Thermodynamic Limit in Vector Lattice Models

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Abstract Classes of Gibbs random fields $\mathbf{u}(x)$, $x \in \mathcal{Z}^d$ on finite sets $\Lambda \subset \mathcal{Z}^d$, $d \in \mathcal{N}$ with values in the space $\mathcal{R}^n$, $n \in \mathcal{N}$ are studied. Each class is connected with the sequence $\langle \Lambda;\ \Lambda \subset \mathcal{Z}^d \rangle$ unboundedly expanding according to the definite rule when $\Lambda \to \mathcal{Z}^d$. Each random field is generated by the Hamiltonian $\mathsf{H}_\Lambda[\mathbf{u}(z)]$. Classes of all functionals $\mathsf{H}_\Lambda[\mathbf{u}(z)]$ corresponding to sequence $\langle \Lambda;\ \Lambda \subset \mathcal{Z}^d \rangle$ form the Banach space H_ν. It is proved the existence of the limit statistical characteristic $\ln\ Z_\Lambda/|\Lambda|$ in each class when $\Lambda \to \mathcal{Z}^d$ which is the continuous functional in H_ν.

Keywords Vector models · Hamiltonian · Gibbs' random field · Free energy · Phase space · Thermo-dynamic limit

1 Introduction

The object of study in this paper is Gibbs random fields on the integer lattice $\mathcal{Z}^d$, $d \in \mathcal{N}$. The importance of studying such mathematical objects is due to the fact that models of statistical mathematical physics are constructed on their basis (about the subject of the study and the terminology used, see, for example, [1–6]). We will call such models as *vector lattice systems*. From the point of view of theoretical physics, these models describe, within the microscopic approach and with appropriate interpretation of the parameters defining theirs, the thermodynamic behavior of single-crystal solid-state structures in a wide temperature range. Despite the fact that a considerable amount of literature is devoted to the mathematical analysis of such theoretical models, in most mathematical works related to their study within the framework of the formalism of statistical mechanics of classical (non-quantum) systems, the greatest attention is paid to such of them which are called *lattice gases*. For

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such mathematical objects, the terminology has been developed that unites them. In terms of this terminology their properties are established at the level of those requirements that are imposed on mathematical texts. The purpose of this work is to extend these basic concepts to a much wider class of models of statistical mechanics of classical systems which we call, as mentioned above, the vector lattice models. For such systems, we will prove, within the framework of accepted general restrictions, the validity of one of the basic provisions of statistical mechanics, namely, we establish the presence of *extensive* asymptotic $F_\Lambda \sim |\,\Lambda\,|$ of the *free energy* F_Λ if the sets Λ tends to $\mathcal{Z}^d$ according to a certain principle dictated by physical considerations.

2 The Gibbs Random Fields

Consider a random field $\tilde{\mathbf{u}}(\Lambda) = \{\tilde{\mathbf{u}}(x)\,;\, x \in \Lambda\}$ on an arbitrary finite subset Λ of the integer lattice $\mathcal{Z}^d$, $d \in \mathcal{N}$, with elements $x = \langle l_1, ..., l_d\rangle$, $l_j \in \mathcal{Z}$, $j = 1 \div d$. We will call the lattice elements as *vertexes*.[1] This means that corresponding probability space $\mathbf{P}_\Lambda = \langle \Omega_\Lambda, \mathbf{B}_\Lambda, \mathsf{P}_\Lambda\rangle$ consists of Ω_Λ elementary random events (random configurations), σ-algebra $\mathbf{B}_\Lambda$ of measurable subsets of Ω_Λ, each element of which is considered as the random event, and the probability distribution P_Λ on $\mathbf{B}_\Lambda$.

For the Gibbs random fields of vector lattice models considered in this paper, the listed components of the probability space $\mathbf{P}_\Lambda$ are defined as follows. Denote the set $\Omega \equiv \mathcal{R}^n$, which we will call the phase space of each vertex in $\mathcal{Z}^d$. The number $n \in \mathcal{N}$ is the dimension of the vector field fixed during the work.

For any subset of $\Lambda \subset \mathcal{Z}^d$, we define the space $\Omega_\Lambda = \Omega^\Lambda$. This means that each vertex $x = \langle l_1, ..., l_d\rangle \in \Lambda$ is mapped to a point of the Ω space which is assigned the label x and, as a result of such an operation, the phase space Ω_x is obtained. Then, for any $\Lambda \subset \mathcal{Z}^d$, the space of elementary events, which we will call *the space of states* (configurations), is represented by the formula

$$\Omega_\Lambda = \bigotimes_{x \in \Lambda} \Omega_x\,. \tag{1}$$

On the space Ω, there is a natural measurability structure defined by the σ-algebra $\mathbf{B}$ of Borel sets in $\mathcal{R}^n$. Then, similarly to the formula (1), by assigning labels, σ-algebras $\mathbf{B}_x$, $x \in \mathcal{Z}$ are introduced on each of the spaces Ω_x and, on the bases of them, the σ-algebra $\mathbf{B}_\Lambda$ is constructed on Ω_Λ

$$\mathbf{B}_\Lambda = \bigotimes_{x \in \Lambda} \mathbf{B}_x\,. \tag{2}$$

In accordance with this structure of measurability, we will also assume that the measure M is defined on the σ-algebra $\mathbf{B}$. For simplicity of further constructions, we

[1] Here and further throughout the text, random variables are marked with the “tilde” sign.

will assume with respect to this measure that it does not contain a *singular* component, that is, it has a derivative $d\mathsf{M}/d\mathbf{u} = D(\mathbf{u}) \geq 0$ of the Lebesgue measure in $\mathcal{R}^n$ with the differential $d\mathbf{u} = du_1...du_n$. This derivative is expressed as a generalized function with respect to the countably normalized space of locally continuous functions on $\mathcal{R}^n$. In particular, in the case of $n = 1$, this means that in the Lebesgue decomposition of the measure M on $\mathcal{R}$, there are only absolutely continuous and discrete components.

On the basis of the measure M, by assigning labels $x \in \mathcal{Z}^d$, we introduce measures M_x on σ-algebras $\mathbf{B}_x$ and, as a result, the measure $\mathbf{B}_\Lambda$ is defined as a product of measures

$$\mathsf{M}_\Lambda = \prod_{x\in\Lambda} \mathsf{M}_x\,, \quad d\mathsf{M}_\Lambda = \prod_{x\in\Lambda} D(\mathbf{u}(x))d\mathbf{u}(x)\,. \tag{3}$$

Each Gibbs random field is $\tilde{\mathbf{u}}(\Lambda)$ is constructed by the definition of the probability distribution P_Λ on a measurable space $\langle \Omega_\Lambda, \mathbf{B}_\Lambda, \mathsf{M}_\Lambda \rangle$. Its random realizations $\tilde{\mathbf{u}}(\Lambda) \in \Omega_\Lambda$ are represented by mappings $\tilde{\mathbf{u}}(\Lambda) : \Lambda \mapsto \mathcal{R}^n$. Due to the finiteness of the set Λ, each such mapping can be considered as a collection of $\{\tilde{\mathbf{u}}(x)\,; x \in \Lambda\}$ of $|\,\Lambda\,|$ (number of vertexes in Λ) random variables taking the value in $\mathcal{R}^n$. The fact that we consider further this set of random variables as a *Gibbs random field* means that the probability distribution P_Λ has a non-negative density on the measure M_Λ defined by the formula

$$d\mathsf{P}_\Lambda = \frac{1}{Z_\Lambda} \exp\big(-\mathsf{H}_\Lambda[\mathbf{u}(z)]\big)d\mathsf{M}_\Lambda\,, \tag{4}$$

where each of the functionals $\mathsf{H}_\Lambda[\mathbf{u}(z)]$, $\Lambda \subset \mathcal{Z}^d$ is called the *Hamiltonian* of the Gibbs random field.

Statistical characteristic Z_Λ of the probability distribution (4) called the *partition function*, is determined on the basis of the normalization condition $\mathsf{P}_\Lambda(\Omega_\Lambda) = 1$ of the distribution P_Λ

$$Z_\Lambda = \int_{\Omega_\Lambda} \exp\big(-\mathsf{H}_\Lambda[\mathbf{u}(z)]\big)d\mathsf{M}_\Lambda\,. \tag{5}$$

Thus, for a fixed measure M, we assume the choice only of such functionals $\mathsf{H}_\Lambda[\mathbf{u}(z)]$ for which this integral is finite.

In order to connect probability spaces $\{\mathbf{P}_\Lambda\,; \Lambda \subset \mathcal{Z}^d\}$ defined at various $\Lambda \subset \mathcal{Z}^d$ by the fixed phase the space Ω and the fixed measure M on it with statistical mechanics models, it is necessary to distribute these spaces by equivalence classes such that one may take into account the property of physical uniformity.

This is done, firstly, taking into account the fact that the translation of the set Λ should not change the physical predictions, that is, it should not change values of statistical averages obtained as a result of calculations on the basis of a mathematical model.

Secondly, it should be taken into account that the sets Λ for statistical mechanics systems consist of an indefinitely large number of vertexes so that each *intensive thermodynamic characteristic*, related to one vertex of the lattice, is practically independent on $|\,\Lambda\,|$.

