

Initial Value Problems for the Abstract Legendre Equation with Two Parameters

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Abstract—Using the concept of a fractional integral of a function over another function, transformation operators are constructed that make it possible to prove the solvability of initial value problems for the abstract singular Legendre equation containing two parameters. Examples are given.

Keywords: initial value problem, singular Legendre differential equation, fractional integral of function over another function, operator of motion with respect to parameter, transformation operator, solvability

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INTRODUCTION

The study of a number of physical processes is based on the solution of equations containing the Laplace operator that, by separating variables in curvilinear coordinate systems, lead to differential equations with a singularity. In the presence of a certain symmetry, these equations are transformed into the Euler–Poisson–Darboux (EPD) or Legendre equations.

Initial problems for the classical EPD equation have been investigated in many works (see the introductions in the monographs [1, 2]). In [3, 4], the present author obtained results for abstract equations.

In this paper, in a Banach space E for $t > 0$, we consider the Legendre equation

$$u''(t) + \frac{(\sinh^k t \sinh^q(2t))'}{\sinh^k t \sinh^q(2t)} u'(t) + \left(\frac{k}{2} + q\right)^2 u(t) = Au(t), \quad (1)$$

containing two real parameters $k, q \in \mathbb{R}$ and a linear closed operator A .

A solution to Eq. (1) is a function $u(t)$ ranging in $D(A)$, twice continuously differentiable and satisfying this equation on the set $(0, \infty)$.

The differential operator on the left-hand side in Eq. (1) is the radial part of the Laplace–Beltrami operator in a two-point homogeneous space and in a symmetric space of rank one (see [2, p. 111]). For $q = 0$, Eq. (1) is the Legendre equation containing one parameter k . A review of publications and studies of a number of uniquely solvable problems for the abstract Legendre equation with one parameter can be found in the author's papers [5–8]. The so-called incomplete Cauchy problem for such an equation, when a solution tending to zero as $t \rightarrow \infty$ is sought and the second initial condition at $t = 0$ is not specified, was considered in the paper [8]. Previously, the incomplete Cauchy problem for the abstract wave equation was studied in [9, 10] and for the Euler–Poisson–Darboux equation, in [11].

Equation (1) is not included in the family of singular differential equations considered in [7] and therefore requires separate study.

One of the methods for studying differential equations is the method of transformation operators. Recall that a nonzero operator $\mathbb{T}$ is called a *transformation operator* intertwining operators L and M if the relation $\mathbb{T}L = M\mathbb{T}$ is satisfied. For a detailed review of the theory of transformation operators see [2, Ch. 2; 12]. In the present paper, the method of transformation operators is applied to establish the solvability of initial value problems for Eq. (1) and explicit formulas for their solutions are obtained.

1. THE TRANSFORMATION OPERATOR BY THE FIRST INDEX k FOR THE CAUCHY PROBLEM AND THE LEGENDRE OPERATOR FUNCTION

In the future, we will need the concepts of the left-sided fractional integral of the function $u(t)$ with respect to a specially chosen function $g(t)$,

$$I_{0+,g}^\alpha u(t) \equiv \left(\frac{d}{dg(t)} \right)_+^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (g(t) - g(s))^{\alpha-1} g'(s) u(s) ds, \quad \alpha > 0, \quad (2)$$

and the left-sided fractional derivative $(\frac{d}{dg(t)})_+^\alpha u(t)$ with respect to the function $g(t)$ (see [13, p. 248]).

The operator on the left-hand side in Eq. (1) is denoted by

$$L_{k,q}u(t) := u''(t) + \frac{(\sinh^k t \sinh^q 2t)'}{\sinh^k t \sinh^q 2t} u'(t) + \left(\frac{k}{2} + q \right)^2 u(t). \quad (3)$$

We will also use the representations

$$\begin{aligned} L_{k,q}u(t) &= u''(t) + \frac{(\sinh^{k+q} t \cosh^q t)'}{\sinh^{k+q} t \cosh^q t} u'(t) + \left(\frac{k}{2} + q \right)^2 u(t) \\ &= u''(t) + ((k+q) \coth t + q \tanh t) u'(t) + \left(\frac{k}{2} + q \right)^2 u(t), \end{aligned} \quad (4)$$

obtained from (3) after elementary transformations.

Introduce the operator

$$\Phi_k u(t) = \sinh^{k-1} t u(t). \quad (5)$$

For $t > 0$ for sufficiently smooth functions $u(t)$, the article [7] established the relations

$$L_{2-k,0} \Phi_k u(t) = \Phi_k L_{k,0} u(t), \quad (6)$$

$$L_{k+2\alpha,0} \left(\frac{1}{\sinh t} \frac{d}{dt} \right)_+^\alpha u(t) = \left(\frac{1}{\sinh t} \frac{d}{dt} \right)_+^\alpha L_{k,0} u(t), \quad \alpha \in \mathbb{R}. \quad (7)$$

Note that formula (2) with $g(t) = \cosh t$ is used in (7).

