion of Theorem 2.6 takes place for $k + \beta \geq 1$; if $1 - k - \beta < b$, then it is valid for $k + \beta < 1$ as well because the spectrum of the operator A can contain only a finite number of zeroes μ_n. If A is a generator of an analytic semigroup, then Theorem 2.6 holds for all positive k and β. Also, taking into account Remark 2.1, one can decrease the smoothness of the elements u_0 and u_1.

Remark 2.3. It is easy to see that Theorems 2.5 and 2.6 about unique solvability hold even if $\beta = 0$ in condition (1.3); then $I^\beta u(t) = u(t)$ and the additional condition is the setting of a final value: $u(1) = u_1$.

Remark 2.4. It is possible to reduce equations of kind

$$u'(t) + \frac{l}{t}\,u(t) = Au(t) + t^m p, \qquad l + m + 1 > 0,$$

to the equation

$$v'(t) = Av(t) + t^{l+m}p,$$

using the following change of variables: $u(t) = t^{-l}v(t)$. Thus, assuming that the operator A is a generator of an $l+m+1$ times IS, we can formulate a similar inverse problem and find conditions for its unique solvability.

3. Inverse Problems with Stationary Inhomogeneous Terms

In the above papers [4, 5, 12, 14, 15] devoted to the inverse problem for Eq. (1.4), the case where A is a generator of an IS is not considered. Now we formulate the corresponding inverse problem and results about its solvability.

Consider the inverse problem

$$u'(t) = Au(t) + p, \quad 0 \le t \le 1, \tag{3.1}$$

$$u(0) = u_0, \tag{3.2}$$

$$\lim_{t\to 1} I^{\beta} u(t) = u_1, \tag{3.3}$$

where $\beta \ge 0$ (I^{β} is the identity operator if $\beta = 0$) and A is the generator of an IS $T_{\gamma}(t)$, $\gamma \ge 0$.

The case where $\gamma = \beta = 0$ and A is a generator of a C_0-semigroup is well-known from [4, 5, 12, 14, 15]; it is a "limiting" case for problem (3.1)–(3.3). The case where $\gamma = 1$ and $\beta > 0$ is covered by Theorem 2.6. Consider problem (3.1)–(3.3) with weaker assumptions on the operator A.

By virtue of [9, Theorem 3], problem (3.1), (3.2) is reduced to the problem to find a function $u(t)$ and an element $p \in D\left(A^{1-[1-\gamma]}\right)$ such that the following relation is valid:

$$u(t) = D^{\gamma}\left(T_{\gamma}(t)u_0 + \int_0^t T_{\gamma}(\tau)p\, d\tau\right), \qquad u_0 \in D\left(A^{2-[1-\gamma]}\right). \tag{3.4}$$

From relation (3.4) and condition (3.3), we obtain the following equation to find p from:

$$\lim_{t\to 1} I^{\beta} D^{\gamma} \int_0^t T_{\gamma}(\tau)p\, d\tau = u_1 - \lim_{t\to 1} I^{\beta} D^{\gamma} T_{\gamma}(t) u_0, \quad u_1 \in D(A);$$

it has the following operator form:

$$B_0 p = q_0,$$

where

$$q_0 = u_1 - \lim_{t\to 1} I^{\beta} D^{\gamma} T_{\gamma}(t) u_0, \qquad q_0 \in D(A),$$

$$B_0 p = \lim_{t\to 1} I^{\beta} D^{\gamma} \int_0^t T_{\gamma}(\tau)p\, d\tau, \quad B_0\colon D(A^{1-[1-\gamma]}) \to D(A). \tag{3.5}$$

The unique solvability for problem (3.1)–(3.3) is equivalent to the existence of an inverse operator for the operator B_0, defined on $D\left(A^{4+[\gamma]-[1-\gamma]}\right)$.

Theorem 3.1. *Let γ and β be nonnegative real numbers, while A is a generator of a γ-times IS $T_{\gamma}(t)$. Then the following representation is valid for any $p \in D\left(A^{[\gamma]+1}\right)$:*

$$B_0 p = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,\beta+2}(z)}{(\lambda - z)^{[\gamma]}} R(z)(\lambda I - A)^{[\gamma]} p\, dz, \tag{3.6}$$

where $\lambda \in \rho(A)$ and $\operatorname{Re}\lambda > \sigma > \omega$.