The first of these requirements can be satisfied by assuming that the collection of all Hamiltonians $\mathsf{H}_\Lambda[\cdot]$, $\Lambda \subset \mathcal{Z}^d$ describing the same physical system, subject to a condition that reflects independence of all statistical averages on the location of the set Λ in $\mathcal{Z}^d$. This is expressed by the property of the *translational invariance*. Let us formulate the simplest version of such a condition. Let z be an arbitrary vertex of $\mathcal{Z}^d$. Then the space Ω_Λ and the measure M_Λ on it have the property

$$\Omega_{\Lambda+z} = \Omega_\Lambda|_{\mathsf{u}(x)\to\mathsf{u}(x+z)}, \quad \mathsf{M}_{\Lambda+z} = \mathsf{M}_\Lambda|_{\mathsf{u}(x)\to\mathsf{u}(x+z)} . \tag{6}$$

Hamiltonian $\mathsf{H}_\Lambda[\mathbf{u}(x)]$ is called the translationally invariant one if it has the following property

$$\mathsf{H}_{\Lambda+z}[\mathbf{u}(x)] = \mathsf{H}_\Lambda[\mathbf{u}(x+z)] . \tag{7}$$

Each Hamiltonian $\mathsf{H}_\Lambda[\mathbf{u}(x)]$ is defined as a function on vector variables $\{x \in \Lambda\}$ for each set Λ. We denote this function as $\mathbf{u}(\Lambda) = \{\mathbf{u}(x)\,;\, x \in \Lambda\}$. Then, the property (7) means that all these functions are the same for all sets $\Lambda + z$, $z \in \mathcal{Z}^d$.

Theorem 1 *If Hamiltonian* $\mathsf{H}[\mathbf{u}(z)]$ *is translationally invariant, then the probability distributions* P_Λ *and* $\mathsf{P}_{\Lambda+z}$ *are equivalent in the sense that*

$$d\mathsf{P}_{\Lambda+z}[\mathbf{u}(y)] = d\mathsf{P}_\Lambda[\mathbf{u}(y+z)] . \tag{8}$$

Proof Statement directly follows from (5)–(7).

Let us now proceed to the discussion of the second requirement for a Gibbs random field with Hamiltonians $\mathsf{H}_\Lambda[\mathbf{u}(z)]$ which allows distribute them into equivalence classes. Let us fix some lattice vertex $\mathcal{Z}^d$ which we will call the zero one. We will consider only Gibbs fields on sets Λ that contain this vertex. Due to the necessity to use a large number of vertexes $|\,\Lambda\,|$ (even for the smallest experimentally studied nanoparticles of a solid state substance $|\,\Lambda\,| \approx 10^6$ and more), it does not make sense to accurately calculate the expectations of $\mathsf{E}_\Lambda(\cdot)$ on the basis of the probability measure P_Λ.

On the contrary, in the practice of using of probability theory methods in theoretical statistical physics, it is necessary only to have confidence the fact that the calculated thermodynamic characteristics have a quite definite asymptotic behavior at unlimited increase of the set Λ occupied by the thermodynamically homogeneous medium under study. In this case, only the main asymptotic terms of expectations $\mathsf{E}_\Lambda(\cdot)$ on the probability measure P_Λ are of interest when Λ is expanded to $\mathcal{Z}^d$ according to a definite rule. Transition to the limit at $\Lambda \to \mathcal{Z}^d$ according to corresponding expanding sequences of statistical characteristics of Gibbs random fields is called the transition to *thermodynamic limit* in statistical mechanics.

In this paper we study so-called the *extensive systems* which are traditional to statistical mechanics when the function $F_\Lambda = \ln Z_\Lambda$, which is named theirs free energy, has the asymptotic

$$F_\Lambda[\mathsf{H}_\Lambda] = |\,\Lambda\,|\big(f(\mathsf{M}, \mathsf{H}_\Lambda) + o(1)\big)\,, \tag{9}$$

that is, this thermodynamic characteristic has the certain density $f(\mathsf{M}, \mathsf{H}_\Lambda)$ which is the functional on the measure M and on the Hamiltonian $\mathsf{H}_\Lambda[\mathbf{u}(z)]$.

The concept of the thermodynamic limit transition needs the serious clarification, since there are some different ways to construct expanding sequences $\langle \Lambda;\, \Lambda \subset \mathcal{Z}^d \rangle$ which are associated with fundamentally different physical situations, and which, generally speaking, should not lead to the same result.

The simplest type of sequences $\langle \Lambda;\, \Lambda \subset \mathcal{Z}^d \rangle$ used in statistical mechanics, whose components serve as geometric models of crystals and which we will consider further is represented by the sets $\Lambda = \{0, 1, ..., L\}^d$ where $L \in \mathcal{N}$ is the size of the "crystal". The number of vertexes in each of such sets is equal $|\,\Lambda\,| = (L+1)^d < \infty$.

Let us consider the equality

$$\mathsf{H}_\Lambda[\mathbf{u}(z)] = \sum_{\Gamma \subset \Lambda\,:\,|\Gamma| > 1} V_\Gamma\big(\mathbf{u}(\Gamma)\big) \tag{10}$$

where each function $V_\Gamma\big(\mathbf{u}(\Gamma)\big)$ at fixed set $\Gamma \subset \Lambda$ of vertexes depends on corresponding collection $\mathbf{u}(\Gamma) = \{\mathbf{u}(x)\,;\, x \in \Gamma\}$. One may consider this equality as the functional equation defining functions $V_\Gamma(\cdot)$. These functions, which we further call *potentials*, are defined by recursively as the solution of this equation, using the induction on the number $|\,\Lambda\,|$ and putting $V_\Gamma\big(\mathbf{u}(\Gamma)\big) = 0$ with $|\Gamma| = 1$. By induction, it is also established that the potentials $V_\Gamma(\cdot)$ have a property similar to (7). Namely, since

$$\sum_{\Gamma \subset \Lambda} V_\Gamma(\mathbf{u}(\Gamma + z)) = \mathsf{H}_\Lambda[\mathbf{u}(x+z)]\,, \qquad \sum_{\Gamma \subset \Lambda + z} V_\Gamma\big(\mathbf{u}(\Gamma)\big) = \mathsf{H}_{\Lambda+z}[\mathbf{u}(x)]\,,$$

then the potentials $V_\Gamma(\mathbf{u}(\Gamma))$ for all sets, which are differed from each other only by shifts with arbitrary vector $z \in \mathcal{Z}^d$, coincides, $V_\Gamma(\mathbf{u}(\Gamma)) = V_{\Gamma+z}(\mathbf{u}(\Gamma))$.

We do not include terms $\Gamma = \{x\}$ with $|\Gamma| = 1$ in $\mathsf{H}_\Lambda[\mathbf{u}(x)]$ and refer them to the definition of M_x, $x \in \Lambda$. At the same time, as already mentioned above, we restrict ourselves to the case when all measures M_x are isomorphic between themselves, that is, they are instances of the same measure M.

Definition 1 The class of Gibbs random fields whose probability spaces $\langle \mathbf{P}_\Lambda,\, \Lambda \subset \mathcal{Z}^d \rangle$ are constructed on the basis of the same measurable phase space $\langle \Omega, \mathbf{B}, \mathsf{M} \rangle$, whose Hamiltonians are determined by the same set of potentials $V_\Gamma(\mathbf{u}(\Gamma))\,;\, |\Gamma| \in \mathcal{N} \setminus \{1\}\rangle$ so that corresponding partition functions are finite when the sequence

$\langle \Lambda \ \ \Lambda \subset \mathcal{Z}^d \rangle$ of sets coincides with $\langle \Lambda(L) = \{0, 1, ..., L\}^d\, ;\, L \in \mathcal{N} \rangle$ with their suitable translation, we will call the limit Gibbs random field on $\mathcal{Z}^d$.

Thus, the limit Gibbs random field is determined by the sequence of Hamiltonians $\langle \mathsf{H}_{\Lambda(L)}[\cdot]\, ;\, \Lambda(L) \subset \mathcal{Z}^d \rangle$ which is constructed on the basis of potentials by decomposition (10) where the sets $\Lambda(L)$ are defined by $L \in \mathcal{N}$.

Note that, accepting this definition, we adhere to a more traditional view about the thermodynamic limit for statistical characteristics of Gibbs random fields within the framework of statistical mechanics, in contrast to the approach known in statistical mathematical physics. It consists of determination of the Gibbs random field on the entire lattice $\mathcal{Z}^d$ by means of a set of conditional probabilities allowed by the fixed set of *relative Hamiltonians* (see, [7]).

3 The Hamiltonians Space $\mathbf{H}_\nu$ of Limit Gibbs Fields

Note that the study of limit Gibbs random fields is sufficient to carry out fixing only their generating family of sets $\{\Lambda(L)\, ;\, L \in \mathcal{N}\}$ without the account of translations, on which the further presentation in this paper is based. Moreover, we will study a family of Gibbs random fields with the fixed measure M. With the account of these remarks, every limit Gibbs random field uniquely characterized by the class of Hamiltonians $\mathbf{H} = \{\mathsf{H}_{\Lambda(L)}[\cdot]\, ;\, L \in \mathcal{N}\}$ which is defined by the fixed set of potentials $\{V_\Gamma(\mathbf{u}(\Gamma))\, ;\, |\Gamma| \in \mathcal{N} \setminus \{1\}\}$. It is obvious that all such classes form a linear manifold with natural linear operations.