Relation (7) is valid for functions satisfying the condition $u(0) = 0$, which can be removed for $\alpha \in \mathbb{N}$. Thus, the operators Φ_k and $(\sinh^{-1} t \frac{d}{dt})_+^\alpha$ perform the movement of the operator $L_{k,0}$ with respect to the first index. Let

$$\mathbb{T}_{k,0} u(t) = \frac{2^{k/2} \Gamma(k/2 + 1/2)}{\sqrt{\pi}} \Phi_{2-k} \left(\frac{1}{\sinh t} \frac{d}{dt} \right)_+^{-k/2} \frac{u(t)}{\sinh t}; \quad (8)$$

then, as established in [8], with $k > 0$ for functions satisfying the condition $u'(0) = 0$, from relations (6) and (7) one has the relation

$$L_{k,0} \mathbb{T}_{k,0} u(t) = \mathbb{T}_{k,0} u''(t), \quad (9)$$

$$\begin{aligned}\mathbb{T}_{k,0} u(t)|_{t=0} &= u(0), \\ (\mathbb{T}_{k,0} u(t))'|_{t=0} &= 0.\end{aligned}\quad (10)$$

Relation (9) means that the transformation operator $\mathbb{T}_{k,0}$ has been constructed, intertwining the operators $L_{k,0}$ and $\frac{d^2}{dt^2}$.

For $k > 0$, $q = 0$ we consider Eq. (1), where the operator A is a generator of the cosine operator function (COF) $C(t; A)$. The terminology and examples of COF generators can be found in [14, Ch. 2.8; 15].

Relations (9) and (10) actually mean that the function $u_{k,0}(t) = \mathbb{T}_{k,0} C(t; A)u_0$ satisfies Eq. (1) for $q = 0$ and the initial conditions

$$\begin{aligned}u(0) &= u_0, \\ u'(0) &= 0.\end{aligned}\quad (11)$$

Theorem 1 [7]. *Let $k > 0$, $q = 0$, $u_0 \in D(A)$ and let an operator A be a generator of the COF $C(t; A)$. Then the function*

$$u_{k,0}(t) = \mathbb{T}_{k,0} C(t; A)u_0 = \frac{2^{k/2}\Gamma(k/2 + 1/2)}{\sqrt{\pi}\Gamma(k/2)} \sinh^{1-k} t \int_0^t (\cosh t - \cosh s)^{k/2-1} C(s; A)u_0 ds \quad (12)$$

satisfies Eq. (1) and the initial conditions (11).

The operator function $P_k(t; A) = \mathbb{T}_{k,0} C(t; A)$ is called the Legendre operator function (LOF) in [5, 7]. In the special case where the operator $A = (a + 1/2)^2$, $a \in \mathbb{R}$ is the operator of multiplication by a number, the LOF $P_k(t; A)$ is expressed via the associated Legendre function of the first kind $P_a^\beta(\cdot)$ (see [16, p. 661]),

$$P_k(t; A) = \frac{\Gamma(1 - \beta) \sinh^\beta t}{2^\beta} P_a^\beta(\cosh t), \quad \beta = \frac{1 - k}{2}.$$

2. TRANSFORMATION OPERATOR BY INDICES k, q FOR THE CAUCHY PROBLEM

Let us further construct a transformation operator, using which and given the known solution $u_{k,0}(t) = \mathbb{T}_{k,0} C(t; A)u_0$ for $q = 0$, we determine the solution $u_{k,q}(t)$ to Eq. (1) for $q > 0$.

Let

$$\Psi_{k,q} u(t) = \sinh^k t \sinh^{q-1}(2t) u(t). \quad (13)$$

In this case, to determine the transformation operator in relation (2) for the left-sided fractional integral of the function $u(t)$, one should choose the function $g(t) = \frac{\cosh(2t)}{2}$, and instead of relations (6), (7), one should, accordingly, use the relations

$$L_{-k,2-q} \Psi_{k,q} u(t) = \Psi_{k,q} L_{k,q} u(t), \quad (14)$$

$$L_{k,q+2\alpha} \left(\frac{1}{\sinh(2t)} \frac{d}{dt} \right)_+^\alpha u(t) = \left(\frac{1}{\sinh(2t)} \frac{d}{dt} \right)_+^\alpha L_{k,q} u(t), \quad \alpha \in \mathbb{R}. \quad (15)$$

Relation (15) is valid for functions satisfying the condition $u(0) = 0$, which can be removed for $\alpha \in \mathbb{N}$. Thus, the operators $\Psi_{k,q}$ and $(\sinh^{-1}(2t) \frac{d}{dt})_+^\alpha$ implement the movement of the operator $L_{k,q}$ with respect to the second index. Let

$$\mathbb{T}_{0,q} u(t) = \frac{2^q \Gamma(q/2 + 1/2)}{\sqrt{\pi}} \Psi_{0,2-q} \left(\frac{1}{\sinh(2t)} \frac{d}{dt} \right)_+^{-q/2} \frac{u(t)}{\sinh(2t)}; \quad (16)$$

then for $k > 0$ and $q > 0$ on functions satisfying the condition $u'(0) = 0$, for the operator $\mathbb{T}_{0,q}$ defined by relation (16) relations (14) and (15) imply the equality

$$L_{k,q} \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t u(t) = \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t L_{k,0} u(t). \quad (17)$$

