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On the Unique Solvability of Nonlocal Problems for Abstract Singular Equations

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Abstract—Sufficient conditions are given for the unique solvability of nonlocal problems for abstract singular equations that are formulated in terms of the zeros of the modified Bessel function and the resolvent of the operator coefficient of the equations under consideration. Examples are presented.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Let E be a complex Banach space, and let A and B be linear closed operators on E whose domains $D(A)$ and $D(B)$ are not necessarily dense in E. Further restrictions on these operators will be indicated in the process of presenting the assertions to be proved. We study nonlocal problems on the finite interval $0 < t < 1$, since the general case of an interval $0 < t < T$ is reduced to the case under consideration by the change of variables $t \to t/T$.

Consider the equation of the form

$$B\left(u''(t) + \frac{k}{t}u'(t)\right) = Au(t), \qquad 0 < t < 1, \tag{1.1}$$

which, in the case of $B \neq I$, generalizes the abstract Euler–Poisson–Darboux equation.

The setting of boundary and nonlocal conditions, due to the singularity (for $k \neq 0$) of the equation under consideration at the point $t = 0$, depends on the parameter $k \in \mathbb{R}$, and these conditions are given below. The nonlocal integral conditions imposed below can be interpreted in the spirit of control theory: it is required to find a solution of the differential equation (1.1) with a given initial state at $t = 0$ and having some prescribed average value. As indicated in [1], conditions of this type arise, for example, when studying the diffusion of particles in turbulent plasma, moisture transfer processes in capillary-porous media, etc.

An equation of the form (1.1) is called an equation of Sobolev type, or a descriptor equation. The Cauchy problem for the singular equation (1.1) with a Fredholm operator B was studied previously in [2], [3]. A detailed survey of the solvability of nonsingular equations of Sobolev type can be found, e.g., in [4]. Nonlocal problems for Eq. (1.1), generally speaking, are not well posed, but the need to solve ill-posed problems is now generally accepted (see the introduction in the book [5] and the extensive bibliography therein). A number of results devoted to nonlocal problems for abstract first-order equations were obtained earlier in [6]–[8], and, for singular second-order equations, but under more rigid conditions than those in the present work for the operator A, can be found in [9]–[11]. Nonlocal problems for partial differential equations containing the Bessel differential operator with respect to a spatial variable were studied in [12]–[15].

We will present sufficient conditions for the unique solvability of diverse nonlocal problems for the singular equation (1.1) on the finite interval $[0, 1]$. A distinctive feature of this paper is that it establishes connections between solutions of nonlocal problems and the corresponding solutions of boundary value problems.

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2. CASE OF $k \geq 0$ AND THE NEUMANN CONDITION AT $t = 0$

Let $k \geq 0$. Consider the problem of determining a function

$$u(t) \in C^1([0,1], E) \cap C^2((0,1], E)$$

belonging together with its derivatives to the domain $D = D(A) \cap D(B)$ for $t \in (0,1)$ and satisfying Eq. (1.1), the Neumann boundary condition

$$u'(0) = 0 \tag{2.1}$$

at $t = 0$, and also a nonlocal condition of the form

$$\lim_{t \to 1} I^{\alpha}_{0+;2,\eta} u(t) = u_0,$$

where $\eta = (k-1)/2$, $\alpha > 0$, $\Gamma(\cdot)$ is the Euler gamma function, and $I^{\alpha}_{0+;2,\eta}$ is the Erdélyi–Kober operator defined by the formula (see [16, p. 246])

$$I^{\alpha}_{0+;2,\eta} u(t) = \frac{2}{\Gamma(\alpha) t^{2(\alpha+\eta)}} \int_0^t s^{2\eta+1} (t^2 - s^2)^{\alpha-1} u(s)\, ds.$$

This nonlocal condition in expanded notation has the form

$$\frac{2}{\Gamma(\alpha)} \int_0^1 t^k (1-t^2)^{\alpha-1} u(t)\, dt = u_0. \tag{2.2}$$

Note that a special case of the nonlocal condition (2.2) for $\alpha = 1$ and with respect to the spatial variable occurred earlier in [1], [12].

Below, we repeatedly use the function

$$Y_k(t;\lambda) = \Gamma\left(\frac{k}{2} + \frac{1}{2}\right) \left(\frac{t\sqrt{\lambda}}{2}\right)^{1/2-k/2} I_{k/2-1/2}\big(t\sqrt{\lambda}\big), \tag{2.3}$$

where $I_\nu(\cdot)$ stands for the modified Bessel function. The function $Y_k(t;\lambda)$ is a solution of the scalar Euler–Poisson–Darboux equation (the case of $B = I$, $A = \lambda I$ in Eq. (1.1)) and, in addition, $Y_k(0;\lambda) = 1$ and $Y_k'(0;\lambda) = 0$.

For problem (1.1), (2.1), (2.2), the following criterion for the uniqueness of a solution of the nonlocal problem was proved in [17].

Theorem 1. *Let $k \geq 0$ and $\alpha > 0$, and let A and B be linear closed operators on E. Let us assume that the nonlocal problem* (1.1), (2.1), (2.2) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that, for every $\lambda_m = \lambda_m(k,\alpha)$, $m \in \mathbb{N}$, that is a zero of the function $\Upsilon_{k,\alpha}(\lambda) = Y_{k+2\alpha}(1;\lambda)$ specified by Eq.* (2.3), *the operator equation*

$$Ah = \lambda Bh \tag{2.4}$$

have no nonzero solution h.

When establishing the solvability of the nonlocal problem (1.1), (2.1), (2.2), we use the statement established in [18] and given below about the unique solvability of some boundary value problem.

Let us denote by $\rho(B,A)$ the set of $\mu \in \mathbb{C}$ such that there exists a bounded inverse $(\mu B - A)^{-1}$; we call this set the resolvent set of the operator A with respect to B.