Let us further assume, throughout the work, that there is a monotone function $\nu(s) > 0$, $s \in (0, \infty)$ such that the integral $\int_{\mathcal{R}^n} \exp\big(a\nu(|\mathbf{u}|)\big) d\mathsf{M}(\mathbf{u}) < \infty$ defined by the density $D(\mathbf{u})$ of the measure M, converges for any $a > 0$. In particular, this takes place if the support of the measure M is compact, that is, it is concentrated on the interval $[0, s_*]$, $s_* < \infty$ and its density $D(\mathbf{u})$ is zero at $|\mathbf{u}| > s_*$. It takes place in the case for the standard vector model (see, for example, [8]). In this case one may consider $\nu = 1$.

We connect the study of Gibbs random fields when their measures M have non-compact supports in order to apply our results for such objects of statistical mathematical physics as, for example, the Berlin-Katz spherical model [9, 10], the Gaussian model and the φ^4 model which play an important role in the fluctuation theory of phase transitions (see, [11]). One may note that the above described Gibbs random field on $\mathcal{Z}^d$ include, in particular, all classical lattice models at $n = 1$ specified in [2]. To see this fact it is sufficient to introduce the measure M with the density $D(u) = \sum_{l=1}^{N} \delta(u - l) e^{\mu(u)}$ on the space $\Omega = \mathcal{R}$.

Further, we fix the function $\nu(\cdot)$ connected with the measure M. Let the potentials $V_\Gamma(\mathbf{u}(\Gamma))$ depend continuously on the values of the field $\mathbf{u}(x)$, $x \in \Lambda$. Then, there exists the function

$$\mathsf{G}(\Gamma) \equiv \sup_{\mathbf{u}(\Gamma)} \frac{\left|V_\Gamma(\mathbf{u}(\Gamma))\right|}{\sum\limits_{x \in \Gamma} \nu(|\mathbf{u}(x)|)} < \infty \tag{11}$$

for each set $\Gamma \subset \mathcal{Z}^d$. Let us additionally assume that the Hamiltonians $\mathsf{H}_{\Lambda(L)}[\mathbf{u}(x)]$ included in each fixed class **H** have such a property that for any vertex $z \in \Lambda$ it takes place

$$\mathsf{N}\big[\mathsf{H}_{\Lambda(L)}\big] \equiv \sum_{\Gamma \subset \mathcal{Z}^d:\ z \in \Gamma,\ |\Gamma|>1} \mathsf{G}(\Gamma) < \infty\,. \tag{12}$$

Due to the translational invariance of potentials, the values of the functional $\mathsf{N}[\cdot]$ on classes **H** of Hamiltonians $\{\mathsf{H}_{\Lambda(L)}\,;\ L \in \mathcal{N}\}$ does not depend on the choice of the vertex $z \in \mathcal{Z}^d$. Then, on the linear manifold of all such classes of Hamiltonians, it is possible to introduce the norm $\mathsf{N}[\cdot]$ that turns this manifold into the Banach space H_ν. In order to simplify the presentation, we omit the proof of the completeness of this space. It is very important that this norm allows also the following definition

$$\|\mathsf{H}_{\Lambda(L)}\| \equiv \sup_{L \in \mathcal{N}}\ \sup_{\mathbf{u}(\Lambda(L))} \frac{\mathsf{W}_{\Lambda(L)}\big[\mathsf{H}_{\Lambda(L)}\big]}{\sum\limits_{x \in \Lambda(L)} \nu(|\mathbf{u}(x)|)}\,,$$

$$\mathsf{W}_{\Lambda(L)}\big[\mathsf{H}_{\Lambda(L)}\big] = \sum_{\Gamma \subset \Lambda(L):\ |\Gamma|>1} |V_\Gamma(\mathbf{u}(\Gamma))|\,. \tag{13}$$

It is valid the following statement.

Theorem 2 *It takes place the equality*

$$\|\mathsf{H}_{\Lambda(L)}\| = \mathsf{N}\big[\mathsf{H}_{\Lambda(L)}\big]\,. \tag{14}$$

Proof Let us consider the inequalities

$$|V_\Gamma(\mathbf{u}(\Gamma))| \le \mathsf{G}(\Gamma) \sum_{x \in \Gamma} \nu(|\mathbf{u}(x)|)\,, \quad \Gamma \subset \Lambda(L)\,.$$

Summing them on all $\Gamma \subset \Lambda(L)$ at $|\Gamma| > 1$, we obtain

$$\mathsf{W}_{\Lambda(L)}\big[\mathsf{H}_{\Lambda(L)}\big] = \sum_{\Gamma \subset \Lambda(L):\ |\Gamma|>1} |V_\Gamma(\mathbf{u}(\Gamma))| \le \sum_{\Gamma \subset \Lambda(L):\ |\Gamma|>1} \mathsf{G}(\Gamma) \sum_{x \in \Lambda(L)} \nu(|\mathbf{u}(x)|) \le$$

$$\le \sum_{x \in \Lambda(L)} \nu(|\mathbf{u}(x)|) \sum_{x \in \Gamma \subset \Lambda(L)} \mathsf{G}(\Gamma) \le \mathsf{N}\big[\mathsf{H}_{\Lambda(L)}\big] \cdot \sum_{x \in \Lambda(L)} \nu(|\mathbf{u}(x)|)$$

and, therefore,

$$\|\mathsf{H}_{\Lambda(L)}\| = \sup_{\Lambda(L)\subset \mathcal{Z}^d} \frac{\mathsf{W}_{\Lambda(L)}[\mathsf{H}_{\Lambda(L)}]}{\sum_{x\in\Lambda(L)} \nu(|\mathbf{u}(x)|)} \le \mathsf{N}[\mathsf{H}_{\Lambda(L)}]. \tag{15}$$

Let us establish the inverse inequality. Choose a value $\varepsilon > 0$. Then, there will be such $L \in \mathcal{N}$ and the field $\mathbf{u}(x)$, $x \in \Lambda(L)$ for which the following inequality

$$\mathsf{W}_{\Lambda(L)}[\mathsf{H}_{\Lambda(L)}] \ge \left(\|\mathsf{H}_{\Lambda(L)}\| - \varepsilon\right) \sum_{x\in\Lambda(L)} \nu(|\mathbf{u}(x)|)$$

takes place. On the other hand, we have

$$\mathsf{W}_{\Lambda(L)}[\mathsf{H}_{\Lambda(L)}] = \sum_{\Gamma\subset\Lambda(L):\ |\Gamma|>1} \mathsf{D}(\Gamma) \sum_{x\in\Gamma} \nu(|\mathbf{u}(x)|) \le \mathsf{N}[\mathsf{H}_{\Lambda(L)}] \sum_{x\in\Lambda(L)} \nu(|\mathbf{u}(x)|)$$

and, therefore, $\|\mathsf{H}_{\Lambda(L)}\| - \varepsilon \le \mathsf{N}[\mathsf{H}_{\Lambda(L)}]$. Due to the arbitrariness of the value $\varepsilon > 0$, there is an inequality $\|\mathsf{H}_{\Lambda(L)}\| \le \mathsf{N}[\mathsf{H}_{\Lambda(L)}]$. The validity of (14) follows from it and from the inequality (15).

We show that if the limit random field defined by the class of Hamiltonians $\{\mathsf{H}_{\Lambda(L)};\ L \in \mathcal{N}\}$ which belongs to the space H_ν with a function $\nu(\cdot)$, then it is correctly defined. Namely, it is valid

Theorem 3 *If the integral* $\int_{\mathcal{R}^n} \exp\left(a\nu(|\mathbf{u}|)\right)d\mathsf{M}(\mathbf{u}) < \infty$ *converges for a monotone function* $\nu(s) > 0$, $s \in (0,\infty)$ *and for any* $a > 0$ *and if the class* $\mathbf{H}$ *of Hamiltonians defined by the set of potentials* $\{V_\Gamma(\mathbf{u}(\Gamma))\,;\ |\Gamma| \in \mathcal{N}\setminus\{1\}\}$ *belongs to* H_ν, *then the partition function* Z_Λ, *defined by (5), is finite and, therefore, the corresponding Gibbs random the field is defined for all* $\Lambda \subset \mathcal{Z}^d$.

