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PARTIAL DIFFERENTIAL EQUATIONS

# Discrete Equations, Discrete Transformations, and Discrete Boundary Value Problems

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**Abstract**—We study the solvability of discrete elliptic pseudodifferential equations in a sector of the plane. Using special factorization of the symbol, the problem is reduced to a boundary value problem for analytic functions of two variables. A periodic analog of one integral transformation is obtained that was used to construct solutions of elliptic pseudodifferential equations in conical domains. The formula for the general solution of the discrete equation under consideration and some boundary value problems are described in terms of this transformation.

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## INTRODUCTION

Discrete equations arise in many fields of study and can look different. Thus, difference schemes [1] and equations that the theory of difference potentials leads to [2] fall under this category. These were developed for the approximate (numerical) solution of boundary value problems for partial differential equations. However, in the 1960s, there appeared the theory of pseudodifferential operators [3] and the corresponding theory of boundary value problems for elliptic pseudodifferential equations [4]. The latter theory was further developed in [5]; this which subsequently led to significant advances. Thus, the solvability of elliptic pseudodifferential equations in conical domains was studied in [6–8]. Solutions were constructed for model pseudodifferential equations in simple canonical domains using special factorization of elliptic symbols. However, from a computational point of view, the obtained formulas are difficult to use and therefore there is a need to discretize the results obtained. The concept of discrete pseudodifferential operators then appeared [9–11], and some results [4] on solvability received a discrete interpretation. In [12, 13], a comparison of discrete and continuous solutions in the case of a half-space was given, but additional difficulties arise in the case of a quarter plane [14, 15]. This is due, in particular, to the fact that it is necessary to impose additional conditions on the symbol of the elliptic pseudodifferential operator, ensuring the unique solvability of the equation or boundary value problem. However, in this case it was also possible to obtain a comparison of discrete and continuous solutions.

In this paper, we extend the methods developed in [8] to the discrete case in order to unify the formula for constructing a general solution of a discrete elliptic pseudodifferential equation. Here we will consider the two-dimensional case and apply discrete analogs of the integral transformations introduced in [7].

## 1. DISCRETE FUNCTIONS, SPACES, AND OPERATORS

To make the presentation as closed as possible, in this section we gather basic information about the object being studied and the mathematical concepts used.

**1.1.** Let $\mathbb{Z}^2$ be an integer grid on the plane, $h > 0, \hbar = h^{-1}$. Let us denote by $K_n = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > a_n|x_1| a_n > 0\}$ the sector with opening angle $2\arctan a_n$, where $a_n$ can take values of the form $n, 1/n, n \in \mathbb{N}$, $K_{n,d} = h\mathbb{Z}^2 \cap K_n$. We work with discrete variable functions $u_d(\tilde{x}), \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in h\mathbb{Z}^2$.

Let us denote $\mathbb{T}^2 = [-\pi, \pi]^2, \hbar = h^{-1}$. We treat functions defined on $\hbar\mathbb{T}^2$ as periodic functions on $\mathbb{R}^2$ with the fundamental period cell $\hbar\mathbb{T}^2$.

On functions $u_d$ of a discrete argument one can define the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^2, \quad \xi \in \hbar\mathbb{T}^2$$

and a discrete analog of the Schwartz space $S(h\mathbb{Z}^2)$ of infinitely differentiable functions rapidly decreasing at infinity. Denote $\zeta^2 = h^{-2}((e^{ih\cdot\xi_1}-1)^2 + (e^{ih\cdot\xi_2}-1)^2)$. Let us introduce the following definition.

**Definition 1.** The space $H^s(h\mathbb{Z}^2)$ consists of discrete functions $u_d$ and is the closure of the space $S(h\mathbb{Z}^2)$ with respect to the norm

$$\|u_d\|_s = \left( \int\limits_{\hbar\mathbb{T}^2} \left(1+|\zeta^2|\right)^s \left|\tilde{u}_d(\xi)\right|^2 d\xi \right)^{1/2}.$$

The space $H^s(K_{n,d})$ consists of discrete functions in $H^s(h\mathbb{Z}^2)$ such that their supports belong to the set $\overline{K}_{n,d}$. The norm in the space $H^s(K_{n,d})$ is induced by the norm in the space $H^s(h\mathbb{Z}^2)$.

**Remark.** In [11], this space was introduced as the space of discrete generalized functions and the main properties of these functions were described. Here we rely on this approach, and by discrete function we mean precisely a discrete generalized function.