Theorem 2. *Suppose that $k \geq 0$, $u_1 \in D(A^2) \cap D(B)$, and A and B are closed linear operators commuting on the elements of $D(A^2) \cap D(B)$. Let also, for every $n \in \mathbb{N}$, the zeros ξ_n of the function $Y_k(1;\lambda)$ defined by Eq.* (2.3) *belong to the resolvent set $\rho(B,A)$ of the operator A with respect to B, and let the following bound hold*:

$$\sup_{n \in \mathbb{N}} |\xi_n| \cdot \|(\xi_n B - A)^{-1}\| < M_0 < \infty.$$

Then the boundary value problem

$$B\left(u''(t)+\frac{k}{t}u'(t)\right)=Au(t),\qquad \lim_{t\to 0+}t^k u'(t)=0,\quad u(1)=u_1,$$

is uniquely solvable, its solution has the form

$$u_k(t)=-2\sum_{n=1}^{\infty}\frac{Y_k(t;\xi_n)}{Y_k'(1;\xi_n)}\xi_n B(\xi_n B-A)^{-1}u_1, \tag{2.5}$$

and moreover, $u_k'(0)=0$.

Let problem (1.1), (2.1), (2.2) satisfy the conditions of Theorem 1 about the uniqueness of the solution of the nonlocal problem, and in addition, for the value $k+2\alpha$ of the parameter of the singular equation (1.1), let the conditions of Theorem 2 established in [18] be satisfied, where the commutativity of the operators A and B, as well as the corresponding estimate for the inverse operator $(\mu B-A)^{-1}$, is required.

If these conditions are satisfied, then there exists a unique solution $u_{k+2\alpha}(t)$ of the boundary value problem

$$B(u''(t)+\frac{k+2\alpha}{t}u'(t))=Au(t),\qquad \lim_{t\to 0+}t^{k+2\alpha}u'(t)=0,\qquad u(1)=u_1. \tag{2.6}$$

It is defined by Eq. (2.5) after the replacement of the parameter k by $k+2\alpha$ in this representation.

Using (e.g., see [19], [20]) the operators of motion of the solutions by the parameter,

$$\Phi_k u(t)=t^{k-1}u(t),\qquad I^{\alpha}_{t^2}u(t)=\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\alpha}u(t),$$

where, for fractional $\alpha>0$ (see [16, p. 248]),

$$\begin{aligned}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\alpha}u(t)&=\left(\frac{1}{t}\,\frac{d}{dt}\right)^{[\alpha]+1}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\{\alpha\}-1}\\ &=\frac{2^{\{\alpha\}}}{\Gamma(1-\{\alpha\})}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{[\alpha]+1}\int_0^t\tau(t^2-\tau^2)^{-\{\alpha\}}u(\tau)\,d\tau,\end{aligned} \tag{2.7}$$

from the solution $u_{k+2\alpha}(t)$ of the boundary value problem (2.6) we construct the solution

$$u_k(t)=t^{1-k}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\alpha}(t^{k+2\alpha-1}u_{k+2\alpha}(t)) \tag{2.8}$$

of Eq. (1.1) and show that this solution $u_k(t)$ satisfies conditions (2.1) and (2.2), which we need.

First, assume that $\alpha\in\mathbb{N}$. Note that the possibility to differentiate the function $u_{k+2\alpha}(t)$ can be provided by the additional requirement that the element u_1 determining this solution belongs to the set $D(A^{2+\alpha})\cap D(B)$.

The solution $u_k(t)$ of Eq. (1.1) satisfies condition (2.1), since this solution can be represented in the form

$$u_k(t)=c_1u_{k+2\alpha}(t)+c_2t^2u'_{k+2\alpha}(t)+\cdots$$

with some constants $c_1,c_2,\dots$, and at the same time we have $u'_{k+2\alpha}(0)=0$, as follows from the representation (2.5).

Let us further verify the validity of condition (2.2) for the same solution. Let us substitute $u_k(t)$ into the left-hand side of condition (2.2). After integration by parts, we obtain

$$\frac{2}{\Gamma(\alpha)}\int_0^1 t^k(1-t^2)^{\alpha-1}u_k(t)\,dt=\frac{2}{\Gamma(\alpha)}\int_0^1 t(1-t^2)^{\alpha-1}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\alpha}t^{k+2\alpha-1}u_{k+2\alpha}(t)\,dt$$

$$= \frac{2^2}{\Gamma(\alpha-1)} \int_0^1 t(1-t^2)^{\alpha-2} \left(\frac{1}{t}\frac{d}{dt}\right)^{\alpha-1} t^{k+2\alpha-1} u_{k+2\alpha}(t)\,dt = \cdots$$
$$= \frac{2^\alpha}{\Gamma(1)} \int_0^1 t\left(\frac{1}{t}\frac{d}{dt}\right) t^{k+2\alpha-1} u_{k+2\alpha}(t)\,dt = 2^\alpha u_{k+2\alpha}(1) = 2^\alpha u_1. \tag{2.9}$$

If in the boundary value problem (2.6) we select u_1 in such a way that $2^\alpha u_1 = u_0$, then, by Eq. (2.9), the solution $u_k(t)$ of Eq. (1.1) also satisfies condition (2.2).