Proof Let Hamiltonians $\mathsf{H}_{\Lambda(L)}$ be satisfied the condition (12). Then, on the basis of definition (5) and according to (14), the following estimates are valid

$$Z_\Lambda \le \int_{\Omega_\Lambda} \exp\left(|\mathsf{H}_\Lambda[\mathbf{u}(z)]|\right)d\mathsf{M}_\Lambda \le \int_{\Omega_\Lambda} \exp\left(\|\mathsf{H}_{\Lambda(L)}[\mathbf{u}(z)]\| \cdot \sum_{x\in\Lambda} \nu(|\mathbf{u}(x)|)\right)d\mathsf{M}_\Lambda \le$$

$$\le \prod_{x\in\Lambda} \int_{\Omega_x} \exp\left(\|\mathsf{H}_{\Lambda(L)}\| \cdot \nu(|\mathbf{u}(x)|)\right)d\mathsf{M}_x(\mathbf{u}(x)) =$$

$$= \left[\int_{\Omega} \exp\left(\|\mathsf{H}_{\Lambda(L)}\| \cdot \nu(|\mathbf{u}|)\right)d\mathsf{M}(\mathbf{u})\right]^{|\Lambda|}. \tag{16}$$

We give the following

Definition 2 Let the class $\{\mathsf{H}_{\Lambda(L)} : \Lambda(L) = \{0, 1, ..., L\}^d; L \in \mathcal{N}\}$ of Hamiltonians determines the limit Gibbs random field with fixed measure M on the phase space Ω. The set of limit Gibbs random fields defined by the set $\beta\mathbf{H} = \{\beta\mathsf{H}_{\Lambda(L)}[\cdot] : L \in \mathcal{N}\}$ of classes of Hamiltonians contained in H_ν where each set is parameterized by $\beta > 0$, is called the lattice classical model of statistical mechanics corresponding to $\beta\mathbf{H}$.

Introduction of the set of classes of Hamiltonians which is represented as a rectilinear ray in the space H_ν, is connected with the fact that the model of equilibrium statistical mechanics is defined by the thermodynamic interpretation of measurable parameters of corresponding limit Gibbs field. First of all, it refers to the main thermodynamic parameter, that is the temperature. According to the canons of statistical mechanics, it is proportional to β^{-1}.

4 The Extensive Asymptotics of Free Energy

Our aim is the proof the asymptotic formula (9) at $\Lambda(L) \to \mathcal{Z}^d$ for each lattice system of statistical mechanics.

Definition 3 The Hamiltonian (10) has the finite range of action if there exists such a finite set $\Delta \subset \mathcal{Z}^d$, $\mathbf{0} \in \Delta$ of vertexes for which $V_\Gamma(\mathbf{u}(\Gamma)) \neq 0$ only in the case when there is such a vertex $z \in \Gamma$ that $\Gamma - z \subset \Delta$.

If the Hamiltonian $\mathsf{H}_{\Lambda(L)}$ has a finite range of action, the pointed out set Δ is named its *support*. It is obvious that all such Hamiltonians form the linear manifold $\mathrm{H}^{(0)}$ in the Banach space H_ν. We begin the proof of the extensiveness of the free energy from the proof of the following statement.

Theorem 4 *For the fixed measure* M *and any finite set* $\Lambda \subset \mathcal{Z}^d$, *the corresponding manifold* $\mathrm{H}^{(0)}$ *of Hamiltonians* H_Λ *is dense in the space* H_ν.

Proof Let us fix the value $\varepsilon > 0$. Since the sum in (12) is finite for the fixed Hamiltonian $\mathsf{H}_\Lambda[\cdot]$, one may choose the finite family Σ of finite subsets $\Gamma \subset \mathcal{Z}^d$ such that each of them contains the vertex $\mathbf{0}$ and it takes place the inequality

$$\sum_{\Gamma \subset \mathcal{Z}^d :\, \mathbf{0} \in \Gamma, \Gamma \notin \Sigma} \mathsf{G}(\Gamma) < \varepsilon\,. \tag{17}$$

Let us introduce the set

$$\Delta = \bigcup_{\Gamma \in \Sigma} \Gamma\,.$$

We add the family Σ such that it should contain all sets $\Gamma \subset \Delta$. The inequality (17) is strengthened only at such an expansion. After that, we define $\widehat{V}_\Gamma(\mathbf{u}(\Gamma)) = V_\Gamma(\mathbf{u}(\Gamma))$, if one may find such a vertex $z \in \mathcal{Z}^d$ for which the inclusion $(\Gamma + z) \in \Sigma$ takes place. In opposite case, we define $\widehat{V}_\Gamma(\mathbf{u}(\Gamma)) = 0$. The latter means that $\widehat{V}_{\Gamma'}(\mathbf{u}(\Gamma')) = 0$ every time when the set $\Gamma' \subset \mathcal{Z}^d$ is such that $\Gamma' + z \not\subset \Lambda$ takes place for any vertex $z \in \mathcal{Z}^d$.

Further, we define the Hamiltonian

$$\widehat{\mathsf{H}}_\Lambda[\mathbf{u}(z)] = \sum_{\Gamma \subset \Lambda\,:\,|\Gamma|>1} \widehat{V}_\Gamma\big(\mathbf{u}(\Gamma)\big)\,. \tag{18}$$

It belongs to the linear manifold $\mathrm{H}^{(0)}$. Then, using the determination of potentials $\widehat{V}_\Gamma\big(\mathbf{u}(\Gamma)\big)$, due to the Theorem 1 the following equality

$$\|\mathsf{H}_\Lambda - \widehat{\mathsf{H}}_\Lambda\| = \mathsf{N}\big[\mathsf{H}_\Lambda - \widehat{\mathsf{H}}_\Lambda\big] = \sum_{\Gamma \subset \mathcal{Z}^d\,:\,\mathbf{0} \in \Gamma,\, \Gamma \notin \Sigma} \mathsf{G}(\Gamma) < \varepsilon$$

takes place that is any Hamiltonian H_Λ may be approximate arbitrarily accurate in the space H_ν by the Hamiltonian $\widehat{\mathsf{H}}_\Lambda$ with finite range of action.

To solve the problem which is set at the beginning of the section, some following supplementary properties of density $D(\cdot)$ should be used. According to the basic supposition, the measure M has the density $D(\cdot)$ which is a generalized function relative to the space of continuous functions. It consists of two summands $D(\cdot) = D_c(\cdot) + D_d(\cdot)$ where $D_c(\cdot)$ is measurable bounded nonnegative function on $\mathcal{R}^n$ and $D_d(\mathbf{u}) = \sum_k \mu_k \delta(\mathbf{u} - \mathbf{v}_k)$; $\mu_k > 0$, $\mathbf{v}_k \in \mathcal{R}^n$. Denote $D^\alpha(\mathbf{u}) = D_c^\alpha(\mathbf{u}) + \sum_k \mu_k^\alpha \delta(\mathbf{u} - \mathbf{v}_k)$ at $0 < \alpha < 1$. We will say that such a density $D(\cdot)$ is bounded by the value K if $\max D_c(\mathbf{u}) \le K$ and $\mu_l \le K, l \in \mathcal{N}$.

In addition to the existence of positive monotone function $\nu(s)$ on $(0, \infty)$ such that the density $D(\mathbf{u})$ possesses the property $\int_\Omega \exp(a\nu(|\mathbf{u}|)) D(\mathbf{u}) d\mathbf{u} < \infty$ at any $a > 0$, we will suppose also the availability of some supplementary more strong restrictions for the density when the basic result of the paper will be obtained in this section.

Lemma 1 *Let the Hamiltonians class* $\{\mathsf{H}_{\Lambda(L)}\,;\, L \in \mathcal{N}\}$ *belongs to the space* H_ν. *Let also the density* $D(\cdot)$ *of measure* M *defines the limit Gibbs random field together with this class. If* $D(\cdot)$ *is bounded by the value* K *and there exists such a nonnegative function* $\nu(s)$, *the value* $\alpha \in (0, 1)$ *for which the integral* $\int_\Omega D^\alpha(\mathbf{u}) e^{a\nu(\mathbf{u})} d\mathbf{u} < \infty$ *is finite and also the function* $\nu(\mathbf{u}) D^{1-\alpha}(\mathbf{u})$ *is bounded by the value* $K_\nu > 0$, *then the following inequality is valid for expectation* $\mathsf{E}_{\Lambda(L)} \nu(|\tilde{\mathbf{u}}(x)|) < K_\nu K^{1-\alpha}$ *and for any vertex* $x \in \mathcal{Z}^d$.

Proof Since the function $\nu(\mathbf{u}) D^{1-\alpha}(\mathbf{u})$ is bounded by the value K_ν, then, for the following integral with any nonnegative weight function $W(\cdot)$ on $\mathcal{R}^n$, the estimate

$$\int_{\mathcal{R}^n} \nu(\mathbf{u}) D(\mathbf{u}) W(\mathbf{u}) d\mathbf{u} < K_\nu \int_{\mathcal{R}^n} D^\alpha(\mathbf{u}) W(\mathbf{u}) d\mathbf{u} \tag{19}$$

takes place. By the same way, since the density $D(\mathbf{u})$ is bounded by the value K, the inequality

$$\int_{\mathcal{R}^n} D(\mathbf{u})W(\mathbf{u})d\mathbf{u} < K^{1-\alpha} \int_{\mathcal{R}^n} D^{\alpha}(\mathbf{u})W(\mathbf{u})d\mathbf{u} \tag{20}$$

is valid.