Introduce the operator

$$\begin{aligned} \mathbb{T}_{k,q} &= \mu_{k,q} \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t \mathbb{T}_{k,0}, \\ \mu_{k,q} &= \frac{\sqrt{\pi} \Gamma(k/2 + q/2 + 1/2)}{\Gamma(k/2 + 1/2) \Gamma(q/2 + 1/2)}, \end{aligned} \quad (18)$$

which is a transformation operator intertwining the operators $L_{k,q}$ and $\frac{d^2}{dt^2}$, i.e., on functions satisfying the condition $u'(0) = 0$, one has the relation

$$L_{k,q} \mathbb{T}_{k,q} u(t) = \mathbb{T}_{k,q} u''(t). \quad (19)$$

Indeed, by virtue of relations (9) and (16)–(18) we have

$$\begin{aligned} L_{k,q} \mathbb{T}_{k,q} u(t) &= \mu_{k,q} L_{k,q} \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t \mathbb{T}_{k,0} u(t) = \mu_{k,q} \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t L_{k,0} \mathbb{T}_{k,0} u(t) \\ &= \mu_{k,q} \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t \mathbb{T}_{k,0} u''(t) = \mathbb{T}_{k,q} u''(t). \end{aligned}$$

Just as relations (6), (7), and (9), using the representation (4) for the operator $L_{k,q}$, one can establish relations (14), (15), and (17) by direct verification, which can be cumbersome in some cases. In Sec. 3, we will show that (like the operators $\mathbb{T}_{k,0}$ and $\mathbb{T}_{0,q}$) the transformation operator $\mathbb{T}_{k,q}$ defined by relation (18) will preserve the initial conditions, i.e.,

$$\begin{aligned} \mathbb{T}_{k,q} u(t) \big|_{t=0} &= u(0), \\ (\mathbb{T}_{k,q} u(t))' \big|_{t=0} &= 0. \end{aligned}$$

3. CAUCHY PROBLEM

Taking into account the established relation (19), we construct a solution to the Cauchy problem (1), (11). For $u_0 \in D(A)$, we introduce the function

$$\begin{aligned} u_{k,q}(t) &= \mathbb{T}_{k,q} C(t; A) u_0 = \mu_{k,q} \sinh^{-k} t \mathbb{T}_{0,q} \sinh^k t \mathbb{T}_{k,0} C(t; A) u_0 \\ &= \frac{4\Gamma(q/2 + 1/2)\mu_{k,q}}{\sqrt{\pi} \Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \int_0^t (\cosh(2t) - \cosh(2s))^{q/2-1} \sinh^k s \mathbb{T}_{k,0} C(s; A) u_0 ds \\ &= \frac{2^{k/2+2} \Gamma(k/2 + q/2 + 1/2)}{\sqrt{\pi} \Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \int_0^t (\cosh(2t) - \cosh(2s))^{q/2-1} \\ &\quad \times \sinh s \int_0^s (\cosh s - \cosh \tau)^{k/2-1} C(\tau; A) u_0 d\tau ds \\ &= \frac{2^{k/2+2} \Gamma(k/2 + q/2 + 1/2)}{\sqrt{\pi} \Gamma(k/2) \Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \\ &\quad \times \int_0^t C(\tau; A) u_0 \int_\tau^t \sinh s (\cosh s - \cosh \tau)^{k/2-1} (\cosh(2t) - \cosh(2s))^{q/2-1} d\tau ds. \end{aligned} \quad (20)$$

Relation (19) means that the function (20) satisfies Eq. (1). Indeed,

$$L_{k,q}u_{k,q}(t) = L_{k,q} \mathbb{T}_{k,q} C(t; A)u_0 = \mathbb{T}_{k,q} C''(t)u_0 = A \mathbb{T}_{k,q} C(t; A)u_0 = Au_{k,q}(t).$$

The inner integral in the representation (20) can be calculated (see formula (3.20) in [17]); then

$$u_{k,q}(t) = \frac{2^{2-k/2}\Gamma(k/2 + q/2 + 1/2)}{\sqrt{\pi}\Gamma(k/2 + q/2) \cosh^{k/2} t \sinh^k t \sinh^{q-1}(2t)} \\ \times \int_0^t (\cosh(2t) - \cosh(2\tau))^{k/2+q/2-1} {}_2F_1\left(\frac{k}{2} + q - 1, \frac{k}{2}; \frac{k+q}{2}; \frac{\cosh t - \cosh \tau}{2 \cosh t}\right) C(\tau; A)u_0 d\tau,$$

where ${}_2F_1(\cdot)$ is the hypergeometric function.

Let us show further that the function (20) satisfies the initial conditions (11). Note that as $t \rightarrow 0$, the operator $\mathbb{T}_{k,q}$ acts on functions as a Poisson-type transformation operator (see [2, pp. 2.1.2, 2.1.3]),

$$\mathbb{P}_{k,q}C(t; A)u_0 = \frac{2\Gamma(k/2 + q/2 + 1/2)}{\Gamma(k/2 + 1/2)\Gamma(q/2)t^{k+q-1}} \int_0^t s^k(t^2 - s^2)^{q/2-1} C(s; A)u_0 ds,$$

which possesses the properties that we need,

$$\begin{aligned} \mathbb{P}_{k,q}u(t)|_{t=0} &= u(0), \\ (\mathbb{P}_{k,q}u(t))'|_{t=0} &= 0. \end{aligned}$$

