On a K-Homogeneous Metric

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Abstract We consider a Riemannian metric which generates the Beltrami-Laplace operator coinciding with the B-elliptic operator up to a factor.

Keywords B-elliptic operator · Riemannian metric · Laplace Beltrami operator · Isometry group · Killing conditions · Lobachevsky geometry

1 K-Homogeneous Metric

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a vector with fixed numbers γ_i, $i = 1 \ldots, n$, which are not equal to zero at the same time. We denote by R^n_+ the set of points $x = (x_1, \ldots, x_n) \in R^n$ such that $x_i \in R$, when $\gamma_i = 0$, $x_i \in (0, +\infty)$, when $\gamma_i \neq 0$.

If $\gamma_i \neq 0$, the variable x_i is called singular. As usual, we will use the notation

$$(x)^\gamma = \prod_{i=1}^{n} x_i^{\gamma_i}, \; x = (x_1, \ldots, x_n) \in R^n_+.$$

Let the function $u(x)$ be twice continuously differentiable in R^n_+.

We define the operator Δ_{B_γ} by the formula

$$\Delta_{B_\gamma} u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\gamma_i}{x_i} \frac{\partial u}{\partial x_i}. \tag{1}$$

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V. Vasilyev (ed.), *Differential Equations, Mathematical Modeling and Computational Algorithms*, Springer Proceedings in Mathematics & Statistics 423,
https://doi.org/10.1007/978-3-031-28505-9_9

Operators of the form (1) have been studied by I. A. Kipriyanov and his disciples (see [1–5]).

The aim of this section is to find a positively defined in R^n_+ symmetric quadratic form (metric)

$$ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\, dx_i\, dx_j\ ,$$

such that the Beltrami–Laplace operator (see [6])

$$\Delta_\omega = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{k=1}^{n} g^{ik}\sqrt{|g|}\frac{\partial}{\partial x_k} \tag{2}$$

would coincide with the operator Δ_{B_γ} up to a multiplier. Here the functions g^{ij}, $i, j = 1, \ldots, n$, are the entries of the matrix $\|g^{ij}\|$, which is the inverse of the matrix $\|g_{ij}\|$ (covariant metric tensor),

$$g = \det \|g_{ij}\|.$$

The study of the properties of elliptic differential operators using Riemannian metrics has a long history (see, for example, [7, 8]).

Theorem 1 *If $n \geq 3$, the entries of the matrix $\|g_{ij}\|$ are defined by formulas*

$$g_{ij} = \delta_{ij} \prod_{i=1}^{n} x_i^{K_i} = \delta_{ij} x^K,\ \ i, j = 1, \ldots, n,\ \ K = (K_1, \ldots, K_n), \tag{3}$$

where

$$K_i = 2\gamma_i/(n-2), \tag{4}$$

δ_{ij} is the Kronecker symbol.

Proof Indeed, since $g_{ij} = 0$ for $i \neq j$, substituting (3) into (2), we get:

$$\Delta_\omega u = \frac{1}{\sqrt{|g|}} \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(g^{kk}\sqrt{|g|}\frac{\partial u}{\partial x_k}\right), \tag{5}$$

where

$$|g| = g = x^{nK} = \prod_{i=1}^{n}\prod_{i=1}^{N} n\, x_i^{K_i} = \prod_{i=1}^{n} x_i^{2n\gamma_i/(n-2)}, \tag{6}$$

$$g^{kk} = x^{-K} = \prod_{i=1}^{n} x_i^{-2\gamma_i/(n-2)}. \tag{7}$$

Taking into account (6) and (7), it is possible to rewrite (5) in the following form:

$$\Delta_\omega u = \frac{1}{x^{nK/2}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(x^{-K} x^{Kn/2} \frac{\partial u}{\partial x_j} \right) =$$

$$= x^{-K} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + x^{-Kn/2} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \left(\prod_{l=1}^{n} x_l^{K_l(n-2)/2} \right) \frac{\partial u}{\partial x_j} =$$