**1.2.** Given a measurable periodic function $A_d(\xi)$ (called a symbol) in $\mathbb{R}^2$ with the fundamental period cell $\hbar\mathbb{T}^2$, we can define a discrete pseudodifferential operator $A_d$ in the discrete sector $K_{n,d}$ by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} A_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} u_d(\tilde{y})\, d\xi, \quad \tilde{x}\in K_{n,d}.$$

The operator $A_d$ is said to be elliptic if

$$\operatorname*{ess\,inf}_{\xi\in\hbar\mathbb{T}^2} \left|A_d(\xi)\right| > 0.$$

We consider symbols satisfying the condition

$$c_1\left(1+|\zeta^2|\right)^{\alpha/2} \le \left|A_d(\xi)\right| \le c_2\left(1+|\zeta^2|\right)^{\alpha/2} \tag{1}$$

with constants $c_1$ and $c_2$ independent of $h$. The number $\alpha\in\mathbb{R}$ is called the order of the discrete pseudodifferential operator $A_d$.

It is well known that a discrete pseudodifferential operator $A_d$ with a symbol $\widetilde{A}_d(\xi)$ satisfying condition (1) is a bounded linear operator $H^s(h\mathbb{Z}^2)\to H^{s-\alpha}(h\mathbb{Z}^2)$ with a norm independent of $h$, and a similar statement is true for the space $H^s(K_{n,d})$.

When studying the solvability of a discrete equation with the operator $A_d$ in the space $H^s(K_{n,d})$, the key role is played by the concept of *periodic wave factorization of the symbol*. For the definition, we need some special domain in the two-dimensional complex space $\mathbb{C}^2$. A domain of type $\mathfrak{T}_h(K_n) = \hbar\mathbb{T}^2 + iK_n$ is called a tubular domain over the sector $K_n$; we will use the properties of analytic functions in the domain $\mathfrak{T}_h(K_n)$. Denote $\overset{*}{K}_n = \{x\in\mathbb{R}^2 : x=(x_1,x_2), a_n x_2 > |x_1|\}$.

**Definition 2.** A periodic wave factorization for the elliptic symbol $A_d(\xi)\in E_\alpha$ is its representation in the form

$$A_d(\xi) = A_{d,\neq}(\xi) A_{d,=}(\xi),$$

where $A_{d,\neq}(\xi)$ and $A_{d,=}(\xi)$ admit analytic continuation in the tubular domains $\mathfrak{T}_h(\overset{*}{K}_n), \mathfrak{T}_h(-\overset{*}{K}_n)$, respectively, with the estimates

$$c_1\left(1+|\hat{\zeta}^2|\right)^{\frac{æ}{2}} \le \left|A_{d,\neq}(\xi+i\tau)\right| \le c_1'\left(1+|\hat{\zeta}^2|\right)^{\frac{æ}{2}},$$
$$c_2\left(1+|\hat{\zeta}^2|\right)^{\frac{\alpha-æ}{2}} \le \left|A_{d,=}(\xi-i\tau)\right| \le c_2'\left(1+|\hat{\zeta}^2|\right)^{\frac{\alpha-æ}{2}},$$

and constants $c_1$, $c_1'$, $c_2$, and $c_2'$ independent of $h$, where

$$\hat{\zeta}^2 \equiv \hbar^2 \left((e^{-ih(\xi_1+i\tau_1)} - 1)^2 + (e^{-ih(\xi_2+i\tau_2)} - 1)^2\right), \quad \xi = (\xi_1, \xi_2) \in \hbar\mathbb{T}^2, \quad \tau - (\tau_1, \tau_2) \in \overset{*}{K}_n\ .$$

The number æ $\in \mathbb{R}$ is called the index of the periodic wave factorization.

## 2. DISCRETE EQUATIONS AND PERIODIC WAVE FACTORIZATION

To simplify the notation, we will consider here the homogeneous equation

$$(A_d u_d)(\tilde{x}) = 0, \quad \tilde{x} \in K_{n,d}, \tag{2}$$

to study the solvability of which we assume the existence of a periodic wave factorization of the symbol $A_d(\xi)$.

The scheme is standard, and we have used it repeatedly. We introduce an auxiliary discrete function

$$v_d(\tilde{x}) = -(A_d u_d)(\tilde{x}), \quad \tilde{x} \in h\mathbb{R}^2,$$

so that $v_d(\tilde{x}) = 0$ for $\tilde{x} \in K_{n,d}$.