Now let $\alpha > 0$, $\{\alpha\} > 0$, and $u_1 \in D(A^{3+[\alpha]}) \cap D(B)$ in problem (2.6). Using the concept of fractional power of the operation of weight differentiation defined by Eq. (2.7), we represent the function $u_k(t)$ in the form

$$\begin{aligned} u_k(t) &= t^{1-k}\left(\frac{1}{t}\frac{d}{dt}\right)^{\alpha} t^{k+2\alpha-1} u_{k+2\alpha}(t) \\ &= t^{1-k}\left(\frac{1}{t}\frac{d}{dt}\right)^{[\alpha]+1} \frac{2^{1-\alpha}}{\Gamma(1-\{\alpha\})} \int_0^t (t^2-\tau^2)^{-\{\alpha\}} \tau^{k+2\alpha} u_{k+2\alpha}(\tau)\,d\tau \\ &= \frac{2^{1-\alpha} t^{1-k}}{\Gamma(1-\{\alpha\})} \left(\frac{1}{t}\frac{d}{dt}\right)^{[\alpha]+1} \left(t^{2[\alpha]+k+1} \int_0^1 (1-s^2)^{-\{\alpha\}} s^{k+2\alpha} u_{k+2\alpha}(ts)\,ds\right) \\ &= d_1 \int_0^1 (1-s^2)^{-\{\alpha\}} s^{k+2\alpha} u_{k+2\alpha}(ts)\,ds \\ &\qquad + d_2 t^2 \int_0^1 (1-s^2)^{-\{\alpha\}} s^{k+2\alpha} u'_{k+2\alpha}(ts)\,ds + \cdots \end{aligned}$$

with some constants $d_1, d_2, \dots$, where we have $u'_{k+2\alpha}(0) = 0$. Consequently, the solution $u_k(t)$ of Eq. (1.1) satisfies condition (2.1) in this case as well.

Let us now verify the validity of condition (2.2) for the same solution. Let us substitute $u_k(t)$ into the left-hand side of condition (2.2). Using the fractional integral I^α_{0+} and the derivative D^α_{0+} in the Riemann–Liouville form [16, p. 41], as well as the formula [16, Theorem 2.4, (2.60)] about the action of the fractional integration operation on the differentiation operation, we have

$$\begin{aligned} \frac{2}{\Gamma(\alpha)} \int_0^1 \tau^k (1-\tau^2)^{\alpha-1} u_k(\tau)\,dt &= \frac{2}{\Gamma(\alpha)} \int_0^1 \tau(1-\tau^2)^{\alpha-1} \left(\frac{1}{\tau}\frac{d}{d\tau}\right)^{\alpha} \tau^{k+2\alpha-1} u_{k+2\alpha}(\tau)\,d\tau \\ &= \frac{2^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left(\frac{d}{ds}\right)^{\alpha} s^{k/2-1/2+\alpha} u_{k+2\alpha}(\sqrt{s})\,ds \\ &= 2^\alpha I^\alpha_{0+} D^\alpha_{0+} \bigl(s^{k/2-1/2+\alpha} u_{k+2\alpha}(\sqrt{s})\bigr)\big|_{s=1} = 2^\alpha u_{k+2\alpha}(1) = 2^\alpha u_1. \end{aligned} \tag{2.10}$$

Here we have used the fact that all summands in [16, (2.60)] that are calculated at the point $t = 0$ are zero.

If in the boundary value problem (2.6) we select u_1 in such a way that $2^\alpha u_1 = u_0$, then, by Eq. (2.10), the solution $u_k(t)$ of Eq. (1.1) also satisfies condition (2.2).

Thus, for any $\alpha > 0$ the solution of problem (1.1), (2.1), (2.2) is unique and has the form

$$u_k(t) = -\frac{t^{1-k}}{2^{\alpha-1}} \left(\frac{1}{t}\frac{d}{dt}\right)^{\alpha} \sum_{n=1}^{\infty} \frac{t^{k+2\alpha-1} Y_{k+2\alpha}(t;\lambda_n)}{Y'_{k+2\alpha}(1;\lambda_n)} \lambda_n B(\lambda_n B - A)^{-1} u_0, \tag{2.11}$$

where $\lambda_n = \lambda_n(k,\alpha)$, $n \in \mathbb{N}$, are the zeros of the function $\Upsilon_{k,\alpha}(\lambda) = Y_{k+2\alpha}(1;\lambda)$ given by Eq. (2.3), $u_0 \in D_\alpha$, $D_\alpha = D(A^{2+\alpha}) \cap D(B)$ for $\alpha \in \mathbb{N}$, and $D_\alpha = D(A^{3+[\alpha]}) \cap D(B)$ for the other $\alpha > 0$.

Let us state the result thus obtained in the form of a theorem.

Theorem 3. *Suppose that* $k \ge 0$, $\alpha > 0$, $u_0 \in D_\alpha$, *and* A *and* B *are closed linear operators commuting on elements of* D_α. *Let also, for all* $n \in \mathbb{N}$, *the zeros* $\lambda_n = \lambda_n(k,\alpha)$ *of the function* $\Upsilon_{k,\alpha}(\lambda) = Y_{k+2\alpha}(1;\lambda)$ *defined by Eq.* (2.3) *belong to the resolvent set* $\rho(B,A)$ *of the operator* A *with respect to* B, *and let the following bound hold*:

$$\sup_{n\in\mathbb{N}} |\lambda_n| \cdot \|(\lambda_n B - A)^{-1}\| < M_0 < \infty. \tag{2.12}$$

Then the nonlocal problem (1.1), (2.1), (2.2) *is uniquely solvable, and its solution* $u_k(t)$ *is determined by Eq.* (2.11).

Note that earlier, in the paper [9], the corresponding theorem about the unique solvability of problem (1.1), (2.1), (2.2) has been established by another method for $B = I$ and under more restrictive conditions than those in the present paper for the operator A.