$$= x^{-K} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + x^{-Kn/2} \sum_{j=1}^{n} \prod_{l=1}^{n} x_l^{K_l(n-2)/2} \frac{K_j(n-2)}{2} x_j^{-1} \frac{\partial u}{\partial x_j} =$$

$$= x^{-K} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + x^{-K} \sum_{j=1}^{n} \frac{K_j(n-2)}{2x_j} \frac{\partial u}{\partial x_j} = x^{-K} \Delta_{B_\gamma} u,$$

so

$$\Delta_\omega u = x^{-K} \Delta_{B_\gamma} u, \tag{8}$$

which was required to be proved.

We will consider the set R_+^n equipped with a Riemannian metric

$$ds^2 = x^K \sum_{i=1}^{n} dx_i^2, \quad K \in R, \tag{9}$$

as a Riemannian space; we will denote it by KI_n, and we will call metric (9) the K-homogeneous Kipriyanov metric.

Theorem 2 *If $n = 2$, the problem of finding a metric satisfying equality (8) has no solution.*

Proof Let

$$g_{11} = E,\ g_{12} = g_{21} = F,\ g_{22} = G.$$

Then

$$g = \det \|g_{ij}\| = EG - F^2, g^{ij} = (-1)^{i+j} \frac{g_{ij}}{EG - F^2}.$$

Hence

$$\Delta_\omega u = G/|g| \frac{\partial^2 u}{\partial x_1^2} + E/|g| \frac{\partial^2 u}{\partial x_2^2} -$$

$$- 2F/|g| \frac{\partial^2 u}{\partial x_1 \partial x_2} + \Phi \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right), \tag{10}$$

where Φ denotes a summand that depends only on the first-order derivatives of the function u. In order for expression (10) to coincide up to a multiplier with (1), it is necessary that $F \equiv 0$. This will entail equalities

$$g = EG,\ g^{11} = 1/E,\ g^{22} = 1/G,\ g_{12} = g_{21} = g^{12} = g^{21} = 0.$$

Therefore,

$$\Delta_\omega u = \frac{1}{\sqrt{|EG|}} \left(\frac{\partial}{\partial x_1} \left(\sqrt{\left|\frac{G}{E}\right|} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\sqrt{\left|\frac{E}{G}\right|} \frac{\partial u}{\partial x_2} \right) \right) =$$

$$= \frac{1}{E} \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{G} \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial}{\partial x_1} \sqrt{\left|\frac{G}{E}\right|} \frac{\partial u}{\partial x_1} + \frac{\partial}{\partial x_2} \sqrt{\left|\frac{E}{G}\right|} \frac{\partial u}{\partial x_2}.$$

The first two terms must have the same coefficients, from where $E = G$. Then the last two terms are equal to zero, which means that it is impossible to find a metric satisfying equality (8) for $n = 2$.

2 Investigation of Isometric Transformations for the K-Homogeneous Kipriyanov Metric

The fulfillment of the Killing requirements

$$\sum_{s=1}^{n} \left(\xi_s \frac{\partial g_{ij}}{\partial x_s} + g_{is} \frac{\partial \xi_s}{\partial x_j} + g_{js} \frac{\partial \xi_s}{\partial x_i} \right) = 0,\ i, j = 1, \dots, n.$$

is a necessary and sufficient condition for a one-parameter group G with an infinitesimal operator

$$X = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}$$

to be an isometry group.

Obviously,

$$\frac{\partial g_{ij}}{\partial x_s} = \delta_{ij} \frac{K_s x^K}{x_s}.$$

Therefore, the Killing equations will take the form

$$\sum_{s=1}^{n} \left(\delta_{ij} \xi_s K_s x^{K-1} + x^K \left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) \right) = 0,\ i, j = 1, \dots, n.$$

By summing and reducing by x^K, we get

$$\delta_{ij}\sum_{s=1}^{N}\frac{\xi_s K_s}{x_s}+\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}=0,\ i,j=1,\dots,n. \tag{11}$$