Expanding the integrands in the representation (20) into the corresponding series and using integrals 2.2.3.1 and 2.2.4.8 in [18], we have

$$\begin{aligned} \lim_{t \rightarrow 0} u_{k,q}(t) &= \lim_{t \rightarrow 0} \frac{2^{q+1}\Gamma(k/2 + q/2 + 1/2)}{\sqrt{\pi}\Gamma(k/2)\Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \int_0^t s(t^2 - s^2)^{q/2-1} \int_0^s (s^2 - \tau^2)^{k/2-1} u_0 d\tau ds \\ &= \lim_{t \rightarrow 0} \frac{2\Gamma(k/2 + q/2 + 1/2)}{\Gamma(k/2 + 1/2)\Gamma(q/2)t^{k+q-1}} \int_0^t s^k(t^2 - s^2)^{q/2-1} u_0 ds = \lim_{t \rightarrow 0} \mathbb{P}_{k,q}C(t; A)u_0 = u_0. \end{aligned}$$

Thus, the following assertion holds.

Theorem 2. *Let $k > 0$, $q > 0$, $u_0 \in D(A)$ and let the operator A be a generator of the COF $C(t; A)$. Then the function (20) is the unique solution to problem (1), (11).*

Proof. To prove the theorem, taking into account what was stated above in Sec. 3, it remains to establish the uniqueness of the solution to problem (1), (11). Let $u_1(t)$ and $u_2(t)$ be two solutions to the problem. Consider the function of two variables

$$w(t, s) = f\left(C(s; A)(u_1(t) - u_2(t))\right),$$

where $f \in E^*$ (E^* is the dual space), $t, s \geq 0$, which obviously satisfies the equation

$$\frac{\partial^2 w(t, s)}{\partial t^2} + \frac{(\sinh^k t \sinh^q(2t))'}{\sinh^k t \sinh^q(2t)} \frac{\partial w(t, s)}{\partial t} + \left(\frac{k}{2} + q\right)^2 w(t, s) = \frac{\partial^2 w(t, s)}{\partial s^2}, \quad t, s > 0, \quad (21)$$

and the conditions

$$\lim_{t \rightarrow 0} w(t, s) = \lim_{t \rightarrow 0} \frac{\partial w(t, s)}{\partial t} = \lim_{s \rightarrow 0} \frac{\partial w(t, s)}{\partial s} = 0. \quad (22)$$

Applying the Laplace transform in the variable s , from (21), (22) we obtain the following problem for the image $\hat{w}(t, \lambda)$:

$$\frac{\partial^2 \hat{w}(t, \lambda)}{\partial t^2} + \frac{(\sinh^k t \sinh^q(2t))'}{\sinh^k t \sinh^q(2t)} \frac{\partial \hat{w}(t, \lambda)}{\partial t} + \left(\frac{k}{2} + q\right)^2 \hat{w}(t, \lambda) = \lambda^2 \hat{w}(t, \lambda), \quad t > 0, \quad (23)$$

$$\lim_{t \rightarrow 0} \hat{w}(t, \lambda) = \lim_{t \rightarrow 0} \frac{\partial \hat{w}(t, \lambda)}{\partial t} = 0. \quad (24)$$

In [19] it was established that the scalar problem (23), (24) has only the zero solution. Consequently, $\hat{w}(t, \lambda) = w(t, s) = 0$ for any $s \geq 0$. Due to the arbitrariness of the functional $f \in E^*$ for $s = 0$ we obtain the equality $u_1(t) \equiv u_2(t)$, thereby establishing the uniqueness of the solution to the problem under consideration. The proof of the theorem is complete.

Example 1. Let $k > 0$, $q = 2$ and let the operator A be a generator of the COF $C(t; A)$. Then after calculating the inner integral in formula (20), the solution to problem (1), (11) will have the form

$$\begin{aligned} u_{k,2}(t) &= \frac{2^{k/2+2} \Gamma(k/2 + 3/2)}{\sqrt{\pi} \Gamma(k/2) \sinh^k t \sinh(2t)} \int_0^t C(\tau; A) u_0 \int_{\tau}^t \sinh s (\cosh s - \cosh \tau)^{k/2-1} ds d\tau \\ &= \frac{2^{k/2+3} \Gamma(k/2 + 3/2)}{k \sqrt{\pi} \Gamma(k/2) \sinh^k t \sinh(2t)} \int_0^t (\cosh s - \cosh \tau)^{k/2} C(\tau; A) u_0 d\tau = \frac{1}{\cosh t} P_{k+2}(t; A) u_0, \end{aligned}$$

where $P_{k+2}(t; A) = \mathbb{T}_{k+2,0} C(t; A)$ is the LOF (see (12)).

