B-subharmonic Functions

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Abstract Considering different problems with Bessel operator we inevitably should obtain the main theorems of harmonic analysis for Laplace–Bessel operator. In this article we obtain condition of B-subharmonicity using the second Green's formula for the Laplace–Bessel operator.

Keywords B-subharmonic functions · Weighted spherical mean · B-harmonic functions · Laplace–Bessel operator

1 Introduction

Subharmonic functions have been introduced in the analysis Hartogs [1]. The systematic study of subharmonic functions began with the work of Riesz [2, 3], Privalov [4] and Radó [5]. It is widely known that subharmonic functions are used in the theory of surfaces of nonpositive Gaussian curvature [6], in solving boundary value problems [7], in the theory of random processes [8] and in studying analytic functions of a complex variable [4]. Now the theory of subharmonic functions is an actively developing area of modern mathematics.

In this article we introduce and proof B-subharmonicity condition. This is a part of B-harmonic analysis which provides a mathematical theory to deal with the singular Bessel differential operator of the form

$$B_{\gamma_j} = \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{\partial}{\partial x_j} = \frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}, \qquad j = 1, \ldots, n.$$

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We will use notation $\triangle_\gamma = (\triangle_\gamma)_x = \sum_{k=1}^{n} (B_{\gamma_k})_{x_k}$. For $\triangle_\gamma$ the term *Laplace–Bessel operator* is used. A function $u = u(x) = u(x_1, \dots, x_n)$ defined in a domain $\Omega \subset R^n$ is said to be *B–harmonic* if $u \in C^2(\Omega)$, $\frac{\partial u}{\partial x_j}|_{x_j=0} = 0$ for all $j = 1, \dots, n$ and satisfies the Laplace–Bessel equation $\triangle_\gamma u = 0$ at every point of the domain Ω.

One can say that a function defined and continuous in some domain is B-subhartnonic if the value of this function at each point of the domain under consideration is less than or equal to its weighted spherical mean. It will be shown that B-subharmonicity of function in some domain follows from inequality $\triangle_\gamma u(x) \geq 0$ which is satisfied at all points of the considered domain.

In classical theory, the definition of subharmonic functions is often given in terms of the positivity of the Laplace operator, and then a generalized mean value theorem is derived with inequality instead of equality. For our case with the Laplace-Bessel operator, we rearrange this order and define subharmonic functions through the generalized mean value theorem with inequalities, and then derive for them a theorem about the non-negativity of the Laplace-Bessel operator.

2 Definitions

Suppose that R^n is the n-dimensional Euclidean space,

$$R^n_+ = \{x = (x_1, \dots, x_n) \in R^n, \quad x_1{>}0, \dots, x_n{>}0\},$$

$$\overline{R}^n_+ = \{x = (x_1, \dots, x_n) \in R^n, \quad x_1{\geq}0, \dots, x_n{\geq}0\},$$

$\gamma = (\gamma_1, \dots, \gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i = 1, \dots, n$, and $|\gamma| = \gamma_1 + \ldots + \gamma_n$.

Let Ω be finite or infinite open set in R^n symmetric with respect to each hyperplane $x_i{=}0$, $i{=}1, ..., n$, $\Omega_+{=}\Omega \cap R^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{R}^n_+$.

We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i = 1, \dots, n$ are continuous up to $x_i{=}0$. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}|_{x_i=0} = 0$ for all non-negative integer $k \leq \frac{m-1}{2}$ (see [9], p. 21).

In the following, we will denote $C^m_{ev}(\overline{R}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+) = \bigcap_{m=0}^{\infty} C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{R}_+) = C^\infty_{ev}$.

The class $C_{ev}(\overline{\Omega}_+)$ is the restriction of the class of even continuous on Ω functions to $\overline{\Omega}_+$.