Now, we note that, due to the lemma conditions relative the integral with the density $D(\cdot)$, the following partition function is finite (see the proof of Theorem 2),

$$Z_{\Lambda(L),\alpha} = \int_{\Omega_x} D^{\alpha}(\mathbf{u}(|x|))d\mathbf{u}(x) \int_{\Omega_{\Lambda(L)\setminus x}} \exp(-\mathsf{H}_{\Lambda(L)}[\mathbf{u}(z)])d\mathsf{M}_{\Lambda(L)\setminus x} <$$

$$< \int_{\Omega} \exp\Big(\|\mathsf{H}_{\Lambda(L)}\| \cdot \nu(|\mathbf{u}|)\Big) D^{\alpha}(\mathbf{u})d\mathbf{u} \cdot \Big[\int_{\Omega} \exp\Big(\|\mathsf{H}_{\Lambda(L)}\| \cdot \nu(|\mathbf{u}|)\Big) d\mathsf{M}(\mathbf{u})\Big]^{|\Lambda|-1},$$

since $|\mathsf{H}_{\Lambda(L)}| \le \mathsf{W}\big[\mathsf{H}_{\Lambda(L)}\big] \le \|\mathsf{H}_{\Lambda(L)}\| \sum_{x\in\Lambda(L)} \nu(|\mathbf{u}|)$.

Then, on the basis of the identity $1 = Z_{\Lambda(L)}/Z_{\Lambda(L)}$, using the inequality (20) for the denominator, we find that

$$Z_{\Lambda(L),\,\alpha} \ge K^{\alpha-1} Z_{\Lambda(L)}\,. \tag{21}$$

By the same way, due to the condition for the integral pointed out and due to the inequality (19), we find the estimate

$$\int_{\Omega_x} \big[\nu(|u(x)|)D^{1-\alpha}(\mathbf{u}(x))\big]D^{\alpha}(\mathbf{u}(x))d\mathbf{u}(x) \int_{\Omega_{\Lambda(L)\setminus x}} \exp(-\mathsf{H}_{\Lambda(L)}[\mathbf{u}(z)])d\mathsf{M}_{\Lambda(L)\setminus x}$$

$$< K_{\nu} Z_{\Lambda(L),\,\alpha}\,. \tag{22}$$

The expression for the expectation $\mathrm{E}_{\Lambda(L)}\big[\,\nu(|\mathbf{u}(x)|)\big]$ is written in the following form

$$\mathrm{E}_{\Lambda(L)}\big[\,\nu(|\mathbf{u}(x)|)\big] = \frac{\displaystyle\int_{\Omega_x} \nu(|\mathbf{u}(x)|)d\mathsf{M}_x \int_{\Omega_{\Lambda(L)\setminus x}} \exp(-\mathsf{H}_{\Lambda(L)}[\mathbf{u}(z)])d\mathsf{M}_{\Lambda(L)\setminus x}}{\displaystyle\int_{\Omega_x} d\mathsf{M}_x \int_{\Omega_{\Lambda(L)\setminus x}} \exp(-\mathsf{H}_{\Lambda(L)}[\mathbf{u}(z)])d\mathsf{M}_{\Lambda(L)\setminus x}}\,.$$

We apply the estimate (22) for the nominator and the estimate (21) for the denominator. Then

$$\mathrm{E}_{\Lambda(L)}\big[\,\nu(|\mathbf{u}(x)|)\big] \le K_{\nu} K^{1-\alpha}\,.$$

Further, we suppose that always the measure M satisfies conditions of Lemma 1.

Let $\Lambda' \subset \Lambda$ and $\mathbf{u}(\Lambda')$ is the restriction of the field $\mathbf{u}(\Lambda)$ on the set Λ'. If the Hamiltonian $\mathsf{H}_\Lambda[\mathbf{u}(z)]$ has the property $\mathsf{H}_\Lambda[\mathbf{u}(\Lambda)] = \mathsf{H}_\Lambda[\mathbf{u}(\Lambda')]$, then we will say that $\mathsf{H}_\Lambda[\mathbf{u}(\Lambda')]$ is the *natural restriction* of the Hamiltonian $\mathsf{H}_\Lambda[\mathbf{u}(z)]$ on the linear manifold $\Omega_{\Lambda'}$ of vector fields $\mathbf{u}(\Lambda')$. We will denote this natural restriction by means of $\mathsf{H}_{\Lambda'}[\mathbf{u}(z)]$.

Lemma 2 *Let $\Lambda \subset \mathbb{Z}^d$. Then, for the partition functions*

$$Z_\Lambda[\mathsf{H}_\Lambda^{(m)}] = \int_{\Omega_\Lambda} \exp\big(-\mathsf{H}_\Lambda^{(m)}[\mathbf{u}(z)]\big) d\mathsf{M}_\Lambda$$

which are defined by Hamiltonians $\mathsf{H}_\Lambda^{(m)}$, $m \in \{1, 2\}$ of the space H_ν such that the difference $\mathsf{H}_{\Lambda'}^{(1)} - \mathsf{H}_{\Lambda'}^{(2)}$ at $\Lambda' \subset \Lambda$ is the natural restriction of the Hamiltonian $\mathsf{H}_\Lambda^{(1)} - \mathsf{H}_\Lambda^{(2)}$ on $\Omega_{\Lambda'}$, the following inequality is valid

$$\big|\ln Z_\Lambda[\mathsf{H}_\Lambda^{(1)}] - \ln Z_\Lambda[\mathsf{H}_\Lambda^{(2)}]\big| \le \Big(\mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|)\Big) \cdot |\Lambda'| \cdot \|\mathsf{H}_{\Lambda'}^{(1)} - \mathsf{H}_{\Lambda'}^{(2)}\| . \tag{23}$$

Proof The Hamiltonian $\mathsf{H}_{\Lambda'}^{(1)} - \mathsf{H}_{\Lambda'}^{(2)}$ possesses the finite norm $\|\cdot\|$. We introduce the family of Hamiltonians $\mathsf{H}[\mathbf{u}(z); t] = \mathsf{H}_{\Lambda'}^{(2)}[\mathbf{u}(z)] + t\big(\mathsf{H}_{\Lambda'}^{(1)}[\mathbf{u}(z)] - \mathsf{H}_{\Lambda'}^{(2)}[\mathbf{u}(z)]\big)$, $t \in [0, 1]$ so that all belong to H_ν, and also we consider the family of corresponding partition functions

$$Z_\Lambda(t) = \int_{\Omega_\Lambda} \exp\big(-\mathsf{H}[\mathbf{u}(z); t]\big) d\mathsf{M}_\Lambda .$$

These functions are finite due to Theorem 2.

Now, we note that the following estimates are valid

$$\left|\frac{d}{dt} \ln Z_\Lambda(t)\right| \le Z_\Lambda^{-1}(t) \int_{\Omega_\Lambda} \left|\frac{d}{dt}\mathsf{H}[\mathbf{u}(z); t]\right| \exp\big(-\mathsf{H}[\mathbf{u}(z); t]\big) d\mathsf{M}_\Lambda \le$$

$$\left\|\frac{d}{dt}\mathsf{H}[\mathbf{u}(z); t]\right\| \cdot Z^{-1}(t) \int_{\Omega_\Lambda} \sum_{x\in\Lambda} \nu(|\mathbf{u}(x)|) \exp\big(-\mathsf{H}[\mathbf{u}(z); t]\big) d\mathsf{M}_\Lambda =$$

$$= \Big(\mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|)\Big)|\Lambda'| \cdot \|\mathsf{H}_{\Lambda'}^{(1)} - \mathsf{H}_{\Lambda'}^{(2)}\| ,$$

if we take into account the definition (13) of the norm and also that the difference $\mathsf{H}_{\Lambda'}^{(1)} - \mathsf{H}_{\Lambda'}^{(2)}$ is the natural restriction on $\Omega_{\Lambda'}$. Here, the expectation $\mathsf{E}\nu(|\tilde{\mathbf{u}}|)$ is finite. Due to Lemma 1, it does not exceed $K_\nu K^{1-\alpha}$. Integrating the obtained inequality from 0 up to 1 and taking into account that $Z_\Lambda(0) = Z_\Lambda[\mathsf{H}_\Lambda^{(2)}]$, $Z_\Lambda(1) = Z_\Lambda[\mathsf{H}_\Lambda^{(1)}]$, the inequality (23) follows.

Lemma 3 *Let the Hamiltonian* $\mathsf{H}_\Lambda \in \mathrm{H}^{(0)}$ *has the finite range of action and* Δ *is the finite subset in* $\mathcal{Z}^d$ *which is its support. Let also* Δ_1 *and* Δ_2 *be any nonintersecting finite subsets in* $\mathcal{Z}^d$, $\Delta_1 \cap \Delta_2 = \emptyset$ *and* $\Sigma_*(\Delta_1, \Delta_2; \Delta)$ *be the set of such vertexes* $z \in \mathcal{Z}^d$ *for which* $(\Delta + z) \cap \Delta_1 \neq \emptyset$, $(\Delta + z) \cap \Delta_2 \neq \emptyset$ *are fulfilled simultaneously.*

Let the Hamiltonians $\mathsf{H}_{\Delta_1 \cup \Delta_2}$, H_{Δ_1}, H_{Δ_2} *are natural restrictions of the Hamiltonian* H_Λ *on* $\Omega_{\Delta_1 \cup \Delta_2}$, Ω_{Δ_1}, Ω_{Δ_2}, *correspondingly. Then, the difference* $(H_{\Delta_1 \cup \Delta_2} - \mathsf{H}_{\Delta_1} - \mathsf{H}_{\Delta_2})$ *has the natural restriction on* $\Omega_{\Sigma_*(\Delta_1, \Delta_2; \Delta)}$ *and the following estimate*

$$|\mathsf{H}_{\Delta_1 \cup \Delta_2}[\mathbf{u}(z)] - \mathsf{H}_{\Delta_1}[\mathbf{u}(z)] - \mathsf{H}_{\Delta_2}[\mathbf{u}(z)]| \leq \|\mathsf{H}_{\Delta_1 \cup \Delta_2}\| \sum_{x \in \Sigma_*(\Delta_1, \Delta_2; \Delta)} \nu(|x|) \quad (24)$$

is valid for it.