Proof. First, let $p \in D\left(A^l\right)$, where $l = [\gamma] + 2$. Then $p = R^l(\lambda)p_0$ and $p_0 \in E$. From (3.5) and (2.11), we have

$$B_0 p = \lim_{t\to 1} I^\beta D^\gamma \int_0^t \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs)}{z^\gamma} R(z) R^l(\lambda) p_0 \, dzds$$

$$= \lim_{t\to 1} I^\beta D^\gamma \int_0^t \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zs)}{z^\gamma (\lambda - z)^l} R(z) p_0 \, dzds \qquad (3.7)$$

$$= \lim_{t\to 1} \frac{1}{2\pi i} I^{\beta+1} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(zt)}{(\lambda - z)^l} R(z) p_0 \, dz = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,\beta+2}(z)}{(\lambda - z)^{l-1}} R(z)(\lambda I - A)^{l-1} p \, dz$$

(this relation is obtained in the same way as relations (2.12), (2.13), and (2.15)).

From relation (3.7), we deduce representation (3.6) for $p \in D\left(A^{[\gamma]+1}\right)$ (similarly to the proof of Theorem 2.2). $\square$

Theorem 3.2. *Let $\beta \geq 0$, A be a linear closed operator in E, and problem* (3.1)–(3.3) *have a solution. If this solution is unique, then the set of zeroes of the entire function $E_{1,\beta+2}(z)$ does not meet the set of eigenvalues of the operator A.*

The proof of Theorem 3.2 is similar to the proof of Theorem 2.4.

Theorem 3.3. *Suppose that $\beta = 0$, $\gamma \geq 0$, $\sigma > \omega$, A is a generator of a γ-times IS $T_\gamma(t)$, $u_0 \in D\left(A^{5+[\gamma]-2[1-\gamma]}\right)$, and $u_1 \in D\left(A^{4+[\gamma]-[1-\gamma]}\right)$. If any zero $\nu_n = 2\pi n i$, $n \in \mathbb{Z} \setminus \{0\}$, of the function $E_{1,2}(z) = (\exp(z) - 1)/z$ belongs to $\rho(A)$ and there exists a positive d such that*

$$\sup_{Re\ \nu_n < \sigma} \left\| \frac{R(\nu_n)}{\nu_n^\gamma} \right\| \leq d, \qquad (3.8)$$

then problem (3.1)–(3.3) *has a unique solution.*

Theorem 3.4. *Suppose that $\beta > 0$, $\gamma \geq 0$, $\sigma > \omega$, A is a generator of a γ-times IS $T_\gamma(t)$, $u_0 \in D\left(A^{5+[\gamma]-2[1-\gamma]}\right)$, and $u_1 \in D\left(A^{4+[\gamma]-[1-\gamma]}\right)$. If any zero ν_n, $n = 1, 2, \ldots, n_0$, of the function $E_{1,\beta+2}(z)$ such that $\operatorname{Re} \nu_n < \sigma$ belongs to $\rho(A)$, then problem* (3.1)–(3.3) *has a unique solution.*

Proof. Proofs of Theorems 3.3 and 3.4 are similar to proofs of Theorems 2.5 and 2.6; the corresponding relations take the following form:

$$\Upsilon_0 q_0 = \frac{-1}{2\pi i} \int_{\Gamma_0} \frac{R(z) q_0 \, dz}{E_{1,\beta+2}(z)(\lambda - z)^{[\gamma]+3}}, \quad \Upsilon_0 : E \to E,$$

where Γ_0 is a contour similar to the contour Γ: it surrounds the zeroes ν_n of the Mittag-Leffler function $E_{1,\beta+1}(z)$;

$$\Upsilon_0 B_0 q_0 = R^{[\gamma]+3}(\lambda) q_0, \quad q_0 \in D\left(A^{1-[1-\gamma]}\right), \quad \Upsilon_0 B_0 : D\left(A^{1-[1-\gamma]}\right) \to D\left(A^{[\gamma]+3}\right);$$