We will use notation $\overset{\circ}{C}{}^{\infty}_{ev}(\overline{\Omega}_+)$ for the space of all functions $f\in C^{\infty}_{ev}(\overline{\Omega}_+)$ with a compact support. We will use notations $\overset{\circ}{C}{}^{\infty}_{ev}(\overline{\Omega}_+)=\mathcal{D}_+(\overline{\Omega}_+)$ and $\overset{\circ}{C}{}^{\infty}_{ev}(\overline{R}_+)=\overset{\circ}{C}{}^{\infty}_{ev}$.

The multidimensional generalized translation is defined by the equality

$$({}^\gamma\mathbf{T}^y_x f)(x) = {}^\gamma\mathbf{T}^y_x f(x) = ({}^{\gamma_1}T^{y_1}_{x_1}\ldots{}^{\gamma_n}T^{y_n}_{x_n} f)(x), \tag{1}$$

where each of one-dimensional generalized translation ${}^{\gamma_i}T^{y_i}_{x_i}$ acts for $i=1,\ldots,n$ according to (see [10])

$$({}^{\gamma_i}T^{y_i}_{x_i} f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)}$$

$$\times \int\limits_0^\pi f(x_1,\ldots,x_{i-1},\sqrt{x_i^2+\tau_i^2-2x_iy_i\cos\varphi_i},x_{i+1},\ldots,x_n)\;\sin^{\gamma_i-1}\varphi_i\,d\varphi_i.$$

Next we will use notation

$$C(\gamma) = \pi^{-\frac{n}{2}}\prod_{i=1}^n \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Part of the sphere of radius r with center at the origin belonging to R^n_+ we will denote $S^+_r(n)$:

$$S^+_r(n) = \{x\in\overline{R}{}^n_+ : |x| = r\}\cup\{x\in\overline{R}{}^n_+ : x_i = 0, |x|\le r, i = 1,\ldots,n\}.$$

For the weighed integral by the $S^+_1(n)$ we have formula [11], formula 107, p. 49

$$|S^+_1(n)|_\gamma = \int\limits_{S^+_1(n)} x^\gamma dS = \frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{2}$$

3 *B*-harmonic Functions

In this section we will consider B-harmonic functions i.e. functions annihilated by the Laplace–Bessel operator in domain $\overline{\Omega}_+ = \Omega\cap\overline{R}{}^n_+$.

A function $u = u(x) = u(x_1,\ldots,x_n)$ defined in a domain $\overline{\Omega}_+$ is said to be B–*harmonic* if $u\in C^2_{ev}(\overline{\Omega}_+)$ and satisfies the Laplace–Bessel equation $\triangle_\gamma u = 0$ at every point of the domain $\overline{\Omega}_+$.

Theorem 1 *Let* $x \in R^n_+$, $n > 1$ *and*

$$E(x)=\begin{cases}\frac{1}{|S_1^+(n)|_\gamma}\ln|x|, & n{+}|\gamma|=2;\\ \frac{|x|^{2-n-|\gamma|}}{(2-n-|\gamma|)|S_1^+(n)|_\gamma}, & n{+}|\gamma|>2,\end{cases}$$

where $|S_1^+(n)|_\gamma$ *is (2). Then for* $|x|>\varepsilon\ \forall\varepsilon>0$ *we obtain that* $E(x)$ *is* B*-harmonic:*

$$\triangle_\gamma E(x)=0.$$

Proof Let consider first the case $n+|\gamma|>2$. We can write

$$\triangle_\gamma E(x)=\sum_{j=1}^n B_{\gamma_j}E(x)=\sum_{j=1}^n\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)=$$

$$=\frac{1}{(2-n-|\gamma|)|S_n^+|_\gamma}\sum_{j=1}^n\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}|x|^{2-n-|\gamma|}=$$

$$=\frac{1}{(2-n-|\gamma|)|S_n^+|_\gamma}\sum_{j=1}^n\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\,\frac{(2-n-|\gamma|)}{2}\,|x|^{-n-|\gamma|}\,2x_j=$$

$$=\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}|x|^{-n-|\gamma|}x_j^{1+\gamma_j}=$$