Example 1. Let $B = I$ in Eq. (1.1), and let the operator $-A$ be the generator of an operator cosine function $C(t;-A)$ of exponential growth ω; the resolvent of‘$-A$ is well known to satisfy the bound (2.12). Then, for the case in which the parameters in problem (1.1), (2.1), (2.2) satisfy the conditions $0 \le k < 2$, $\alpha = 1 - k/2$, and $k + 2\alpha = 2$, the solution of the boundary value problem (2.6) has the form (see [18, Example 2])

$$u_2(t) = \frac{\sin \pi t}{t} \int_0^\infty \frac{C(s;-A)u_1\, ds}{\cosh \pi s + \cos \pi t}, \qquad \omega < \pi.$$

Further, for $u_0 \in D(A^3)$, using formula (2.8), we write out the solution of the nonlocal problem (1.1), (2.1), (2.2),

$$\begin{aligned} u_k(t) &= t^{1-k}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{1-k/2} (t u_2(t)) \\ &= 2^{k/2-1} t^{1-k} \left(\frac{1}{t}\,\frac{d}{dt}\right)^{1-k/2} \left(\sin \pi t \int_0^\infty \frac{C(s;-A)u_0\, ds}{\cosh \pi s + \cos \pi t}\right). \end{aligned}$$

In particular, for $k = 0$ we obtain

$$u_0(t) = \frac{1}{2}\,\frac{d}{dt}\left(\sin \pi t \int_0^\infty \frac{C(s;-A)u_0\, ds}{\cosh \pi s + \cos \pi t}\right).$$

If in problem (1.1), (2.1), (2.2) we have $0 < k < 2$, $\alpha = 1 - k/2 < 1$, then, owing to (2.7), the solution of the nonlocal problem (1.1), (2.1), (2.2) has the form

$$u_k(t) = \frac{1}{\Gamma(k/2)t^k}\,\frac{d}{dt}\left(\int_0^t \tau(t^2-\tau^2)^{k/2-1} \sin \pi\tau \int_0^\infty \frac{C(s;-A)u_1 ds}{\cosh \pi s + \cos \pi\tau}\, d\tau\right).$$

Example 2. In problem (1.1), let the operator A be the multiplication by a scalar $A < 0$, and let $B = 1$. Then, for the case in which the parameters in problem (1.1), (2.1), (2.2) satisfy the conditions $k \ge 0$ and $\alpha > 0$, again by [18, Example 2], the solution of the boundary problem (2.6) has the form

$$u_{k+2\alpha}(t) = \frac{t^{1/2-k/2-\alpha} J_{k/2-1/2+\alpha}(t\sqrt{-A})u_0}{2^\alpha J_{k/1-1/2+\alpha}(\sqrt{-A})}.$$

If $\{\alpha\} > 0$, then, taking into account (2.7), we see that, according to formula (2.8), the solution of the nonlocal problem (1.1), (2.1), (2.2) can be represented in the form

$$\begin{aligned} u_k(t) &= t^{1-k}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\alpha} (t^{k-1+2\alpha} u_{k+2\alpha}(t)) \\ &= \frac{t^{1-k}}{2^\alpha J_{k/1-1/2+\alpha}(\sqrt{-A})u_0}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\alpha} \left(t^{k/2-1/2+\alpha} J_{k/2-1/2+\alpha}(t\sqrt{-A})\right) \end{aligned}$$

$$= \frac{2^{-\alpha}t^{1-k}u_0}{\Gamma(1-\{\alpha\})J_{k/2-1/2+\alpha}(\sqrt{-A})}$$
$$\times\left(\frac{1}{t}\,\frac{d}{dt}\right)^{[\alpha]+1}\int_0^t \tau^{k/2+1/2+\alpha}(t^2-\tau^2)^{-\{\alpha\}}J_{k/2-1/2+\alpha}\big(t\sqrt{-A}\big)\,d\tau.$$

The last integral can be calculated (see [21, 2.12.4.6]), and therefore,

$$u_k(t) = \frac{t^{1/2-k/2}(\sqrt{-A})^{\alpha}J_{k/2-1/2}(t\sqrt{-A})u_0}{2^{\alpha}J_{k/2-1/2+\alpha}(\sqrt{-A})}.$$

It can readily be seen that the same representation holds for the solutions $u_k(t)$ in the case of $\{\alpha\}=0$. Note that the validity of the nonlocal condition (2.2) for the functions $u_k(t)$ can readily be verified using the integral [21, 2.12.4.6].

In particular, if $k=1$ and $\alpha=1/2$, then the solution of the nonlocal problem (1.1), (2.1), (2.2) has the form

$$u_1(t) = \frac{\sqrt{-\pi A}}{2\sin(\sqrt{-A})}J_0\big(t\sqrt{-A}\big)u_0.$$

3. CASE OF $k<1$, THE DIRICHLET CONDITION AT $t=0$

Let $k<1$. Consider the problem of determining a function

$$u(t)\in C([0,1],E)\cap C^2((0,1],E)$$

belonging together with its derivatives to the domain $D=D(A)\cap D(B)$ for $t\in(0,1)$ and satisfying Eq. (1.1), the Dirichlet boundary condition

$$u(0)=0 \tag{3.1}$$

at $t=0$, and a nonlocal condition of the form

$$\frac{2}{\Gamma(\beta)}\int_0^1 t(1-t^2)^{\beta-1}u(t)\,dt = u_0, \qquad \beta>0, \tag{3.2}$$

or, using the Erdélyi–Kober operator,

$$\lim_{t\to 1} I^{\beta}_{0+;2,0}u(t) = u_0.$$

For problem (1.1), (3.1), (3.2), the criterion given below for the uniqueness of the solution of the nonlocal problem was proved in [17].

Theorem 4. *Let $k<1$, let $\beta>0$, and let A and B be linear closed operators in E. Let us assume that the nonlocal problem* (1.1), (3.1), (3.2) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that, for any zero $\lambda_m=\lambda_m(k,\beta)$, $m\in\mathbb{N}$, of the function $\Psi_{k,\beta}(\lambda)=Y_{2\beta+2-k}(1;\lambda)$, where the function $Y_{2\beta+2-k}(t;\lambda)$ is given by Eq.* (2.3)*, the operator equation* (2.4) *has no nonzero solution h.*

When establishing the solvability of the nonlocal problem (1.1), (3.1), (3.2), we will also use the statement, established in [18], about the unique solvability of a boundary value problem.