In a special case, if $k = q = 2$, $A \in \mathbb{R}$, $C(t; A) = \cosh(t\sqrt{A})$ in Eq. (1), then the integral can be calculated,

$$\begin{aligned} u_{2,2}(t) &= \frac{6}{\sinh^2 t \sinh(2t)} \int_0^t (\cosh t - \cosh \tau) \cosh(\tau\sqrt{A}) u_0 d\tau \\ &= \frac{6\sqrt{A} \sinh t \cosh(t\sqrt{A}) u_0 - 6 \cosh t \sinh(t\sqrt{A}) u_0}{(A-1)\sqrt{A} \sinh^2 t \sinh(2t)} \quad \text{if } A \neq 0, A \neq 1. \end{aligned} \quad (25)$$

For $A = 0$ or $A = 1$, one should calculate the corresponding limits on the right-hand side in (25) to obtain

$$u_{2,2}(t) = \begin{cases} \frac{6(t \cosh t - \sinh t) u_0}{\sinh^2 t \sinh(2t)}, & A = 0, \\ \frac{3(\sinh 2t - 2t) u_0}{2 \sinh^2 t \sinh(2t)}, & A = 1. \end{cases}$$

4. THE TRANSFORMATION OPERATOR BY THE FIRST INDEX k FOR THE INCOMPLETE CAUCHY PROBLEM AND THE ASSOCIATED LEGENDRE OPERATOR FUNCTION

In what follows, we will be interested in a solution to Eq. (1) that tends to zero as $t \rightarrow \infty$, so in this section, instead of the left-sided fractional integral of the function $u(t)$ over a specially chosen

function $g(t)$, we will use the concepts of the right-sided fractional integral

$$I_{\infty-,g}^{\alpha} u(t) = \left(\frac{d}{dg(t)} \right)^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (g(s) - g(t))^{\alpha-1} g'(s) u(s) ds, \quad \alpha > 0, \quad (26)$$

and the right-sided fractional derivative $\left(\frac{d}{dg(t)} \right)^{\alpha} u(t)$.

For $t > 0$ with sufficiently smooth and exponentially decreasing functions $u(t)$, the paper [8] established the relation

$$L_{k+2\alpha,0} \left(\frac{1}{\sinh t} \frac{d}{dt} \right)^{\alpha} u(t) = \left(\frac{1}{\sinh t} \frac{d}{dt} \right)^{\alpha} L_{k,0} u(t), \quad \alpha \in \mathbb{R}. \quad (27)$$

Thus, the operator Φ_k defined by relation (5) and the operator $(\sinh^{-1} t \frac{d}{dt})_{-}^{\alpha}$ implement the movement of the operator $L_{k,0}$ with respect to the first index.

Let

$$\hat{\mathbb{T}}_{k,0} u(t) = \frac{\sqrt{\pi}}{\Gamma(k/2)} \Phi_{2-k} \left(\frac{1}{\sinh t} \frac{d}{dt} \right)^{-k/2} \frac{u(t)}{\sinh t}; \quad (28)$$

then [8] for $k > 0$ for exponentially decreasing functions, relations (6), (27) imply the relation

$$L_{k,0} \hat{\mathbb{T}}_{k,0} u(t) = \hat{\mathbb{T}}_{k,0} u''(t), \quad (29)$$

which means that formula (28) defines a transformation operator $\hat{\mathbb{T}}_{k,0}$ intertwining the operators $L_{k,0}$ and $\frac{d^2}{dt^2}$.

Since we are interested in a solution to Eq. (1) that tends to zero as $t \rightarrow \infty$, the required condition is provided by the form of the operator A . We assume that the operator $A = B^2$, where B is a generator of an exponentially decreasing C_0 -semigroup $T(t; B)$ admitting the estimate

$$\|T(t; B)\| \leq \Upsilon e^{-\omega t}, \quad \Upsilon \geq 1, \quad \omega > 0. \quad (30)$$

For $k > 0$, $\omega > |k/2 - 1|$, $v_0 \in E$, following [8], we introduce the operator function

$$\begin{aligned} Q_k(t; B)v_0 &= \hat{\mathbb{T}}_{k,0} T(t; B)v_0 = \frac{\sqrt{\pi}}{\Gamma(k/2)} \Phi_{2-k} \left(\frac{1}{\sinh t} \frac{d}{dt} \right)^{-k/2} \frac{T(t; B)v_0}{\sinh t} \\ &= \frac{\sqrt{\pi}}{\Gamma(k/2)} \sinh^{1-k} t \int_t^{\infty} (\cosh s - \cosh t)^{k/2-1} T(s; B)v_0 ds. \end{aligned} \quad (31)$$

The convergence of the integral in (31) for $\omega > |k/2 - 1|$ is ensured by the condition $k > 0$ and inequality (30). Indeed, taking into account integral 2.4.14.1 in [18], we obtain the estimate

$$\begin{aligned} \|Q_k(t; B)\| &\leq \frac{\Upsilon \sqrt{\pi}}{\Gamma(k/2)} \sinh^{1-k} t \int_t^{\infty} (\cosh t - \cosh s)^{k/2-1} e^{-\omega t} ds \\ &\leq \Upsilon \sqrt{2} \sinh^{1/2-k/2} t Q_{\omega-1/2}^{k/2-1/2}(\cosh t), \end{aligned} \quad (32)$$

where $Q_{\omega-1/2}^{k/2-1/2}(\cdot)$ is the associated Legendre function of the second kind, which tends to zero as $t \rightarrow +\infty$, since as $t \rightarrow \infty$ the leading term of the asymptotics of the function $Q_{\omega-1/2}^{k/2-1/2}(\cosh t)$ has the form $c_0 \cosh^{-\omega-1/2} t$ with some constant c_0 (see, for example, [20, Sec. 7]).