$$=\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n\frac{1}{x_j^{\gamma_j}}\left[\frac{(-n-|\gamma|)}{2}|x|^{-n-|\gamma|-2}\,2x_j^{2+\gamma_j}+(1+\gamma_j)|x|^{-n-|\gamma|}x_j^{\gamma_j}\right]=$$

$$=\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n[(-n-|\gamma|)|x|^{-n-|\gamma|-2}\,x_j^2+(1+\gamma_j)|x|^{-n-|\gamma|}]=$$

$$=\frac{1}{|S_n^+|_\gamma}[(-n-|\gamma|)|x|^{-n-|\gamma|}+(n+|\gamma|)|x|^{-n-|\gamma|}]=0.$$

Now consider the case $n+|\gamma|=2$:

$$\triangle_\gamma E(x)=\sum_{j=1}^n B_{\gamma_j}E(x)=\sum_{j=1}^n\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)=$$

$$= \frac{1}{|S_n^+|_\gamma} \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{\partial}{\partial x_j} \ln|x| = \frac{1}{|S_n^+|_\gamma} \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} |x|^{-2} x_j^{1+\gamma_j} =$$

$$= \frac{1}{|S_n^+|_\gamma} \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}} [-2|x|^{-4} x_j^{2+\gamma_j} + (1+\gamma_j)|x|^{-2} x_j^{\gamma_j}] =$$

$$= \frac{1}{|S_n^+|_\gamma} \sum_{j=1}^n [-2|x|^{-4} x_j^2 + (1+\gamma_j)|x|^{-2}] =$$

$$= \frac{1}{|S_n^+|_\gamma} [-2|x|^{-2} + (n+|\gamma|)|x|^{-2}] = 0,$$

because $n + |\gamma| = 2$.

4 Weighted Spherical Mean

In *B*-harmonic analysis when constructing a weighted spherical mean, instead of the usual shift, a multidimensional generalized translation (1) is used.

Weighted spherical mean (see [11–13]) of function $u(x)$, $x \in \overline{R}^n_+$ for $n \geq 2$ is

$$(M_r^\gamma u)(x) = (M_r^\gamma)_x[u(x)] = \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} {}^\gamma \mathbf{T}_x^{r\theta} u(x) \theta^\gamma dS, \tag{3}$$

where $\theta^\gamma = \prod\limits_{i=1}^n \theta_i^{\gamma_i}$.

Weighted spherical mean has properties

$$(M_r^\gamma u)(x)|_{r=0} = u(x), \qquad \frac{\partial}{\partial r} (M_r^\gamma u)(x) \bigg|_{r=0} = 0. \tag{4}$$

In the classical case, the transition from integration over a unit sphere centered at the origin to a sphere centered at a point x^0 of radius r is carried out by a simple linear change of coordinates. In our case, the presence of a generalized translation significantly complicates such a transition. Let's consider this point in more detail.

We will transform $(M_t^\gamma u)(x)$ so that the center of the part of the sphere over which the integration takes place moves. In this case, the dimension of the space will double. We have

$$(M_r^\gamma u)(x) = \frac{C(\gamma)}{|S_1^+(n)|_\gamma} \times$$

$$\times \int\limits_{S_1^+(n)} \int\limits_0^\pi ... \int\limits_0^\pi u(\sqrt{x_1^2 - 2rx_1\theta_1 \cos\beta_1 + r^2\theta_1^2}, ..., \sqrt{x_n^2 - 2rx_n\theta_n \cos\beta_1 + r^2\theta_n^2}) \times$$

$$\times \prod_{i=1}^n \sin^{\gamma_i - 1} \beta_i \, d\beta \theta^\gamma dS.$$

One can convert this integral into integral by the part of sphere in R^{2n} by using formulas

$$\widetilde{\theta}_1 = r\theta_1 \cos\beta_1, \qquad \widetilde{\theta}_2 = r\theta_1 \sin\beta_1,$$

$$\widetilde{\theta}_3 = r\theta_2 \cos\beta_2, \qquad \widetilde{\theta}_4 = r\theta_2 \sin\beta_2, \ldots, \tag{5}$$

$$\widetilde{\theta}_{2n-1} = r\theta_n \cos\beta_n, \qquad \widetilde{\theta}_{2n} = r\theta_n \sin\beta_n.$$