Theorem 5. *Suppose that $k<1$, $u_1\in D(A^2)\cap D(B)$, and A and B are closed linear operators commuting on the elements of $D(A^2)\cap D(B)$. Let also, for all $n\in\mathbb{N}$, the zeros η_n of the function $Y_{2-k}(1;\lambda)$ defined by Eq.* (2.3) *belong to the resolvent set $\rho(B,A)$ of the operator A with respect to B, and let the following bound hold*:

$$\sup_{n\in\mathbb{N}}|\eta_n|\cdot\|(\eta_nB-A)^{-1}\| < M_0 < \infty.$$

Then the boundary value problem

$$B\left(u''(t)+\frac{k}{t}u'(t)\right)=Au(t),\qquad u(0)=0,\quad u(1)=u_1$$

is uniquely solvable, and its solution has the form

$$u_k(t)=-2t^{1-k}\sum_{n=1}^{\infty}\frac{Y_{2-k}(t;\eta_n)}{Y'_{2-k}(1;\eta_n)}\eta_n B(\eta_n B-A)^{-1}u_1. \tag{3.3}$$

Let problem (1.1), (3.1), (3.2) satisfy the conditions of Theorem 4 about the uniqueness of a solution of a nonlocal problem, and in addition, for the parameter value $k-2\beta$ of the singular equation (1.1), let the conditions of Theorem 5 be satisfied; then there exists a unique solution $u_{k-2\beta}(t)$ of the boundary value problem

$$B\left(u''(t)+\frac{k-2\beta}{t}u'(t)\right)=Au(t),\qquad u(0)=0,\quad u(1)=u_1, \tag{3.4}$$

defined by Eq. (3.3) after the replacement of the parameter k in this representation by $k-2\beta$.

Let us assume that $\beta\in\mathbb{N}$. As was done in Sec. 2, using the operator of motion of the solutions by the parameter, from the solution $u_{k-2\beta}(t)$ of the boundary value problem (3.4), we construct a solution of Eq. (1.1) in the form

$$u_k(t)=\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\beta}u_{k-2\beta}(t) \tag{3.5}$$

and show that this solution $u_k(t)$ satisfies the desired conditions (3.1) and (3.2).

The solution $u_k(t)$ of Eq. (1.1) satisfies condition (3.1), since this solution can be written in the form

$$u_k(t)=c_1t^{1-k}u_{k-2\beta}(t)+c_2t^{3-k}u'_{k-2\beta}(t)+\cdots$$

with some constants $c_1,c_2,\dots$.

Let us further verify the validity of condition (3.2) for the same solution. Let us substitute $u_k(t)$ into the left-hand side of condition (3.2). By analogy with (2.9), after integration by parts we have

$$\frac{2}{\Gamma(\beta)}\int_0^1 t(1-t^2)^{\beta-1}u_k(t)\,dt=\frac{2}{\Gamma(\beta)}\int_0^1 t(1-t^2)^{\beta-1}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\beta}u_{k-2\beta}(t)\,dt=2^{\beta}u_1. \tag{3.6}$$

Choosing u_1 in the boundary value problem(3.4) in such a way that $2^{\beta}u_1=u_0$, we establish, owing to Eq. (3.6), the validity of condition (3.2).

Now let $\beta>0$ and $\{\beta\}>0$, and let $u_1\in D(A^{3+[\beta]})\cap D(B)$ in problem (3.4). Then let us write out the function $u_k(t)$ defined by Eq. (3.5) in the form

$$u_k(t)=\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\beta}(t^{1-k+2\beta}v_{k-2\beta}(t)),$$

where

$$v_k(t)=-2\sum_{n=1}^{\infty}\frac{Y_{2\beta+2-k}(t;\lambda_n)}{Y'_{2\beta+2-k}(1;\lambda_n)}\lambda_n B(\lambda_n B-A)^{-1}u_1$$

and $\lambda_n=\lambda_n(k,\beta)$ are the zeros of the function $\Psi_{k,\beta}(\lambda)=Y_{k-2\beta}(1;\lambda)$ defined by Eq. (2.3).

After this, the further verification of the validity of conditions (3.1) and (3.2) is carried out in the same way as this was done in Sec. 2 for the case of a fractional α.

Thus we can claim that, in the case of $\beta>0$, the solution of problem (1.1), (3.1), (3.2) is unique and has the form

$$u_k(t)=-\frac{1}{2^{\beta-1}}\left(\frac{1}{t}\,\frac{d}{dt}\right)^{\beta}\sum_{n=1}^{\infty}\frac{t^{1-k+2\beta}Y_{2\beta+2-k}(t;\lambda_n)}{Y'_{2\beta+2-k}(1;\lambda_n)}\lambda_n B(\lambda_n B-A)^{-1}u_0, \tag{3.7}$$

where $u_0 \in D_\beta$, $D_\beta = D(A^{2+\beta}) \cap D(B)$ for $\beta \in \mathbb{N}$, and $D_\beta = D(A^{3+[\beta]}) \cap D(B)$ for the remaining $\beta > 0$.

Let us formulate the result thus obtained in the form of a theorem.

Theorem 6. *Let $k < 1$, $\beta > 0$, and $u_0 \in D_\beta$, and let A and B be closed linear operators on E commuting on the elements of D_β. Let also, for every $n \in \mathbb{N}$, the zeros $\lambda_n = \lambda_n(k, \beta)$ of the function $\Psi_{k,\beta}(\lambda) = Y_{2\beta+2-k}(1;\lambda)$ defined by Eq.* (2.3) *belong to the resolvent set $\rho(B, A)$ of the operator A with respect to B, and let the following bound hold*:

$$\sup_{n\in\mathbb{N}} |\lambda_n| \cdot \|(\lambda_n B - A)^{-1}\| < M_0 < \infty. \tag{3.8}$$

Then the nonlocal problem (1.1), (3.1), (3.2) *is uniquely solvable, and its solution $u_k(t)$ is determined by Eq.* (3.7).