Equation (2) is rewritten as a pair equation in the discrete space $H^s(h\mathbb{Z}^2)$, and the discrete Fourier transform is applied to it,

$$A_d(\xi)\tilde{u}_d(\xi) + \tilde{v}_d(\xi) = 0.$$

Next, periodic wave factorization is applied to the symbol $A_d(\xi)$ and the last equation is written as

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) = -A_{d,=}^{-1}(\xi)\tilde{v}_d(\xi). \tag{3}$$

Denote by $\widetilde{H}^s(K)$ the Fourier image of the space $H^s(K)$. A detailed study of the left- and right-hand sides of relation (3) leads to the following conclusion:

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-æ}(K_{n,d}),$$
$$A_{d,=}^{-1}(\xi)\tilde{v}_d(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^2 \setminus K_{n,d}),$$

or, having applied the inverse discrete Fourier transform,

$$w_d^{(1)}(\tilde{x}) \equiv (F_d^{-1})_{\xi\to\tilde{x}}\big(A_{d,\neq}(\xi)\tilde{u}_d(\xi)\big) \in H^{s-æ}(K_{n,d}),$$
$$w_d^{(2)}(\tilde{x}) \equiv (F_d^{-1})_{\xi\to\tilde{x}}\big(A_{d,=}^{-1}(\xi)\tilde{v}_d(\xi)\big) \in H^{s-æ}(h\mathbb{Z}^2 \setminus K_{n,d}).$$

Thus, one of the functions $w_d^{(1)}(\tilde{x})$ vanishes in $h\mathbb{Z}^2 \setminus K_{n,d}$, and the other, $w_d^{(2)}(\tilde{x})$, vanishes in $K_{n,d}$. For discrete functions $u_d(\tilde{x})$ defined on $h\mathbb{Z}^2$, we introduce the change of variables

$$\tilde{y}_1 = \tilde{x}_1,$$
$$\tilde{y}_2 = \tilde{x}_2 - a_n|\tilde{x}_1|.$$

and the corresponding variable replacement operator $T_{a_n}$ acting according to the rule

$$(T_{a_n}u_d)(\tilde{x}) = u_d\big(\tilde{x}_1, \tilde{x}_2 - a_n|\tilde{x}_1|\big).$$

Applying successively the inverse discrete Fourier transform and the operator $T_{a_n}$ to relation (3), we obtain

$$(T_{a_n}w_d^{(1)})(\tilde{x}) = -(T_{a_n}w_d^{(2)})(\tilde{x}),$$

where $(T_{a_n}w_d^{(1)})(\tilde{x})$ vanishes on $\mathbb{Z}^2_-$ and $(T_{a_n}w_d^{(2)})(\tilde{x})$, on $\mathbb{Z}^2_+$. Therefore, this can only be a discrete function concentrated on a discrete line $h\mathbb{Z}$. The form of such a function is given in [11], according to which

$$(T_{a_n}w_d^{(1)})(\tilde{x}) = \sum_{k=0}^{m-1} c_k(x_1)\Delta^{(k)}\delta_d(x_2),$$

under the condition that $æ - s = m + \varepsilon, m \in \mathbb{N}, |\varepsilon| < 1/2$. Here $\delta_d(x_2)$ is a discrete delta function, and $\Delta^{(k)}$ is the difference quotient of the $k$th order [11].

Applying the discrete Fourier transform to the last formula, we obtain

$$(F_d T_{a_n} w_d^{(1)})(\xi) = \sum_{k=0}^{m-1} \tilde{c}_k(\xi_1)\zeta_2^k, \quad \zeta_2 = \frac{e^{ih\xi_2} - 1}{h},$$

where $\tilde{c}_k$ are arbitrary functions in $H^{s_k}(\hbar\mathbb{T})$, $s_k = s - æ + k + 1/2$, $k = 0, 1, \dots, m-1$.

Introduce an operator $V_{a_n} \equiv F_d T_{a_n} F_d^{-1}$. It is obvious that the operator $V_{a_n}$ is invertible and $V_{a_n}^{-1} = V_{-a_n}$.

Since the solution of Eq. (2) will be constructed in terms of Fourier images, it is desirable to know what the operator $F_d T_{a_n}$ looks like in Fourier images. Calculations lead to the following conclusions:

$$\begin{aligned}(F_d T_{a_n} u_d)(\xi) &= \lim_{\tau\to+0} \sum_{\tilde{x}\in h\mathbb{Z}^2} e^{i\tilde{x}\cdot\xi} u_d\big(\tilde{x}_1, \tilde{x}_2 - a_n|\tilde{x}_1|\big)h^2 \\ &= \sum_{\tilde{x}_1\in h\mathbb{Z}} he^{i\tilde{x}_1\cdot\xi_1}\left(\sum_{\tilde{x}_2\in h\mathbb{Z}} e^{i\tilde{x}_2\cdot\xi_2} u_d\big(\tilde{x}_1, \tilde{x}_2 - a_n|\tilde{x}_1|\big)h\right) \\ &= \sum_{\tilde{x}_1\in h\mathbb{Z}} he^{i\tilde{x}_1\cdot\xi_1} \sum_{\tilde{y}_2\in h\mathbb{Z}} e^{i(\tilde{y}_2 + a_n|x_1|)\cdot\xi_2} u_d(\tilde{x}_1, \tilde{y}_2)h\end{aligned}$$