We obtain

$$(M_r^\gamma u)(x) = \frac{C(\gamma)}{|S_1^+(n)|_\gamma r^{n+|\gamma|-1}} \times$$

$$\times \int\limits_{\widetilde{S}_r^+(2n)} u(\sqrt{(x_1 - \widetilde{\theta}_1)^2 + \widetilde{\theta}_2^2}, ..., \sqrt{(x_n - \widetilde{\theta}_{2n-1})^2 + \widetilde{\theta}_{2n}^2}) \prod_{i=1}^n \widetilde{\theta}_{2i}^{\gamma_i - 1} d\widetilde{S} =$$

$$= \frac{C(\gamma)}{|S_1^+(n)|_\gamma r^{n+|\gamma|-1}} \int\limits_{\widetilde{S}_{r,x}^+(2n)} u(\sqrt{z_1^2 + \widetilde{\theta}_2^2}, ..., \sqrt{z_{2n-1}^2 + \widetilde{\theta}_{2n}^2}) \prod_{i=1}^n \widetilde{\theta}_{2i}^{\gamma_i - 1} d\widetilde{S}',$$

where we put $\{\widetilde{\theta}_{2i-1} - x_i = z_{2i-1}, i = 1, ..., n\}$. Here $\widetilde{\theta}_{2i} > 0, i = 1, ..., n,$

$$\widetilde{S}_r^+(2n) = \{\widetilde{\theta} \in R^{2n} : |\widetilde{\theta}| = r\}$$

and

$$\widetilde{S}_{r,x}^+(2n) =$$

$$= \{(z_1, \widetilde{\theta}_2, ..., z_{2n-1}, \widetilde{\theta}_{2n}) \in R^{2n} : (z_1 - x_1)^2 + \widetilde{\theta}_2^2 + \cdots + (z_{2n-1} - x_n)^2 + \widetilde{\theta}_{2n}^2 = r^2\},$$

differentials $d\widetilde{S}$ and $d\widetilde{S}'$ mean that we are integrating over a surfaces $\widetilde{S}_r^+(2n)$ and $\widetilde{S}_{r,x}^+(2n)$ respectively.

Let now $z_{2i-1} = \theta_i \cos\beta_i \; \widetilde{\theta}_{2i} = \theta_i \sin\beta_i, i = 1, ..., n$. We can write

$$(M_r^\gamma u)(x) = \frac{C(\gamma)}{|S_1^+(n)|_\gamma r^{n+|\gamma|-1}} \int\limits_0^\pi ... \int\limits_0^\pi \left(\int\limits_{\widetilde{\widetilde{S}}_{r,x}^+(n)} u(\theta)\theta^\gamma dS \right) \prod_{i=1}^n \sin^{\gamma_i - 1} \beta_i \, d\beta, \tag{6}$$

where $\widetilde{\widetilde{S^+_{r,x}}}(n)$ is a sphere (or a part of sphere) $(\theta_1 \cos\beta_1 - x_1)^2 + \theta_1^2 \sin^2\beta_1 + \cdots + (\theta_n \cos\beta_n - x_n)^2 + \theta_n^2 \sin^2\beta_n = r^2$. To simplify the right part of (6) we introduce the next notation

$$\int\limits_{{}^\gamma \mathbf{T}^x_\theta S^+_r(n)} u(\theta)\theta^\gamma dS = C(\gamma)\int\limits_0^\pi \ldots \int\limits_0^\pi \left(\int\limits_{\widetilde{\widetilde{S^+_{r,x}}}(n)} u(\theta)\theta^\gamma dS \right) \prod_{i=1}^n \sin^{\gamma_i - 1}\beta_i \, d\beta$$

so we can write

$$(M^\gamma_r u)(x) = \frac{1}{|S^+_1(n)|_\gamma r^{n+|\gamma|-1}} \int\limits_{{}^\gamma \mathbf{T}^x_\theta S^+_{r,x}(n)} u(\theta)\theta^\gamma dS. \tag{7}$$

5 *B*-subharmonic Functions

In this section we define the *B*-subharmonic function and prove that if Laplace-Bessel operator of a sufficiently smooth function is non-negative in domain then this function is *B*-subharmonic.