Example 3. Let $B = I$ in Eq. (1.1), and let the operator $-A$ be the generator of an operator cosine function $C(t; -A)$ of exponential growth ω; as is known, the bound (3.8) holds for the resolvent of $-A$. Then, for the case in which the parameters in problem (1.1), (3.1), (3.2) satisfy the conditions $0 < k < 1$, $\beta = k/2$, and $k - 2\beta = 0$, the solution of the boundary value problem (3.4) has the form [18, Example 1]

$$u_0(t) = \frac{\sin \pi t}{2^\beta} \int_0^\infty \frac{C(s; -A)u_0\, ds}{\cosh \pi s + \cos \pi t}, \qquad \omega < \pi.$$

Further, for $u_0 \in D(A^3)$, taking into account (2.7) and using formula (3.5), we obtain the solution of the nonlocal problem (1.1), (3.1), (3.2) in the form

$$\begin{aligned} u_k(t) &= \left(\frac{1}{t}\frac{d}{dt}\right)^{k/2} u_0(t) = \left(\frac{1}{t}\frac{d}{dt}\right)^{k/2} \frac{\sin \pi t}{2^{k/2}} \int_0^\infty \frac{C(s; -A)u_0\, ds}{\cosh \pi s + \cos \pi t} \\ &= \frac{1}{\Gamma(k/2)t}\frac{d}{dt}\left(\int_0^t \tau(t^2-\tau^2)^{k/2-1} \sin \pi\tau \int_0^\infty \frac{C(s;-A)u_0 ds}{\cosh \pi s + \cos \pi\tau}\, d\tau\right). \end{aligned}$$

Example 4. In Eq. (1.1), let the operator A be the multiplication by a scalar $A < 0$, and let $B = 1$. Then, for the case in which the parameters in problem (1.1), (3.1), (3.2) satisfy the conditions $k < 1$ and $\beta > 0$, we write the solution of the boundary value problem (3.4) according to [18, Example 1]:

$$u_{k-2\beta}(t) = \frac{t^{1/2-k/2+\beta} J_{1/2-k/2+\beta}(t\sqrt{-A})}{2^\beta J_{1/2-k/2+\beta}(\sqrt{-A})} u_0.$$

Let $\{\beta\} > 0$; taking into account (2.7) and using formula (3.5) we obtain the solution of the nonlocal problem (1.1), (3.1), (3.2) in the form

$$\begin{aligned} u_k(t) &= \left(\frac{1}{t}\frac{d}{dt}\right)^{\beta} u_{k-2\beta}(t) = \left(\frac{1}{t}\frac{d}{dt}\right)^{\beta} \frac{t^{1/2-k/2+\beta} J_{1/2-k/2+\beta}(t\sqrt{-A})}{2^\beta J_{1/2-k/2+\beta}(\sqrt{-A})} u_0 \\ &= \frac{2^{-[\beta]}u_0}{\Gamma(1-\{\beta\}) J_{1/2-k/2+\beta}(\sqrt{-A})} \\ &\qquad \times \left(\frac{1}{t}\frac{d}{dt}\right)^{[\beta]+1} \int_0^t \tau^{3/2-k/2+\beta}(t^2-\tau^2)^{-\{\beta\}} J_{1/2-k/2+\beta}(t\sqrt{-A})\, d\tau. \end{aligned}$$

The last integral can be calculated (see [21, 2.12.4.6]); therefore,

$$u_k(t) = \frac{t^{1/2-k/2}(\sqrt{-A})^\beta J_{1/2-k/2}(t\sqrt{-A}) u_0}{2^\beta J_{1/2-k/2+\beta}(\sqrt{-A})}.$$

It can readily be proved that, in the case of $\{\beta\} = 0$, the same representation holds for the solutions $u_k(t)$. Note that the validity of the nonlocal condition (3.2) for the functions $u_k(t)$ can be verified using the integral [21, 2.12.4.6].

In particular, for $k = 0$, the solution of the nonlocal problem (1.1), (3.1), (3.2) has the form

$$u_0(t) = \frac{(\sqrt{-A})^{\beta-1/4} \sin(t\sqrt{-A}) u_0}{2^{\beta-1/2}\sqrt{\pi} J_{\beta+1/2}(\sqrt{-A})}.$$

4. NONLOCAL CONDITION OF THE SECOND KIND. CASE OF $k \geq 0$, THE NEUMANN CONDITION AT $t = 0$

Instead of the nonlocal condition (2.2) for Eq. (1.1), let us pose a condition of the form

$$a \int_0^1 t^k u(t)\,dt + bu'(1) = u_0, \qquad a \neq 0, \quad b \neq 0. \tag{4.1}$$

This type of nonlocal conditions for partial differential equations occurred earlier in [13] and [14]. In this section, we establish the corresponding solvability theorems for problem (1.1), (2.1), (4.1). For this problem, the following uniqueness criterion for solutions was proved in [17].

Theorem 7. *Let $k \geq 0$, and let A and B be closed linear operators in E. Assume that the nonlocal problem* (1.1)*,* (2.1)*,* (4.1) *has a solution $u(t)$. For this solution to be unique, it is necessary and, in the case of $u(t) \in C^3((0,1], D)$, also sufficient that, for any zero λ_m, $m \in \mathbb{N}_0$, of the function $\Phi_{k,a,b}(\lambda) = (a + b\lambda)Y_{k+2}(1;\lambda)$, where the function $Y_{k+2}(t;\lambda)$ is given by Eq.* (2.3)*, the operator equation* (2.4) *has no nonzero solution.*

Let problem (1.1), (2.1), (4.1) satisfy the conditions of Theorem 7 (the uniqueness theorem for the solution of a nonlocal problem), and in addition, let the conditions of Theorem 2 be satisfied for the parameter value $k+2$ of the singular equation (1.1). Then there exists a unique solution $u_{k+2}(t)$ of the boundary value problem (2.6) for $\alpha = 1$, which is defined by Eq. (2.5) after the replacement of the parameter k by $k+2$ in this representation. Note that we have not yet used the fact that there exists no nonzero solution of Eq. (2.4) for $\lambda = -a/b$. This will be taken into account below.