In particular, if $B \in \mathbb{R}$, $B < 0$, then $T(t; B) = e^{Bt}$ and (see integral 2.4.14.1 in [18])

$$Q_k(t; B) = -i\sqrt{2} \exp\left\{\frac{k\pi}{2}i\right\} \sinh^{1/2-k/2} t Q_{-B-1/2}^{k/2-1/2}(\cosh t). \quad (33)$$

The operator function $Q_k(t; B)$ is the result of applying the transformation operator $\hat{\mathbb{T}}_{k,0}$ to the decreasing semigroup $T(t; B)$ and is called the adjoint Legendre operator function (ALOF), and relation (29) means that for $q = 0$ the function $u(t) = Q_k(t; B)v_0$ satisfies Eq. (1). Thus, the following assertion holds.

Theorem 3 [8]. *Let $k > 0$, $q = 0$, $v_0 \in D(A)$ and let the operator A be representable as $A = B^2$, where B is a generator of the C_0 -semigroup $T(t; B)$ that admits the estimate (30) for $\omega > |k/2 - 1|$. Then the function $u_{k,0}(t) = Q_k(t; B)v_0$ defined by relation (31) satisfies Eq. (1).*

5. THE TRANSFORMATION OPERATOR BY INDICES k, q FOR THE INCOMPLETE CAUCHY PROBLEM

Let us further construct a transformation operator, using which and given the known solution $u_{k,0}(t) = Q_k(t; B)v_0$, we determine the solution $u_{k,q}(t)$ to Eq. (1) for $q > 0$. Let

$$\Psi_{k,q}u(t) = \sinh^k t \sinh^{q-1}(2t)u(t). \quad (34)$$

In this case, to determine the transformation operator, instead of relations (6), (7), one should use the relations

$$L_{-k,2-q}\Psi_{k,q}u(t) = \Psi_{k,q}L_{k,q}u(t), \quad (35)$$

$$L_{k,q+2\alpha}\left(\frac{1}{\sinh(2t)}\frac{d}{dt}\right)_-^\alpha u(t) = \left(\frac{1}{\sinh(2t)}\frac{d}{dt}\right)_-^\alpha L_{k,q}u(t), \quad \alpha \in \mathbb{R}. \quad (36)$$

Relation (36) is valid for exponentially decreasing functions $u(t)$. Thus, the operator $\Psi_{k,q}$ defined by formula (34) and the operator $(\sinh^{-1}(2t)\frac{d}{dt})_-^\alpha$ defined by relation (26) for $g(t) = \frac{\cosh(2t)}{2}$ implement the movement of the operator $L_{k,q}$ with respect to the second index. Let

$$\hat{\mathbb{T}}_{0,q}u(t) = \Psi_{0,2-q}\left(\frac{1}{\sinh(2t)}\frac{d}{dt}\right)_-^{-q/2} \frac{u(t)}{\sinh(2t)}; \quad (37)$$

then for $k > 0$, $q > 0$ with exponentially decreasing functions $u(t)$ relations (35), (36) entail the equality

$$L_{k,q} \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t u(t) = \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t L_{k,0}u(t), \quad (38)$$

which means that we have constructed the transformation operator

$$\hat{\mathbb{T}}_{k,q} = \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t \hat{\mathbb{T}}_{k,0}, \quad (39)$$

intertwining the operators $L_{k,q}$ and $\frac{d^2}{dt^2}$, i.e., for exponentially decreasing functions $u(t)$ one has the relation

$$L_{k,q} \hat{\mathbb{T}}_{k,q} u(t) = \hat{\mathbb{T}}_{k,q} u''(t). \quad (40)$$

Indeed, by virtue of relations (29), (37)–(40), we have

$$\begin{aligned} L_{k,q} \hat{\mathbb{T}}_{k,q} u(t) &= L_{k,q} \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t \hat{\mathbb{T}}_{k,0} u(t) = \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t L_{k,0} \hat{\mathbb{T}}_{k,0} u(t) \\ &= \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t \hat{\mathbb{T}}_{k,0} u''(t) = \hat{\mathbb{T}}_{k,q} u''(t). \end{aligned}$$

Just as in the case of relations (6), (7), and (9), relations (35), (36), and (38) are established by direct verification using the representation (4) for the operator $L_{k,q}$.

6. INCOMPLETE CAUCHY PROBLEM

Relation (40) means that for $v_0 \in D(B^2)$ the function

$$\begin{aligned} u_{k,q}(t) &= \hat{\mathbb{T}}_{k,q} T(t; B)v_0 = \sinh^{-k} t \hat{\mathbb{T}}_{0,q} \sinh^k t \hat{\mathbb{T}}_{k,0} T(t; B)v_0 \\ &= \frac{2^{1-q/2}}{\Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \int_t^\infty (\cosh(2s) - \cosh(2t))^{q/2-1} \sinh^k s Q_k(s; B)v_0 ds, \end{aligned} \quad (41)$$

which, considering (31), can be represented in the form

$$\begin{aligned} u_{k,q}(t) &= \frac{2^{1-q/2} \sqrt{\pi}}{\Gamma(k/2) \Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \\ &\quad \times \int_t^\infty (\cosh(2s) - \cosh(2t))^{q/2-1} \sinh s \int_s^\infty (\cosh \tau - \cosh s)^{k/2-1} T(\tau; B)v_0 d\tau ds \\ &= \frac{2^{1-q/2} \sqrt{\pi}}{\Gamma(k/2) \Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \\ &\quad \times \int_t^\infty T(\tau; B)v_0 \int_t^\tau \sinh s (\cosh \tau - \cosh s)^{k/2-1} (\cosh(2s) - \cosh(2t))^{q/2-1} ds d\tau, \end{aligned} \quad (42)$$

satisfies Eq. (1), since, by virtue of (40), we have

$$L_{k,q} u_{k,q}(t) = L_{k,q} \hat{\mathbb{T}}_{k,q} T(t; B)v_0 = \hat{\mathbb{T}}_{k,q} T''(t)v_0 = B^2 \hat{\mathbb{T}}_{k,q} T(t; B)v_0 = A u_{k,q}(t).$$