Let $u \in C_{ev}(\overline{\Omega}_+)$. We say that a function u is B-subharmonic if

$$u(x^0) \le (M^\gamma_r u)(x^0) = \frac{1}{|S^+_1(n)|_\gamma} \int\limits_{S^+_1(n)} {}^\gamma \mathbf{T}^{r\theta}_{x^0} u(x^0)\theta^\gamma dS$$

whenever the part of the sphere $\{x \in R^n_+ : |x - x_0| \le r\}$ is contained in $\overline{\Omega}_+$.

Theorem 2 *Suppose* $u \in C^2_{ev}(\overline{\Omega}_+)$ *and* $\triangle_\gamma u(x) \ge 0$ *for all* $x \in \overline{\Omega}_+$*, then* $u(x)$ *B-subhartnonic at all points of* $\overline{\Omega}_+$.

Proof Let x^0 is any point of $\overline{\Omega}_+$,

$$v(x) = \begin{cases} -\ln|x - x^0| + \ln r, & \text{n+}|\gamma| = 2s; \\ |x - x^0|^{2-n-|\gamma|} - r^{2-n-|\gamma|}, & \text{n+}|\gamma| > 2, \end{cases}$$

is *B*-harmonic function by Theorem 1 in $\overline{\Omega}_+$: $\triangle_\gamma v = 0$, $v(x) \ge 0$.

We consider $\theta \in R^n_+$,

$$I(x) = C(\gamma)\int\limits_0^\pi \ldots \int\limits_0^\pi \left(\int\limits_{\widetilde{G}^+} \left(u(\theta)\triangle_\gamma v(\theta) - v(\theta)\triangle_\gamma u(\theta)\right)\theta^\gamma d\theta \right) \prod_{i=1}^n \sin^{\gamma_i - 1}\beta_i \, d\beta,$$

where $\widetilde{G}^+$ the shell domain between

$$(\theta_1 \cos\beta_1 - x_1^0)^2 + \theta_1^2 \sin^2\beta_1 + \dots + (\theta_n \cos\beta_n - x_n^0)^2 + \theta_n^2 \sin^2\beta_n = \varepsilon^2$$

and

$$(\theta_1 \cos\beta_1 - x_1^0)^2 + \theta_1^2 \sin^2\beta_1 + \dots + (\theta_n \cos\beta_n - x_n^0)^2 + \theta_n^2 \sin^2\beta_n = r^2.$$

Numbers ε and r satisfy inequalities $0 < \varepsilon < r$ chosen so that set $\widetilde{G}^+$ lies entirely in $\overline{\Omega}_+$. The boundary of $\widetilde{G}^+$ can include parts of the coordinate plains.

Since $\triangle_\gamma v = 0$, $v(x) \geq 0$ and $\triangle_\gamma u(x) \geq 0$ for all $x \in \overline{\Omega}_+$ and $\widetilde{G}^+ \subseteq \overline{\Omega}_+$ we get

$$0 \geq I(x) = C(\gamma) \int\limits_0^\pi \dots \int\limits_0^\pi \left(\int\limits_{\widetilde{G}^+} (u(\theta)\triangle_\gamma v(\theta) - v(\theta)\triangle_\gamma u(\theta))\, \theta^\gamma d\theta \right) \prod_{i=1}^n \sin^{\gamma_i - 1} \beta_i \, d\beta,$$

The second Green's formula for the Laplace–Bessel operator (see [14]) is

$$0 \geq I = C(\gamma) \int\limits_0^\pi \dots \int\limits_0^\pi \left(\int\limits_{\partial\widetilde{G}^+} \left(u\frac{\partial v}{\partial \boldsymbol{\nu}} - v\frac{\partial u}{\partial \boldsymbol{\nu}} \right) \theta^\gamma \, dS \right) \prod_{i=1}^n \sin^{\gamma_i - 1} \beta_i \, d\beta,$$

where $\partial\widetilde{G}^+$ the boundary of $\widetilde{G}^+$, $\boldsymbol{\nu}$ is a normal vector of the surface $\partial\widetilde{G}^+$.