For $i\neq j$, Eq. (11) can be written as

$$\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}=0,\ i,j=1,\dots,n,\ i\neq j. \tag{12}$$

For $i=j$, Eq. (11) can be written in the form

$$2\frac{\partial\xi_j}{\partial x_j}+\sum_{s=1}^{n}\frac{K_s\xi_s}{x_s}=0,\ i=1,\dots,n. \tag{13}$$

The vector

$$\xi=(\xi_1,\dots,\xi_n),\quad \xi_j=C\,x^p x_j, \tag{14}$$

where

$$p=(p_1,\dots,p_n),\quad p_1=p_2=\dots=p_n=\beta=-\sum_{l=1}^{n}K_l/2-1, \tag{15}$$

is a solution to system (13), which can be checked by direct verification. Substituting (14) into (12), taking into account (15), we obtain

$$0\equiv\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}=\beta x^p\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}\right),\ i,j=1,\dots,n,\ i\neq j.$$

Hence we get

$$p_1=p_2=\dots=p_n=\beta=-\sum_{l=1}^{n}K_l/2-1=0, \tag{16}$$

or, what is the same,

$$\sum_{l=1}^{n}K_l=-2, \tag{17}$$

and considering (4),

$$\sum_{i=1}^{n}\gamma_i=2-N. \tag{18}$$

3 Characteristics of the K-Homogeneous Kipriyanov Metric in the Case of a Single Singular Variable

There is a well-known case of fulfillment of condition (16), or, what is the same, (18). When $\gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = 0$, $\gamma_n = 2 - n$, $K = -2$, the space KI_n is the Poincare model of the $n-$dimensional Lobachevsky geometry. Next, we will consider the case of $\gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = 0$, $\gamma_n = 0$. Metric (3) will now take the form

$$g_{ij} = \delta_{ij} x_n^K,\ i, j = 1, \ldots, n, \tag{19}$$

where

$$K = 2\gamma/(n-2), \tag{20}$$

δ_{ij} is the Kronecker symbol.

The following facts are established by direct calculation.

Theorem 3 *The Christoffel symbols of the first kind, corresponding to metric (19), have the form*

$$\Gamma_{ij,k} = \frac{K x_n^{K-1}}{2} (\delta_{ik}\delta_{jn} + \delta_{jk}\delta_{in} - \delta_{ij}\delta_{kn}).$$

Proof From the definition of the Christoffel symbols of the first kind, taking into account (19)–(20), we obtain:

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right) =$$

$$= \frac{1}{2}(\delta_{ik}\delta_{jn} K x_n^{K-1} + \delta_{jk}\delta_{in} K x_n^{K-1} - \delta_{ij}\delta_{kn} K x_n^{K-1}),$$

which was required to be proved.

Theorem 4 *The Christoffel symbols of the second kind, corresponding to metric (9), have the form*

$$\Gamma_{ij}^k = \frac{K}{2x_n}(\delta_{ik}\delta_{jn} + \delta_{jk}\delta_{in} - \delta_{ij}\delta_{kn}).$$

Proof From the definition of the Christoffel symbols of the second kind and the previous theorem, we obtain:

$$\Gamma_{ij}^k = \sum_{h=1}^{n} g^{kh}\Gamma_{ij,h} = \frac{K}{2}\sum_{h=1}^{n} \delta_{kh} x_n^{-K} x_n^{K-1} (\delta_{ih}\delta_{jn} + \delta_{jh}\delta_{in} - \delta_{ij}\delta_{hn}) =$$

$$\frac{K}{2x_n}\,(\delta_{ki}\delta_{jn} + \delta_{kj}\delta_{in} - \delta_{ij}\delta_{kn}).$$

The theorem is proved.