Just as when proving Theorem 3, from the solution $u_{k+2}(t)$ of the boundary value problem (2.6) we construct the following solution of Eq. (1.1):

$$u_k(t) = t^{-k}\frac{d}{dt}(t^{k+1}u_{k+2}(t)). \tag{4.2}$$

As above, this solution satisfies condition (2.1). Let us show that the solution $u_k(t)$ satisfies condition (4.1) as well. Substituting $u_k(t)$ into the left-hand side of condition (4.1), we obtain

$$a \int_0^1 t^k u_k(t)\,dt + bu_k'(1) = a \int_0^1 \frac{d}{dt}(t^{k+1}u_{k+2}(t))\,dt + bu_k'(1) = au_1 + bu_k'(1),$$

and the nonlocal condition (4.1) is satisfied if the boundary value u_1 in problem (2.6) satisfies the relation

$$au_1 + bu_k'(1) = u_0.$$

Since, by formula (4.2),

$$Bu_k'(t) = B\big((k+2)u_{k+2}'(t) + tu_{k+2}''(t)\big) = tB\left(u_{k+2}''(t) + \frac{k+2}{t}u_{k+2}'(t)\right) = tAu_{k+2}(t),$$

it follows that the value u_1 in the boundary value problem (2.6) must be chosen in such a way that

$$(aB + bA)u_1 = Bu_0.$$

The possibility of this choice can be ensured by the invertibility of the operator $aB + bA$, i.e., by requiring that the number $\lambda = -a/b$ belongs to the resolvent set $\rho(B, A)$, and then we have $u_1 = (aB + bA)^{-1}Bu_0$.

Thus, the solution of problem (1.1), (2.1), (4.1) is unique and has the form

$$u_k(t) = t^{-k}\frac{d}{dt}\sum_{n=1}^{\infty}\frac{t^{k+1}Y_{k+2}(t;\lambda_n)}{Y_{k+2}'(1;\lambda_n)}\lambda_n B^2(\lambda_n B - A)^{-1}(aB + bA)^{-1}u_0, \tag{4.3}$$

where $u_0 \in D(A^3) \cap D(B^2)$ and λ_n, $n \in \mathbb{N}$, are the zeros of the function $\Phi_{k,a,b}(\lambda) = (a + b\lambda)Y_{k+2}(1;\lambda)$ specified by Eq. (2.3).

Let us state the result thus obtained in the form of a theorem.

Theorem 8. *Suppose that $k \geq 0$, $u_0 \in D(A^3) \cap D(B^2)$, and A and B are closed linear operators commuting on the elements of $D(A^3) \cap D(B^2)$. Let also, for every $n \in \mathbb{N}$, the zeros λ_n of the function $\Phi_{k,a,b}(\lambda) = (a + b\lambda)Y_{k+2}(1; \lambda)$ defined by Eq.* (2.3) *belong to the resolvent set $\rho(B, A)$ of the operator A with respect to B, and let the following bound hold*:

$$\sup_{n\in\mathbb{N}} |\lambda_n| \cdot \|(\lambda_n B - A)^{-1}\| < M_0 < \infty. \tag{4.4}$$

Then the nonlocal problem (1.1), (2.1), (4.1) *is uniquely solvable, and its solution $u_k(t)$ is determined by Eq.* (4.3).

Example 5. Assume that in problem (1.1), (2.1), (4.1) we have $k = 0$, $-a/b \in \rho(A)$, and $B = I$, and let the operator $-A$ be the generator of an operator cosine function $C(t; -A)$ of exponential growth ω; it is well known that the resolvent of $-A$ satisfies the bound (4.4). Then, according to formula (4.2), the solution of the nonlocal problem (1.1), (2.1), (4.1) has the form

$$u_0(t) = \frac{d}{dt}(tu_2(t)),$$

where $u_2(t)$ is defined in Example 1, and therefore,

$$u_0(t) = \frac{d}{dt}\biggl(\sin \pi t \int_0^\infty \frac{C(s; -A)(aI + bA)^{-1}u_0\, ds}{\cosh \pi s + \cos \pi t}\biggr).$$

Example 6. In Eq. (1.1), let the operator A be the multiplication by a scalar $A < 0$, and let $B = 1$. Then, for the case in which the parameters in problem (1.1), (2.1), (4.1) satisfy the conditions $k \geq 0$ and $a + bA \neq 0$, by Example 2, the solution of the boundary value problem (2.6) has the form

$$u_{k+2}(t) = \frac{t^{-1/2-k/2} J_{k/2+1/2}(t\sqrt{-A})(a + bA)^{-1}u_0}{J_{k/1+1/2}(\sqrt{-A})},$$

and by formula (4.2), the solution of the nonlocal problem (1.1), (2.1), (4.1) is defined by the relation

$$u_k(t) = t^{1-k}\biggl(\frac{1}{t}\,\frac{d}{dt}\biggr)(t^{k+1}u_{k+2}(t)) = \frac{t^{1/2-k/2}\sqrt{-A}J_{k/2-1/2}(t\sqrt{-A})u_0}{(a + bA)J_{k/2+1/2}(\sqrt{-A})}.$$

In particular, for $k = 0$ we have the representation

$$u_0(t) = \frac{\sqrt{-A}\cos(t\sqrt{-A})u_0}{(a + bA)\sin(\sqrt{-A})}.$$

5. NONLOCAL CONDITION OF THE SECOND KIND. CASE OF $k < 1$, DIRICHLET CONDITION AT $t = 0$

Let $k < 1$. In this case, for Eq. (1.1), instead of the nonlocal condition of the second kind (4.1), one should specify a condition of the form

$$a\int_0^1 tu(t)\, dt + b\lim_{t\to 1}(t^{k-1}u(t))' = 0, \qquad a \neq 0, \quad b \neq 0. \tag{5.1}$$

The following criterion for the uniqueness of a solution was proved for problem (1.1), (3.1), (5.1) in [17].