In new coordinates

$$z_1 = \theta_1 \cos\beta_1, \qquad z_2 = \theta_1 \sin\beta_1,$$

$$z_3 = \theta_2 \cos\beta_2, \qquad z_4 = \theta_2 \sin\beta_2, \dots,$$

$$z_{2n-1} = \theta_n \cos\beta_n, \qquad z_{2n} = \theta_n \sin\beta_n,$$

such that $z_{2i} > 0$, $i = 1, ..., n$, we can write

$$0 \geq I = C(\gamma) \int\limits_{\partial\widetilde{W}^+} \left(\widetilde{u}\frac{\partial\widetilde{v}}{\partial\widetilde{\boldsymbol{\nu}}} - \widetilde{v}\frac{\partial\widetilde{u}}{\partial\widetilde{\boldsymbol{\nu}}} \right) \prod_{i=1}^n z_{2i}^{\gamma_i - 1} \, d\widetilde{S},$$

where $\widetilde{u} = u\left(\sqrt{z_1^2 + z_2^2}, ..., \sqrt{z_{2n-1}^2 + z_{2n}^2}\right)$, $\widetilde{v} = v\left(\sqrt{z_1^2 + z_2^2}, ..., \sqrt{z_{2n-1}^2 + z_{2n}^2}\right)$, $\partial\widetilde{W}^+$ is a surface consisted of two spheres (or a parts of spheres in R^{2n}) with center at $\xi \in R^{2n}$, $\xi = (x_1, 0, x_2, 0, ..., x_{2n-1}, 0)$ of radii ε and r such that $0 < \varepsilon < r$:

$$\widetilde{S}^+_{\varepsilon,\xi}(2n) =$$

$$= \{z \in R^{2n} : (z_1 - x_1)^2 + z_2 + \dots + (z_{2n-1} - x_n)^2 + z_{2n}^2 = \varepsilon^2\},$$

$$\widetilde{S}^{+}_{r,\xi}(2n) =$$

$$= \{z \in R^{2n} : (z_1 - x_1)^2 + z_2 + \cdots + (z_{2n-1} - x_n)^2 + z_{2n}^2 = r^2\}$$

and possibly parts of coordinate plains, $\widetilde{\boldsymbol{\nu}}$ is is a normal vector of the surface $\partial \widetilde{W}^{+}$, $d\widetilde{S}$ is the element of the surface $\partial \widetilde{W}^{+}$. Therefore,

$$0 \geq I = C(\gamma)\left[\left(\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}(2n)} + \int\limits_{\widetilde{S}^{+}_{r,\xi}(2n)}\right)\widetilde{u}\frac{\partial \widetilde{v}}{\partial \widetilde{\boldsymbol{\nu}}}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S} - \right.$$

$$\left. - \left(\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}(2n)} + \int\limits_{\widetilde{S}^{+}_{r,\xi}(2n)}\right)\widetilde{v}\frac{\partial \widetilde{u}}{\partial \widetilde{\boldsymbol{\nu}}}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S}\right].$$

On $\widetilde{S}^{+}_{r,\xi}(2n)$ we have $\widetilde{v} = 0$. Also, since $\triangle_{\gamma} u \geq 0$ and $\widetilde{\boldsymbol{\nu}}$ is directed toward the center of the $\widetilde{S}^{+}_{\varepsilon,\xi}$ we get that $\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}} \widetilde{v}\frac{\partial \widetilde{u}}{\partial \widetilde{\boldsymbol{\nu}}}\prod\limits_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S} \leq 0$.