Theorem 5 *The components of the Riemann tensor, corresponding to metric (9), have the form*

$$R^l_{ijk} = (\frac{K^2}{4x_n^2} - \frac{K}{2x_n^2})(\delta_{li}\delta_{in}\delta_{kn} + \delta_{ik}\delta_{jn}\delta_{ln} - \delta_{ij}\delta_{kn}\delta_{ln} - \delta_{lk}\delta_{in}\delta_{jn}) +$$
$$+ \frac{K^2}{4x_n^2}(\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}).$$

Proof In accordance to definition, the components of the Riemann tensor are calculated by the formulas

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x_j} - \frac{\partial \Gamma^l_{ij}}{\partial x_k} + \sum_{m=1}^{n} (\Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}).$$

We will calculate the partial derivatives included in these formulas. We have

$$\frac{\partial \Gamma^k_{ij}}{\partial x_s} = -\frac{K}{2x_n^2}\delta_{sn}(\delta_{ki}\delta_{jn} + \delta_{kj}\delta_{in} - \delta_{ij}\delta_{kn}),$$

from where, we obtain

$$\frac{\partial \Gamma^l_{ik}}{\partial x_j} = -\frac{K}{2x_n^2}\delta_{jn}(\delta_{li}\delta_{kn} + \delta_{lk}\delta_{in} - \delta_{ik}\delta_{ln}),$$

$$\frac{\partial \Gamma^l_{ij}}{\partial x_k} = -\frac{K}{2x_n^2}\delta_{kn}(\delta_{li}\delta_{jn} + \delta_{lj}\delta_{in} - \delta_{ij}\delta_{ln}).$$

Therefore,

$$\frac{\partial \Gamma^l_{i}}{\partial x_j} - \frac{\partial \Gamma^l_{ij}}{\partial x_s} = -\frac{K}{2x_n^2}(\delta_{jn}\delta_{lk}\delta_{in} - \delta_{jn}\delta_{ik}\delta_{ln} - \delta_{kn}\delta_{lj}\delta_{in} + \delta_{kn}\delta_{ij}\delta_{ln}).$$

Now we will calculate the last term in the definition. Taking into account Theorem 4, we find:

$$\sum_{m=1}^{n} (\Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}) =$$

$$= \frac{K^2}{4x_n^2}\sum_{m=1}^{n} (\delta_{mi}\delta_{kn}\delta_{lm}\delta_{jn} + \delta_{mi}\delta_{kn}\delta_{lj}\delta_{mn} -$$

$$-\delta_{mi}\delta_{kn}\delta_{mj}\delta_{ln} + \delta_{mk}\delta_{in}\delta_{lm}\delta_{jn} +$$

$$+\delta_{mk}\delta_{in}\delta_{lj}\delta_{mn} - \delta_{mk}\delta_{in}\delta_{mj}\delta_{ln} - \delta_{ik}\delta_{mn}\delta_{lm}\delta_{jn} - \delta_{ik}\delta_{mn}\delta_{lj}\delta_{mn} +$$

$$\delta_{ik}\delta_{mn}\delta_{mj}\delta_{ln} - \delta_{mi}\delta_{jn}\delta_{lm}\delta_{kn} - \delta_{mi}\delta_{jn}\delta_{lk}\delta_{mn} + \delta_{mi}\delta_{jn}\delta_{mk}\delta_{ln} -$$

$$-\delta_{mj}\delta_{in}\delta_{lm}\delta_{kn} - \delta_{mj}\delta_{in}\delta_{lk}\delta_{mn} + \delta_{mj}\delta_{in}\delta_{mk}\delta_{ln} + \delta_{ij}\delta_{mn}\delta_{lm}\delta_{kn} +$$

$$+\delta_{ij}\delta_{mn}\delta_{lk}\delta_{mn} - \delta_{ij}\delta_{mn}\delta_{mk}\delta_{ln}).$$

Hence, taking into account the properties of the Kronecker symbol, in particular, the formulas

$$\delta_{il} = \delta_{li},$$

$$\sum_{m=1}^{n} \delta_{mi}\delta_{lm} = \delta_{il},$$

after identical transformations, we obtain a statement of the theorem.