Theorem 9. *Let $k < 1$ and let A and B be closed linear operators in E. Assume that the nonlocal problem* (1.1), (3.1), (5.1) *has a solution $u(t)$. For this solution to be unique, it is necessary and, in the case of $u(t) \in C^3((0, 1], D)$, also sufficient that, for any zero λ_m, $m \in \mathbb{N}_0$, of the function $\Theta_{k,a,b}(\lambda) = (a + b\lambda)Y_{4-k}(1; \lambda)$, where the function $Y_{4-k}(t; \lambda)$ is given by Eq.* (2.3), *the operator equation* (2.4) *has no nonzero solution.*

Let problem (1.1), (3.1), (5.1) satisfy the conditions of Theorem 9 on the uniqueness of a solution of the nonlocal problem, and in addition, let the conditions of Theorem 5 be satisfied for the parameter value $k-2$ of the singular equation (1.1). Then there exists a unique solution $u_{k-2}(t)$ of the boundary value problem (3.4) for $\beta=1$, which is defined by Eq. (3.3) after the replacement of the parameter k by $k-2$ in this representation. We have not yet used the fact that there exists no nonzero solution of Eq. (2.4) for $\lambda=-a/b$. It will be taken into account below.

As in the proof of Theorem 6, from the solution $u_{k-2}(t)$ of the boundary value problem (3.4) we construct the following solution of Eq. (1.1):

$$u_k(t)=\frac{1}{t}u'_{k-2}(t). \tag{5.2}$$

The solution defined by Eq. (5.2) satisfies condition (3.1). Let us show that this solution $u_k(t)$ also satisfies condition (5.1). Substituting $u_k(t)$ into the left-hand side of condition (5.1), we have

$$\begin{aligned} a\int_0^1 tu_k(t)\,dt+b\lim_{t\to1}(t^{k-1}u(t))' &= a\int_0^1 u'_{k-2}(t)\,dt+b\lim_{t\to1}(t^{k-2}u'_{k-2}(t))' \\ &= au_1+b\lim_{t\to1}\biggl(u''_{k-2}(t)+\frac{k-2}{t}u'_{k-2}(t)\biggr), \end{aligned}$$

and by analogy with Sec. 4, the nonlocal condition (5.1) is satisfied if the boundary value u_1 in problem (3.4) satisfies the relation

$$(aB+bA)u_1=Bu_0.$$

The possibility of this choice can be ensured by the invertibility of the operator $aB+bA$, assuming that the number $\lambda=-a/b$ belongs to the resolvent set $\rho(B,A)$, and then

$$u_1=(aB+bA)^{-1}Bu_0.$$

Thus, the solution of problem (1.1), (3.1), (5.1) is unique and has the form

$$u_k(t)=\frac{1}{t}\frac{d}{dt}\sum_{n=1}^{\infty}\frac{t^{3-k}Y_{4-k}(t;\lambda_n)}{Y'_{4-k}(1;\lambda_n)}\lambda_nB^2(\lambda_nB-A)^{-1}(aB+bA)^{-1}u_0, \tag{5.3}$$

where $u_0\in D(A^4)\cap D(B^2)$ and λ_n, $n\in\mathbb{N}$, are the zeros of the function $\Theta_{k,a,b}(\lambda)=(a+b\lambda)Y_{4-k}(1;\lambda)$ given by Eq. (2.3)

Let us state the result thus obtained in the form of a theorem.

Theorem 10. *Suppose that $k<1$, $u_0\in D(A^4)\cap D(B^2)$, and A and B are closed linear operators commuting on the elements of $D(A^4)\cap D(B^2)$. Suppose also that, for every $n\in\mathbb{N}$, the zeros λ_n of the function*

$$\Theta_{k,a,b}(\lambda)=(a+b\lambda)Y_{4-k}(1;\lambda)$$

defined by Eq. (2.3) *belong to the resolvent set $\rho(B,A)$ of the operator A with respect to B and the following bound holds*:

$$\sup_{n\in\mathbb{N}}|\lambda_n|\cdot\|(\lambda_nB-A)^{-1}\|<M_0<\infty.$$

Then the nonlocal problem (1.1), (3.1), (5.1) *is uniquely solvable, and its solution $u_k(t)$ is determined by Eq.* (5.3).

Example 7. In Eq. (1.1), let the operator A be the multiplication by a scalar $A<0$, and let $B=1$. Then, for the case in which the parameters in problem (1.1), (3.1), (5.1) satisfy the conditions $k<1$ and $a+bA\neq0$, by Example 4 we have a solution of the boundary value problem (3.4)

$$u_{k-2}(t)=\frac{t^{3/2-k/2}J_{3/2-k/2}(t\sqrt{-A})(a+bA)^{-1}u_0}{J_{3/2-k/2}(\sqrt{-A})},$$

and by formula (5.2), the solution of the nonlocal problem (1.1), (3.1), (5.1) has the form

$$u_k(t) = \frac{1}{t} u'_{k-2}(t) = \frac{t^{1/2-k/2}\sqrt{-A}\,J_{1/2-k/2}(t\sqrt{-A})u_0}{(a+bA)J_{3/2-k/2}(\sqrt{-A})}.$$

In particular, for $k = 0$, we have the representation

$$u_0(t) = -\frac{A\sin(t\sqrt{-A})u_0}{(a+bA)(\sin(\sqrt{-A}) - \sqrt{-A}\cos(\sqrt{-A}))}.$$

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The author of this work declares that he has no conflicts of interest.

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