Proof Let us estimate the left-hand side of the inequality (24)

$$\left|\mathsf{H}_{\Delta_1 \cup \Delta_2}[\mathbf{u}(z)] - \mathsf{H}_{\Delta_1}[\mathbf{u}(z)] - \mathsf{H}_{\Delta_2}[\mathbf{u}(z)]\right| \leq$$

$$\leq \Bigg(\sum_{\substack{\Gamma \subset \Delta_1 \cup \Delta_2 \\ |\Gamma| > 1}} - \sum_{\substack{\Gamma \subset \Delta_1 : \\ |\Gamma| > 1}} - \sum_{\substack{\Gamma \subset \Delta_2 : \\ |\Gamma| > 1}} \Bigg) \left|V_\Gamma(\mathbf{u}(\Gamma))\right| = \sum_{\substack{\Gamma \subset \Delta_1 \cup \Delta_2 : |\Gamma| > 1 \\ \Gamma \cap \Delta_1 \neq \emptyset,\ \Gamma \cap \Delta_2 \neq \emptyset}} \left|V_\Gamma(\mathbf{u}(\Gamma))\right| \leq$$

$$\leq \sum_{x \in \Delta_1 \cup \Delta_2} \sum_{\substack{\Gamma \subset \Delta_1 \cup \Delta_2 : \, x \in \Gamma,\, \Gamma - x \subset \Delta, \\ \Gamma \cap \Delta_1 \neq \emptyset,\ \Gamma \cap \Delta_2 \neq \emptyset, |\Gamma| > 1}} \left|V_\Gamma(\mathbf{u}(\Gamma))\right| \leq$$

$$\leq \sum_{\Gamma \subset \mathcal{Z}^d : \mathbf{0} \in \Gamma, |\Gamma| > 1} \mathsf{G}(\Gamma) \sum_{x \in \Sigma(\Delta_1, \Delta_2; \Delta)} \nu(|x|) .$$

Here, we take into account that $V_\Gamma(\mathbf{u}(\Gamma)) \neq 0$ only in the case when there exists such a vertex $x \in \Gamma$ for which the relation $\Gamma - x \subset \Delta$ is valid and, therefore, we introduce the set $\Sigma(\Delta_1, \Delta_2; \Delta)$ of vertexes $x \in \Delta_1 \cup \Delta_2$. For each vertex in this set there exists a subset Γ with the following properties $\Gamma \subset \Delta_1 \cup \Delta_2$, $x \in \Gamma$, $\Gamma - x \subset \Delta$, $\Gamma \cap \Delta_1 \neq \emptyset$, $\Gamma \cap \Delta_2 \neq \emptyset$.

Now, we show that the inclusion $\Sigma(\Delta_1, \Delta_2; \Delta) \subset \Sigma_*(\Delta_1, \Delta_2; \Delta)$ takes place. Indeed, from two last inclusions we conclude $(\Gamma - x) \cap (\Delta_1 - x) \neq \emptyset$ and $(\Gamma - x) \cap (\Delta_2 - x) \neq \emptyset$. Then, combining these inclusions with the following $\Gamma - x \subset \Delta$, we may assert that relationships $\Delta \cap (\Delta_1 - x) \neq \emptyset$ and $\Delta \cap (\Delta_2 - x) \neq \emptyset$ are realized. Thus, $(\Delta + x) \cap \Delta_1 \neq \emptyset$, $(\Delta + x) \cap \Delta_2 \neq \emptyset$ and, therefore, the last inequality leads to the inequality (24) if we take into account the statement of Theorem 2.

The following lemma is the consequence of Lemmas 2 and 3.

Lemma 4 *Let Hamiltonian* $\mathsf{H}_\Lambda \in \mathrm{H}^{(0)}$ *has the finite range of action and* Δ *is the finite subset in* $\mathcal{Z}^d$ *which is its support. If* Δ_1 *are* Δ_2 *are nonintersecting finite subsets in* $\mathcal{Z}^d$, $\Delta_1 \cap \Delta_2 = \emptyset$, *then the following estimate takes place*

$$\left|\ln Z_{\Delta_1\cup\Delta_2} - \ln Z_{\Delta_1} - \ln Z_{\Delta_2}\right| \le \Big(\mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|)\Big)\cdot \|\mathsf{H}_\Lambda\|\cdot|\Sigma_*(\Delta_1,\,\Delta_2;\,\Delta)|\,. \quad (25)$$

Proof We define the following Hamiltonians $\mathsf{H}^{(1)}_\Lambda = \mathsf{H}_{\Delta_1\cup\Delta_2}$ and $\mathsf{H}^{(2)}_\Lambda = \mathsf{H}_{\Delta_1} + \mathsf{H}_{\Delta_2}$. Then, using this definition, we have $Z_{\Delta_1\cup\Delta_2} = Z_\Lambda[\mathsf{H}_{\Delta_1\cup\Delta_2}]$, $Z_{\Delta_1} = Z_\Lambda[\mathsf{H}_{\Delta_1}]$, $Z_{\Delta_2} = Z_\Lambda[\mathsf{H}_{\Delta_2}]$. Due to $\Delta_1\cap\Delta_2=\emptyset$, the Hamiltonians H_{Δ_1}, H_{Δ_2} act in linear manifolds which have the empty intersection. Consequently,

$$\left|\ln Z_{\Delta_1\cup\Delta_2} - \ln Z_{\Delta_1} - \ln Z_{\Delta_2}\right| = \left|\ln Z_\Lambda[\mathsf{H}^{(1)}_\Lambda] - \ln Z_\Lambda[\mathsf{H}^{(2)}_\Lambda]\right|.$$

Further, we apply the inequality (24) to partition functions $Z_\Lambda[\mathsf{H}^{(m)}_\Lambda]$, $m=1,2$,

$$\left|\ln Z_\Lambda[\mathsf{H}^{(1)}_\Lambda] - \ln Z_\Lambda[\mathsf{H}^{(2)}_\Lambda]\right| \le \Big(\mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|)\Big)\cdot|\Sigma_*(\Delta_1,\,\Delta_2;\,\Delta)|\cdot\|\mathsf{H}^{(1)}_\Lambda - \mathsf{H}^{(2)}_\Lambda\|\,,$$

where we take into account that the difference $\mathsf{H}^{(1)}_\Lambda - \mathsf{H}^{(2)}_\Lambda$ has the natural restriction on $\Sigma_*(\Delta_1,\,\Delta_2;\,\Delta)$. Since $\mathsf{H}^{(1)}_\Lambda - \mathsf{H}^{(2)}_\Lambda = \mathsf{H}_{\Sigma_*(\Delta_1,\,\Delta_2;\,\Delta)}$, then $\|\mathsf{H}^{(1)}_\Lambda - \mathsf{H}^{(2)}_\Lambda\| = \|\mathsf{H}_{\Sigma_*(\Delta_1,\,\Delta_2;\,\Delta)}\| \le \|\mathsf{H}_\Lambda\|$ because of the nondecreasing of the Hamiltonian norm when the set Λ is expanded. From here, it follows the inequality (25).

Corollary 1 *Let Hamiltonian* $\mathsf{H}_\Lambda \in \mathrm{H}^{(0)}$ *has the finite rang of action and* Δ *is finite subset in* $\mathcal{Z}^d$ *which is its support. If* Δ_j, $j = 1\div m$ *are finite subsets in* $\mathcal{Z}^d$ *such that* $\Delta_j\cap\Delta_k=\emptyset$ *at* $j\neq k$, *then the following estimate*

$$\left|\ln Z_{\Upsilon_m} - \sum_{j=1}^m \ln Z_{\Delta_j}\right| < \Big(\mathsf{E}_{\Upsilon_m}\nu(|\tilde{\mathbf{u}}|)\Big)\cdot\|\mathsf{H}_{\Upsilon_m}\|\cdot\sum_{j=2}^m|\Sigma_*(\Upsilon_{j-1},\,\Delta_j;\,\Delta)| \quad (26)$$

takes place where $\Upsilon_l = \bigcup_{j=1}^l \Delta_j$ *and* $\Sigma_*(\Upsilon_{j-1},\Delta_j;\,\Delta)$ *is the set of such vertexes* $z\in\mathcal{Z}^d$ *for which the relationships* $(\Delta+z)\cap\Upsilon_l\neq\emptyset$, $(\Delta+z)\cap\Delta_l\neq\emptyset$ *follow simultaneously for each* $l = 2\div m$.

Proof The proof is carried out by induction according to $m\in\mathcal{N}$ with the use of the inequality (25), starting out $m=2$.

Let us proceed to the proof of the main result of this work. It is carried out according to the same scheme that is proposed in [2], and it is based on the representation of a lattice model as the sum of a large number of isomorphic disjoint identical "weakly interacting" lattice models.