$$B_0^{-1} q_0 = (\lambda I - A)^{[\gamma]+3} \Upsilon_0 q_0, \quad q_0 \in D\left(A^{[\gamma]+3}\right);$$

$$B_0 B_0^{-1} q_0 = B_0 (\lambda I - A)^{[\gamma]+3} \Upsilon_0 q_0 = q_0, \quad q_0 \in D\left(A^{[\gamma]+3}\right);$$

$$B_0^{-1} B_0 q_0 = (\lambda I - A)^{[\gamma]+3} \Upsilon_0 B_0 q_0 = q_0, \quad q_0 \in D\left(A^{1-[1-\gamma]}\right);$$

$$p = (\lambda I - A)^{[\gamma]+3} \Upsilon_0 q_0, \quad q_0 \in D\left(A^{4+[\gamma]-[1-\gamma]}\right), \quad p \in D\left(A^{1-[1-\gamma]}\right);$$

finally, the function $u(t)$ is defined by relation (3.4).

To conclude, we note that if $\gamma = 0$, then Theorem 3.3 is [4, Theorem 4]. We know from [4] that the belonging of the points $\nu_n = 2\pi ni$ to the resolvent set is not a sufficient condition of the solvability. In [4, Theorem 1], condition (3.8) is replaced (for $\gamma = 0$) by the following condition: for any $p \in E$, the series $\sum\limits_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} R(\nu_n)\, p$ is summable in average.

Thus, the conditions formulated in [5] for the case where $\beta = \gamma = 0$ and $u_0, u_1 \in D(A)$ are necessary and sufficient for the unique solvability of problem (3.1)–(3.3).

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REFERENCES

1. W. Arendt, "Vector valued Laplace transforms and Cauchy problems," *Israel J. Math.*, **59**, 327–352 (1987).
2. N. Danford and Dž. Švarc, *Linear operators. Part I: General theory* [Russian translation], Inostr. Lit., Moscow (1962).
3. M. M. Dzhrbashyan, *Integral Transformations and the Presentation of Functions in a Complex Domain* [in Russian], Nauka, Moscow (1966).
4. Yu. S. Èĭdel'man, "Two-point boundary value problem for a differential equation with a parameter," *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, No. 4, 15–18 (1983).
5. Yu. S. Èĭdel'man, "An inverse problem for an evolution equation," *Math. Notes*, **49**, No. 5-6, 535–540 (1991).
6. A. V. Glushak, "On a relation between the integrated operator cosine function and the operator Bessel function," *Differ. Equ.*, **42**, No. 5, 619–626 (2006).
7. E. Hille and R. Phillips, *Functional Analysis and Semigroups* [Russian translation], Inostr. Lit., Moscow (1962).
8. I. V. Mel'nikova and A. I. Filinkov, "Integrated semigroups and C-semigroups. Well-posedness and regularization of operator-differential problems," *Russian Math. Surveys*, **49**, No. 6, 115–155 (1994).
9. M. Mijatović, S. Pilipović, and F. Vejzović, "α-Times integrated semigroups ($\alpha \in \mathbb{R}^+$)," *J. Math. Anal. Appl.*, **210**, 790–803 (1997).
10. F. Neubrander, "Integrated semigroups and their applications to the abstract Cauchy problem," *Pacific J. Math.*, **135**, 111–155 (1988).
11. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York (1983).
12. A. I. Prilepko, D. G. Orlovsky, and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, Basel–New York (2000).
13. A. M. Sedletskiĭ, "On the zeros of a function of Mittag-Leffler type," *Math. Notes*, **68**, No. 5-6, 602–613 (2000).
14. I. V. Tikhonov and Yu. S. Èĭdel'man, "An inverse problem for a differential equation in a Banach space and the distribution of zeros of an entire function of Mittag-Leffler type," *Differ. Equ.*, **38**, No. 5, 669–677 (2002).
15. I. V. Tikhonov and Yu. S. Èĭdel'man, "A uniqueness criterion in an inverse problem for an abstract differential equation with a nonstationary inhomogeneous term," *Math. Notes*, **77**, No. 1-2, 246–262 (2005).