Just as in (20), the inner integral in the representation (42) can be calculated, and then the solution to Eq. (1) is written as

$$\begin{aligned} u_{k,q}(t) &= \frac{2^{1-k/2-q/2} \sqrt{\pi}}{\Gamma(k/2 + q/2) \cosh^{k/2} t \sinh^k t \sinh^{q-1}(2t)} \int_t^\infty (\cosh(2\tau) - \cosh(2t))^{k/2+q/2-1} \\ &\quad \times {}_2F_1\left(\frac{k}{2} + q - 1, \frac{k}{2}; \frac{k+q}{2}; \frac{\cosh t - \cosh \tau}{2 \cosh t}\right) T(\tau; B)v_0 d\tau. \end{aligned} \quad (43)$$

As already noted, the leading term in the asymptotics of the function $Q_{\omega-1/2}^{k/2-1/2}(\cosh t)$ has the form $c_0 \cosh^{-\omega-1/2} t$ with some constant c_0 ; therefore for $\omega > \max\{k/2 + q - 2, |1 - k/2|\}$, as follows from the representation (20) and relations (12) and (13), the function $u_{k,q}(t)$ satisfies the limit relations

$$\lim_{t \rightarrow \infty} \|u(t)\| = \lim_{t \rightarrow \infty} \|u'(t)\| = \lim_{t \rightarrow \infty} \|u''(t)\| = 0.$$

Now let us find out what initial condition the function (43) satisfies at $t = 0$. If $0 < k + q < 1$, $v_0 \in E$, then by virtue of the estimate (32) and Lebesgue's theorem, after obvious transformations we obtain

$$\lim_{t \rightarrow 0} u_{k,q}(t) = \lim_{t \rightarrow 0} \frac{2^{1-k/2-q/2} \sqrt{\pi} t}{\Gamma(k/2 + q/2) \cosh^{k/2} t \sinh^k t \sinh^{q-1}(2t)} \int_1^\infty (\cosh(2t\xi) - \cosh(2t))^{k/2+q/2-1}$$

$$\begin{aligned} & \times {}_2F_1\left(\frac{k}{2} + q - 1, \frac{k}{2}; \frac{k+q}{2}; \frac{\cosh t - \cosh(t\xi)}{2 \cosh t}\right) T(t\xi; B)v_0 d\xi \\ &= \frac{2^{1-q}\sqrt{\pi}}{\Gamma(k/2 + q/2)} \int_1^\infty (\xi^2 - 1)^{k/2+q/2-1} v_0 d\xi = \frac{\Gamma(1/2 - k/2 - q/2)}{2^q} v_0. \end{aligned}$$

Thus, the following assertion holds true.

Theorem 4. Let $k > 0$, $q > 0$, $k + q < 1$ and let the operator A be representable as $A = B^2$, where B is a generator of the C_0 -semigroup $T(t; B)$ admitting the estimate (30) for $\omega > 1 - k/2$. Then for any choice of $u_0 \in D(A)$ the function

$$\begin{aligned} u_{k,q}(t) &= \frac{2^{q/2+1}}{\Gamma(q/2)\Gamma(1 - k/2 - q/2) \sinh^k t \sinh^{q-1}(2t)} \\ &\quad \times \int_t^\infty (\cosh(2s) - \cosh(2t))^{q/2-1} \sinh^k s Q_k(s; B) u_0 ds \end{aligned}$$

is a solution to Eq. (1) satisfying the conditions

$$u(0) = u_0, \quad \lim_{t \rightarrow \infty} \|u(t)\| = \lim_{t \rightarrow \infty} \|u'(t)\| = \lim_{t \rightarrow \infty} \|u''(t)\| = 0.$$

Next, for $k > 0$, $q > 0$, $k + q \geq 1$ with $v_0 \in D(A)$ we similarly find the limit as $t \rightarrow 0$ of the derivative of the function $u_{k,q}(t)$ with weight t^{k+q} . We have

$$\begin{aligned} \lim_{t \rightarrow 0} t^{k+q} u'_{k,q}(t) &= \frac{2^{1-q}\sqrt{\pi}}{\Gamma(k/2 + q/2)} \lim_{t \rightarrow 0} t^{k+q} \frac{d}{dt} \int_1^\infty (\xi^2 - 1)^{k/2+q/2-1} T(t\xi; B)v_0 d\xi \\ &= \frac{2^{1-q}\sqrt{\pi}}{(k+q)\Gamma(k/2 + q/2)} \lim_{t \rightarrow 0} t^{k+q+1} \int_1^\infty (\xi^2 - 1)^{k/2+q/2} T(t\xi; B) B^2 v_0 d\xi. \end{aligned}$$

Taking into account the existence of a fractional power of the operator $-B$ (see, for example, [14, p. 96]), the limit of the last integral was found in [8]; as a result, we obtain

$$\lim_{t \rightarrow 0} t^{k+q} u'_{k,q}(t) = -2^{k/2} \Gamma(k/2 + q/2 + 1/2) (-B)^{1-k-q} v_0.$$

Based on the previous reasoning, we have the following statement.