Theorem 5 *If* $\mathsf{H}_\Lambda\in\mathrm{H}^{(0)}$, *then there exists the finite limit*

$$f(\mathsf{M},\mathsf{H}_\Lambda) = \lim_{L\to\infty}\frac{\ln Z_\Lambda}{|\Lambda(L)|}\,. \quad (27)$$

Proof On the basis of the set $\Lambda(a-1)$, $a\in\mathcal{N}$, $a\ge 2$ and vertexes $y\in\mathcal{Z}^d$, we define the sets $\Lambda_y = \Lambda(a-1)+ay$. Let $L = aN-1$. Consider the set $\Lambda(aN-1)$

which contains $a^d N^d$ vertexes. We represent it in the form

$$\Lambda(aN-1) = \bigcup_{y \in \Lambda(N-1)} \Lambda_y$$

where $\Lambda_{y_1} \cap \Lambda_{y_2} = \emptyset$ for any pair of vertexes $\{y_1, y_2\} \subset \Lambda(N-1)$.

Let us introduce lexicographical order of the set $\Lambda(N-1)$ containing N^d vertexes, we demote the fact that the vertex y_2 follows the vertex y_1 by $y_1 < y_2$. It means that for each pair of such vertexes $y_1 = \langle y_1^{(1)}, ..., y_d^{(1)} \rangle$, $y_2 = \langle y_1^{(2)}, ..., y_d^{(2)} \rangle$; $y_j^{(m)} = 0 \div N-1$; $j = 1 \div d, m \in \{1, 2\}$ there exists such a number $k = 1 \div d$ for which $y_j^{(1)} = y_j^{(2)}$, $j = 1 \div k-1$, $y_k^{(1)} < y_k^{(2)}$.

The values of functionals $\ln Z_{\Lambda_{y_j}}$ do not depend on $j = 1 \div N^d$ due to translational invariance. Then

$$\left|\ln Z_{\Lambda(L)} - N^d \ln Z_{\Lambda(a-1)}\right| = \left|\ln Z_{\Lambda(L)} - \sum_{j=1}^{N^d} \ln Z_{\Lambda_{y_j}}\right|. \tag{28}$$

To estimate the right-hand side of this equality we apply the inequality (26) connected with sets $\Delta_j = \Lambda_{y_j}$, $j = 1 \div N^d$, $\Upsilon_{N^2} = \Lambda(L)$ in the sense of the introduced order,

$$\left|\ln Z_{\Lambda(L)} - \sum_{j=1}^{N^d} \ln Z_{\Lambda_{y_j}}\right| < \left(\mathsf{E}_{\Lambda(L)}\, \nu(|\tilde{\mathbf{u}}|)\right) \cdot \|\mathsf{H}_{\Lambda(L)}\| \cdot \sum_{j=2}^{N^d} \left|\Sigma_*(\Upsilon_{j-1}, \Lambda_{y_j}; \Delta)\right| \tag{29}$$

where $\Upsilon_l = \bigcup_{j=1}^{l} \Lambda_{y_j}$, $\Upsilon_{y_1} = \Lambda(a-1)$. We choose the number $a \in \mathcal{N}$ so large that the inclusion $\Lambda(a-1) \supset (\Delta + z)$ is fulfilled for a vertex z.

Suppose there are two sets Λ_{y_j} and Λ_{y_k} such that there exists such a vertex $x \in \mathcal{Z}^d$ for them when the relationships $(\Delta + x) \cap \Lambda_{y_j} \neq \emptyset$ and $(\Delta + x) \cap \Lambda_{y_k} \neq \emptyset$ are valid. Then from the inclusion $\Lambda(a-1) \supset (\Delta + z)$ it follows that $(\Lambda(a-1) + x - z) \cap \Lambda_{y_j} \neq \emptyset$ and $(\Lambda(a-1) + x - z) \cap \Lambda_{y_k} \neq \emptyset$. Such a situation is possible only in the case when Λ_{y_j} and Λ_{y_k} are "neighboring" sets, namely, $y_j = \langle y_1^{(j)}, ..., y_d^{(j)} \rangle$, $y_k = \langle y_1^{(j)} + \alpha_1, ..., y_d^{(j)} + \alpha_d \rangle$, $\alpha_i \in \{-1, 0, 1\}$, $i = 1 \div d$. For each set Λ_{y_j} there exists no more that $3^d - 1$ neighboring sets among all $\Lambda_{y_k}, k \neq j, k = 1 \div N^d$. In this case, if the vertex x is contained in anything set Λ_{y_j}, then there are $3^d - 1$ sets Λ_{y_k} such that $(\Delta + x) \cap \Lambda_{y_k} \neq \emptyset$. Consequently, the number of vertexes $\left|\Sigma_*(\Upsilon_{j-1}, \Lambda_{y_j}; \Delta)\right|$ does not exceed $(3^d - 1) \max_{k \in \Xi_j} \left|\Sigma_*(\Lambda_{y_k}, \Lambda_{y_j}; \Delta)\right|$ for any $j = 1 \div N^d$ where Ξ_j is the set which consists of those $3^d - 1$ numbers $k \in \{1, ..., N^d\}$ for which Λ_{y_k} is a neighbor with Λ_{y_j}.

Let us estimate the number $\left|\Sigma_*(\Lambda_{y_k}, \Lambda_{y_j}; \Delta)\right|$ for two neighboring sets Λ_{y_j} and Λ_{y_k}. It is obvious that it is maximal in the case when there is the face of Λ_{y_j} with the dimension $d-1$ which divides them. It contains a^{d-1} vertexes. Let x_0 is the fixed vertex in this face. Then, we find the number of vertexes x for which simultaneous feasibility of relationships $(\Delta + x) \cap \Lambda_{y_j} \neq \emptyset$ and $(\Delta + x) \cap \Lambda_{y_k} \neq \emptyset$ is possible. In this case $x_0 \in \Delta + x$ does not exceed $|\Delta|$. Then, it is valid $\left|\Sigma_*(\Lambda_{y_k}, \Lambda_{y_j}; \Delta)\right| \leq a^{d-1}|\Delta|$. On the basis of this estimate, we obtain the following inequality

$$\left|\Sigma_*(\Upsilon_{j-1}, \Lambda_{y_j}; \Delta)\right| < (3^d - 1)a^{d-1}|\Delta| \,.$$

Using it and also (28) and (29), we conclude that the inequality

$$\left|\frac{\ln Z_{\Lambda(aN-1)}}{|\Lambda(aN-1)|} - \frac{\ln Z_{\Lambda(a-1)}}{|\Lambda(a-1)|}\right| < (3^d - 1)\Big(\mathsf{E}_{\Lambda(L)}\, \nu(|\tilde{\mathbf{u}}|)\Big) \cdot \|\mathsf{H}_{\Lambda(L)}\| \cdot \frac{|\Delta|}{a} \,. \quad (30)$$

takes place at $|\Lambda(aN-1)| = (aN)^d$, $|\Lambda(a-1)| = a^d$. Since the right-hand side of the inequality (30) tends to zero at $a \to \infty$, then, to complete the proof of the theorem, we show that the sequence $\langle |\Lambda(L)|^{-1} \ln\, Z_{\Lambda(L)};\, L \in \mathcal{N}\rangle$ is the fundamental one. For this, we will prove that, for each $\varepsilon > 0$, there is such a sufficiently large number L, for which there are values a and N when for any $L' > L$ we may find $a' > a$, $N' > N$ when the following inequality

$$\left|\frac{\ln Z_{\Lambda(L')}}{|\Lambda(L')|} - \frac{\ln Z_{\Lambda(a'-1)}}{|\Lambda(a'-1)|}\right| < \varepsilon$$

takes place. It is obvious that the sequence under consideration is fundamental in this case since

$$\left|\frac{\ln Z_{\Lambda(L)}}{|\Lambda(L)|} - \frac{\ln Z_{\Lambda(L')}}{|\Lambda(L')|}\right| < 2\varepsilon\,, \quad L' > L\,. \quad (31)$$

We introduce the sets $\Lambda(aN-1)$ and $\partial\Lambda(aN-1) = \Lambda(L) \setminus \Lambda(aN-1)$. Let us estimate the expression in left-hand side of (31) at $L' = L$, $a' = a$ on the basis of

$$\left|\frac{\ln Z_{\Lambda(L)}}{|\Lambda(L)|} - \frac{\ln Z_{\Lambda(a-1)}}{|\Lambda(a-1)|}\right| \leq \frac{1}{|\Lambda(L)|}\left|\ln Z_{\Lambda(L)} - \ln Z_{\Lambda(aN-1)} - \ln Z_{\partial\Lambda(aN-1)}\right|$$

$$+ \frac{\ln Z_{\partial\Lambda(aN-1)}}{\ln Z_{\Lambda(L)}} + \left|\frac{\ln Z_{\Lambda(aN-1)}}{|\Lambda(aN-1)|} - \frac{\ln Z_{\Lambda(a-1)}}{|\Lambda(a-1)|}\right| . \quad (32)$$

To estimate first summand, we apply (25) with $\Delta_1 = \Lambda(aN-1)$ and $\Delta_2 = \partial\Lambda(aN-1)$, taking into account that $\Lambda(L) = \Lambda(aN-1) \cup \partial\Lambda(aN-1)$,

$$\left|\ln\, Z_{\Lambda(L)} - \ln\, Z_{\Lambda(aN-1)} - \ln\, Z_{\partial\Lambda(aN-1)}\right| \leq$$

$$\leq \Big(\mathsf{E}_{\Lambda}\nu(|\tilde{\mathbf{u}}|)\Big) \cdot \|\mathsf{H}_{\Lambda}\| \cdot |\Sigma_*(\Lambda(aN-1),\ \partial\Lambda(aN-1); \Delta)| \,.$$

Here, $\Sigma_*(\Lambda(aN-1), \partial\Lambda(aN-1); \Delta)$ is the set of such vertexes x for which the set $\Delta + x$ contains the vertex in $\partial\Lambda(aN-1)$. Then, it follows that $\left|\Sigma_*(\Lambda(aN-1), \partial\Lambda(aN-1); \Delta)\right| \le |\partial\Lambda(aN-1)| \cdot |\Delta|$. Consequently, the inequality

$$\frac{1}{|\Lambda(L)|}\left|\ln Z_{\Lambda(L)} - \ln Z_{\Lambda(aN-1)} - \ln Z_{\partial\Lambda(aN-1)}\right| \le \delta \tag{33}$$

takes place at sufficiently large number N.