That means that

$$0 \geq C(\gamma)\left(\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}(2n)} + \int\limits_{\widetilde{S}^{+}_{r,\xi}(2n)}\right)\widetilde{u}\frac{\partial \widetilde{v}}{\partial \widetilde{\boldsymbol{\nu}}}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S}.$$

For $n + |\gamma| = 2$ we get

$$0 \geq C(\gamma)\left(\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}(2n)} - \int\limits_{\widetilde{S}^{+}_{r,\xi}(2n)}\right)\frac{\widetilde{u}(z)}{|z-\xi|}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S}$$

and for $n + |\gamma| > 2$ we get

$$0 \geq C(\gamma)(n + |\gamma| - 2)\left(\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}(2n)} - \int\limits_{\widetilde{S}^{+}_{r,\xi}(2n)}\right)\frac{\widetilde{u}(z)}{|z-\xi|^{n+|\gamma|-1}}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S},$$

where $\xi \in R^{2n}$, $\xi = (x_1, 0, x_2, 0, ..., x_{2n-1}, 0)$. In either case,

$$C(\gamma)\int\limits_{\widetilde{S}^{+}_{\varepsilon,\xi}(2n)} \frac{\widetilde{u}(z)}{|z-\xi|^{n+|\gamma|-1}}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S} \leq C(\gamma)\int\limits_{\widetilde{S}^{+}_{r,\xi}(2n)} \frac{\widetilde{u}(z)}{|z-\xi|^{n+|\gamma|-1}}\prod_{i=1}^{n} z_{2i}^{\gamma_i - 1}\, d\widetilde{S}$$

or

$$\frac{C(\gamma)}{\varepsilon^{n+|\gamma|-1}} \int\limits_{\widetilde{S}^+_{\varepsilon,\xi}(2n)} \widetilde{u}(z) \prod_{i=1}^{n} z_{2i}^{\gamma_i-1}\, d\widetilde{S} \leq \frac{C(\gamma)}{r^{n+|\gamma|-1}} \int\limits_{\widetilde{S}^+_{r,\xi}(2n)} \widetilde{u}(z) \prod_{i=1}^{n} z_{2i}^{\gamma_i-1}\, d\widetilde{S}.$$

Returning to coordinates $\theta_1, ..., \theta_n$ by formulas $z_{2i-1} = \theta_i \cos\beta_i\ \widetilde{\theta}_{2i} = \theta_i \sin\beta_i$, $i = 1, ..., n$ we obtain

$$\frac{1}{|S_1^+(n)|_\gamma \varepsilon^{n+|\gamma|-1}} \int\limits_{{}^\gamma\mathbf{T}^x_\theta S^+_{\varepsilon,x^0}(n)} u(\theta)\theta^\gamma dS \leq \frac{1}{|S_1^+(n)|_\gamma r^{n+|\gamma|-1}} \int\limits_{{}^\gamma\mathbf{T}^x_\theta S^+_{r,x^0}(n)} u(\theta)\theta^\gamma dS$$

or, using (7),

$$(M_\varepsilon^\gamma u)(x^0) = \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} {}^\gamma\mathbf{T}^{\varepsilon\theta}_{x^0} u(x^0)\theta^\gamma dS \leq$$

$$\leq \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} {}^\gamma\mathbf{T}^{r\theta}_{x^0} u(x^0)\theta^\gamma dS = (M_r^\gamma u)(x^0).$$

Letting ε tend to 0 the left side tends to $u(x^0)$ by (4) and we obtain inequality

$$u(x^0) \leq (M_r^\gamma u)(x^0).$$

Notes and Comments. There are a lot of properties of B-subharmonic functions need to prove. For example, it is interesting to consider the maximum principle, criterion of B-harmonicity in terms of B-subharmonic functions, the Perron method for solving the Dirichlet problem for Laplace-Bessel operator, the connection to the B-potential theory (for the B-potential theory see [15, 16]), Harnack inequality for singular equations and other.

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