Theorem 6 *The components of the Ricci tensor, corresponding to metric (9), have the form*

$$R_{ij} = \frac{K}{4x_n^2}\left((K-2)(2-n)\delta_{in}\delta_{jn} + (K(n-2)+2)\delta_{ij}\right).$$

Proof Directly from the definition of the components of the Ricci tensor

$$R_{ij} = \sum_{k=1}^{n} R^k_{ijk},$$

after identical transformations, we come to the validity of the theorem.

Theorem 7 *The curvature of the space KI_n is calculated by the formula*

$$R = \frac{Kn(n-2)}{x_n^{K+2}} = \frac{2\gamma n}{x_n^{(2\gamma+2n-4)/(n-2)}}.$$

Proof From the definition of curvature

$$R = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij} R_{ij},$$

we come to the statement of the theorem by performing summation and identical transformations.

4 Investigation of Geodesic Lines for a K-Homogeneous Kipriyanov Metric

Theorem 8 *The system of equations for geodesic lines in the space* KI_n *can be reduced to a system of the first order*

$$\frac{dx_k}{ds} = \frac{C_k}{x_n^K}\ , \quad k = 1, \ldots, n-1, \tag{21}$$

$$\left(\frac{dx_n}{ds}\right)^2 = \frac{C_n}{x_n^K} - \frac{B^2}{x_n^{2K}}\ , \tag{22}$$

where

$$B = \sqrt{\sum_{k=1}^{n-1} C_k^2}. \tag{23}$$

Proof The system of equations for geodesic lines in a given metric $\|g_{ij}\|$ has the form

$$\frac{d^2x_k}{ds^2} + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_{ij}^k \frac{dx_i}{ds}\frac{dx_j}{ds} = 0,\ k = 1, 2, \ldots, n,$$

where s is the natural parameter (arc length). In our case, using the calculated Christoffel symbols, we can write this system as

$$\frac{d^2x_k}{ds^2} + \frac{K}{x_n}\frac{dx_n}{ds}\frac{dx_k}{ds} = 0,\ k = 1, \ldots, n-1, \tag{24}$$

$$\frac{d^2x_n}{ds^2} - \frac{K}{2x_n}\sum_{i=1}^{n}\left(\frac{dx_i}{ds}\right)^2 + \frac{K}{2x_n}\left(\frac{dx_n}{ds}\right)^2 = 0. \tag{25}$$

Equation (24) can be written as

$$x_n^{-K}\frac{d}{ds}\left(x_n^K\frac{dx_k}{ds}\right) = 0,\ k = 1, \ldots, n-1. \tag{26}$$

Multiplying (26) by x_n^K, integrating and dividing by x_n^K, we get

$$\frac{dx_k}{ds} = \frac{C_k}{x_n^K},\ k = 1, \ldots, n-1. \tag{27}$$

Substituting (27) into (25), we get

$$\frac{d^2x_n}{ds^2} - \frac{KB^2}{2x_n^{2K+1}} + \frac{K}{2x_n}\left(\frac{dx_n}{ds}\right)^2 = 0. \tag{28}$$

Equation (28) admits a reduction of the order in a standard way. Suppose

$$p = p(x_n) = \frac{dx_n}{ds},\ v = p^2.$$

Then

$$\frac{d^2x_n}{ds^2} = p'p = \frac{1}{2}v'.$$

After that, Eq. (28) will be reduced to the form

$$v' + \frac{K}{x_n}v = \frac{B^2K}{x_n^{2K+1}},$$

which is equivalent to the equation

$$\frac{d}{dx_n}(x_n^K v) = \frac{B^2K}{x_n^{K+1}}.$$

Integrating and dividing by x_n^K, we get

$$v = p^2 = \left(\frac{dx_n}{ds}\right)^2 = \frac{C_n}{x_n^K} - \frac{B^2}{x_n^{2K}}.$$

It is known [9], that geodesic lines have the property

$$\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} = const.$$

In the case under consideration, this will lead to equality

$$\sum_{i=1}^{n} x_n^K\left(\frac{dx_i}{ds}\right)^2 = const. \tag{29}$$

From (21), it is easily deduced that the constant in equality (29) coincides with C_n.

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