The estimate of second summand is given by the inequality

$$\frac{\ln Z_{\partial\Lambda(aN-1)}}{|\Lambda(L)|} \le \frac{|\partial\Lambda(aN-1)|}{|\Lambda(L)|}\int_{\Omega} \exp\Big(\|\mathsf{H}_{\Lambda(L)}\| \cdot \nu(|\mathbf{u}|)\Big) d\mathsf{M}(\mathbf{u}) < \delta\,, \tag{34}$$

which should be valid at sufficiently large L at fixed number N.

Finally, last summand at right-hand side of (32) is estimated by choice a sufficiently large value a in the inequality (30) for any $N \in \mathcal{N}$ so that its right-hand side may be done less that δ. Thus, by selecting $\delta < \varepsilon/3$ and, at first, choosing a suitable value a, and then choosing a sufficiently large number N so that the inequalities (33) and (34) are satisfied, we will ensure the satisfiability of the inequality (31).

Theorem 6 *If* $\mathsf{H}_\Lambda \in \mathrm{H}_\nu$*, then there exists the finite limit*

$$f(\mathsf{M}, \mathsf{H}_\Lambda) = \lim_{L\to\infty} \frac{\ln Z_{\Lambda(L)}}{|\Lambda(L)|}\,. \tag{35}$$

The limit function $f(\mathsf{M}, \mathsf{H}_\Lambda)$ *is the continuous functional in the space of* H_ν.

Proof The inequality (23) points out that the estimate

$$\frac{1}{|\Lambda|}\left|\ln Z_\Lambda[\mathsf{H}^{(1)}_{\Lambda(L)}] - \ln Z_\Lambda[\mathsf{H}^{(2)}_{\Lambda(L)}]\right| \le \Big(\mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|)\Big) \cdot \|\mathsf{H}^{(1)}_{\Lambda(L)} - \mathsf{H}^{(2)}_{\Lambda(L)}\|\,, \tag{36}$$

takes place for any pair of classes $\{\mathsf{H}^{(1)}_{\Lambda(L)}; L \in \mathcal{N}\}$, $\{\mathsf{H}^{(2)}_{\Lambda(L)}; L \in \mathcal{N}\}$ of Hamiltonians in the space H_ν.

Since the manifold $\mathrm{H}^{(0)}$ is dense in H_ν, then, for a given class of Hamiltonians $\mathsf{H}_{\Lambda(L)} \equiv \mathsf{H}^{(1)}_{\Lambda(L)} \in \mathrm{H}_\nu$, $L \in \mathcal{N}$ and for the value $\varepsilon > 0$, choosing such a class $\mathsf{H}^{(0)} \equiv \mathsf{H}^{(2)}_{\Lambda(L)}$, $L \in \mathcal{N}$ in $\mathrm{H}^{(0)}$ for which $\|\mathsf{H}_{\Lambda(L)} - \mathsf{H}^{(0)}_{\Lambda(L)}\| < \varepsilon$, we get

$$\varepsilon\mathsf{E}_\Lambda\nu(|\tilde{\mathbf{u}}|) + \frac{1}{|\Lambda|}\ln Z_\Lambda[\mathsf{H}^{(0)}_{\Lambda(L)}] > \frac{1}{|\Lambda|}\ln Z_\Lambda[\mathsf{H}_{\Lambda(L)}] > \frac{1}{|\Lambda|}\ln Z_\Lambda[\mathsf{H}^{(0)}_{\Lambda(L)}] - \varepsilon\mathsf{E}_\Lambda\nu(|\tilde{\mathbf{u}}|)\,.$$

Since, according to Theorem 5, the sequence of functions $|\Lambda(L)|^{-1}\ln Z_{\Lambda(L)}[\mathsf{H}^{(0)}_{\Lambda(L)}]$, $L \in \mathcal{N}$ converges to a fixed limit, then, going to the limit $L \to \infty$, we get an estimate for the difference between the upper and lower limits of the sequences of functions

$$\limsup_{L\to\infty} \frac{\ln Z_\Lambda[\mathsf{H}_{\Lambda(L)}]}{|\Lambda|} - \liminf_{L\to\infty} \frac{\ln Z_\Lambda[\mathsf{H}_{\Lambda(L)}]}{|\Lambda|} < \varepsilon \mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|)\,.$$

Taking into account the arbitrariness of the value $\varepsilon > 0$, we find that the first statement of the theorem is true.

The limit function $f(\mathsf{M}, \mathsf{H}_\Lambda)$ (35) depends functionally on the set of potentials $V_\Gamma(\mathbf{u}(\Gamma))$, $\Gamma \subset \mathcal{Z}^d$, $1 < |\Gamma| < \infty$, that is, on the class of Hamiltonians $\{\mathsf{H}_{\Lambda(L)}; L \in \mathcal{N}\}$. Since such limiting at $L \to \infty$ functions $f(\mathsf{M}, \mathsf{H}^{(m)}_{\Lambda(L)})$ exist for every pair $\{\mathsf{H}^{(m)}_{\Lambda(L)}; L \in \mathcal{N}\}$, $m \in \{1, 2\}$ of Hamiltonians classes in H_ν, then, going to the limit when $L \to \infty$ in (36), for of these limit functions, we obtain

$$\left| f(\mathsf{M}, \mathsf{H}^{(2)}_{\Lambda(L)}) - f(\mathsf{M}, \mathsf{H}^{(1)}_{\Lambda(L)}) \right| \le \left(\mathsf{E}_\Lambda \nu(|\tilde{\mathbf{u}}|) \right) \cdot \| \mathsf{H}^{(1)}_{\Lambda(L)} - \mathsf{H}^{(2)}_{\Lambda(L)} \|\,.$$

From here, it follows that the limit functional $f(\mathsf{M}, \mathsf{H}_\Lambda)$ is continuous on the space of Hamiltonians H_ν that proves the second part of the statement.

5 Conclusion

In the paper it is proved the extensiveness of the free energy $F_\Lambda[\mathsf{H}_\Lambda]$ of classical vector lattice models in statistical mechanics, that is, the presence of asymptotic behavior (9) at $\Lambda \to \mathcal{Z}^d$ for this thermodynamic function. The proved statement is valid for any classes of translationally invariant Hamiltonians of the space H_ν and for any dimension d of the immersion space of the specified type models.

It is necessary to note that investigated models are used in statistical physics only at $d = 3$ for bulk physical samples of a solid and at $d = 2$ in the study of thermodynamic phenomena on the boundaries of macroscopic physical bodies (in particular, the surface tension). Besides, in practical calculations within the framework of statistical mechanics, as a rule, Hamiltonians of *pair interaction* are used that is $V_\Gamma(\mathbf{u}(\Gamma)) \neq 0$ only when $|\Gamma| = 2$ with a summable potential.

At the same time, it should be noted that we have proved the presence of extensive asymptotic only in the special case, which is used when applying models of statistical mechanics in problems of theoretical statistical physics. Namely, the sets Λ which serve as geometric models of crystals, have the form $\Lambda = \Lambda(L)$. So, it would be desirable to extend the constructions proposed in this paper to the case when Λ sets have a more general form. It may be done if it is permissible to determine the so-called thermodynamic Van Hove limit transition (see [2]). Such a generalization is important as from the viewpoint of development of the general theory of the Gibbs random fields and as from the physical viewpoint because of the development of theoretical physics. The latter is connected with the fact that different constructions of thermodynamic limit transition may describe different physical reality. For example, if it is violated the so-called Fisher condition (see [2, Sect. 2]) when the thermodynamic limit transition is fulfilled, in particular, there are violated those conditions that are inherent in the definition of the Van Hove limit transition, then it seems that one may describe fractal solid-state structures within the framework of statistical mechanics.

In conclusion, we note that, from our opinion, the development of an alternative approach in the theory of Gibbs random fields proposed by Dobrushin [7], despite its undoubted general theoretical importance, will not lead to the elimination of the concept of thermodynamic limit transition in the traditional sense in statistical mechanics.

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