Theorem 5. Let $k > 0$, $q > 0$, $k + q \geq 1$, and let the operator A be representable as $A = B^2$, where B is a generator of the C_0 -semigroup $T(t; B)$ admitting the estimate (30) for $\omega > \max\{k/2 + q - 2, |1 - k/2|\}$, and, in addition, $u_1 \in D((-B)^{k+q+1})$. Then the function

$$\begin{aligned} u_{k,q}(t) &= \frac{2^{1-q/2-k}}{\Gamma(q/2)\Gamma(k/2 + q/2 + 1/2) \sinh^k t \sinh^{q-1}(2t)} \\ &\quad \times \int_t^\infty (\cosh(2s) - \cosh(2t))^{q/2-1} \sinh^k s Q_k(s; B) (-B)^{k+q-1} u_1 ds \end{aligned} \tag{44}$$

is a solution to Eq. (1) satisfying the conditions

$$\lim_{t \rightarrow 0} t^k u'(t) = u_1, \quad \lim_{t \rightarrow \infty} \|u(t)\| = \lim_{t \rightarrow \infty} \|u'(t)\| = \lim_{t \rightarrow \infty} \|u''(t)\| = 0. \tag{45}$$

Example 2. Let $k > 0$, $q = 2$ and the operator A be representable as $A = B^2$, where B is a generator of the C_0 -semigroup $T(t; B)$ admitting the estimate (30) for $\omega > \max\{k/2, |1 - k/2|\}$. Then, according to formula (44), after calculating the inner integral, we have

$$\begin{aligned} u_{k,2}(t) &= \frac{\sqrt{\pi}}{\Gamma(k/2) \sinh^k t \sinh(2t)} \int_t^\infty T(\tau; B) v_0 \int_t^\tau \sinh s (\cosh \tau - \cosh s)^{k/2-1} ds d\tau \\ &= \frac{\sqrt{\pi}}{\Gamma(k/2 + 1) \sinh^k t \sinh(2t)} \int_t^\infty (\cosh \tau - \cosh t)^{k/2} T(\tau; B) v_0 d\tau \\ &= \frac{1}{2 \cosh t} Q_{k+2}(t; B) v_0, \end{aligned} \quad (46)$$

where $Q_{k+2}(t; B)$ is the ALOF (see (11)),

$$v_0 = -\frac{(-B)^{k+1} u_1}{2^{k/2+1} \Gamma(k/2 + 3/2)}. \quad (47)$$

The function (46) is a solution to problem (1), (45).

Example 3. Let $k > 0$, $q > 0$, $k+q \geq 1$, $B \in \mathbb{R}$, $B < -\max\{k/2+q-2, |1-k/2|\}$, $T(t; B) = e^{Bt}$; then the ALOF has the form (see (33))

$$Q_k(t; B) = \mu_k \sinh^{1/2-k/2} t Q_{-B-1/2}^{k/2-1/2}(\cosh t), \quad \mu_k = -i\sqrt{2} \exp\left\{\frac{k\pi}{2}i\right\}.$$

Taking into account integral 2.18.4.3 in [16], after obvious transformations from relations (41), (47) we obtain

$$\begin{aligned} u_{k,q}(t) &= \frac{2^{1-q/2} \mu_k}{\Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \\ &\quad \times \int_t^\infty (\cosh(2s) - \cosh(2t))^{q/2-1} \sinh^{k/2+1/2} s Q_{-B-1/2}^{k/2-1/2}(\cosh s) v_0 ds \\ &= \frac{\mu_k}{\Gamma(q/2) \sinh^k t \sinh^{q-1}(2t)} \int_t^\infty (\xi^2 - \cosh^2 t)^{q/2-1} (\xi^2 - 1)^{k/4-1/4} Q_{-B-1/2}^{k/2-1/2}(\xi) v_0 d\xi \\ &= -\frac{\Gamma(-B/2 - k/4 + 1/2) \Gamma(-B/2 - k/4 - q/2 + 1)}{2^{3k/2+q} \Gamma(k/2 + q/2 + 1/2) \Gamma(1 - B) \sinh^k t \sinh^{q-1} 2t \cosh^{k/2-q-B+1} t} \\ &\quad \times {}_2F_1(-B/2 - k/4 + 1/2, -B/2 - k/4 - q/2 + 1; 1 - B; \cosh^{-2} t) (-B)^{k+q-1} u_1. \end{aligned} \quad (48)$$

The function (48) is a solution to problem (1), (45).

In a particular case, if $k = q = 2$, $B < -1$ in Eq. (1), then the solution to problem (1), (45) has the form

$$u_{2,2}(t) = \frac{B^2(B \sinh t - \cosh t)}{3(B^2 - 1) \sinh^2 t \sinh(2t)} e^{Bt} u_1.$$

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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