after the change $\tilde{y}_2 = \tilde{x}_2 - a_n|\tilde{x}_1|$. Then

$$(F_d T_{a_n} u_d)(\xi) = \sum_{\tilde{x}_1\in h\mathbb{Z}} he^{i\tilde{x}_1\cdot\xi_1} e^{ia_n|\tilde{x}_1|\cdot\xi_2} \hat{u}_d(x_1, \xi_2),$$

where $\hat{u}_d(x_1, \xi_2)$ is the discrete Fourier transform with respect to the second variable. We set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$ and divide the last sum into two terms. Then

$$(F_d T_{a_n} u_d)(\xi) = \sum_{\tilde{x}_1\in h\mathbb{Z}_+} he^{i\tilde{x}_1\cdot(\xi_1 + a_n\xi_2)} \hat{u}_d(x_1, \xi_2) + \sum_{\tilde{x}_1\in h\mathbb{Z}_-} he^{i\tilde{x}_1\cdot(\xi_1 - a_n\xi_2)} \hat{u}_d(x_1, \xi_2)$$

The last two sums were calculated in [9] for $h = 1$ using regularization by introducing a complex parameter and applying the convolution property of the Fourier transform. As a result, we have the expression

$$\begin{aligned}(F_d T_{a_n} u_d)(\xi) = \frac{\tilde{u}_d(\xi_1 + a_n\xi_2, \xi_2) + \tilde{u}_d(\xi_1 - a_n\xi_2, \xi_2)}{2} \\ + v.p.\frac{ih}{4\pi} \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 + a_n\xi_2 - \eta_1)}{2} \tilde{u}_d(\eta_1, \xi_2)\, d\eta_1 \\ - v.p.\frac{ih}{4\pi} \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 - a_n\xi_2 - \eta_1)}{2} \tilde{u}_d(\eta_1, \xi_2)\, d\eta_1.\end{aligned}$$

Using the last formula, we can construct a solution of the discrete equation (2). Indeed, taking into account the calculations carried out, we obtain

$$\tilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi) V_{-a_n}\left(\sum_{k=0}^{m-1} \tilde{c}_k(\xi_1)\zeta_2^k\right). \tag{4}$$

## 3. SOLVABILITY THEOREM

In this section, we rigorously formulate the solvability theorem in order to subsequently move on to a suitable boundary value problem.

**Theorem 1.** *Let the symbol $A_d(\xi)$ admit periodic wave factorization with respect to $K_n$ with an index æ such that æ $- s = m + \varepsilon$, $m \in \mathbb{N}$, $|\varepsilon| < 1/2$. Then the general solution Eq. (2) in the space $H^s(K_{n,d})$ is given by formula (4), where $c_k$ are arbitrary functions in $H^{s_k}(h\mathbb{Z})$, $s_k = s -$ æ $+ k + 1/2$.*

*One has the a priori estimate*

$$\|u_d\|_s \le b \sum_{k=0}^{m-1} [c_k]_{s_k}$$

*with a constant $b$ independent of $h$, where $[c_k]_{s_k}$ denotes the $H^{s_k}$-norm on $h\mathbb{Z}$.*

## 4. DISCRETE BOUNDARY VALUE PROBLEM

Here we will consider the simplest case of $m = 1$, where there is only one arbitrary function in the general solution. In expanded form, formula (4) looks as follows:

$$\begin{aligned}\tilde{u}_d(\xi) = \frac{\tilde{c}_0(\xi_1 - a_n\xi_2) + \tilde{c}_0(\xi_1 + a_n\xi_2)}{2A_{d,\neq}(\xi)} &+ v.p.\frac{ih}{4\pi A_{d,\neq}(\xi)} \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 - a_n\xi_2 - \eta)}{2}\tilde{c}_0(\eta)\,d\eta \\ &- v.p.\frac{ih}{4\pi A_{d,\neq}(\xi)} \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 + a_n\xi_2 - \eta)}{2}\tilde{c}_0(\eta)\,d\eta.\end{aligned} \tag{5}$$

An additional condition must be imposed for uniqueness. The following observation can be made here. The right-hand side of formula (5) with $\xi_2 = 0$ takes on a very simple form and formula (5) looks as

$$\tilde{u}_d(\xi_1, 0) = \tilde{c}_0(\xi_1) A^{-1}_{d,\neq}(\xi_1, 0). \tag{6}$$

It immediately follows that, knowing $\tilde{u}_d(\xi_1, 0)$, we can uniquely determine $\tilde{c}_0(\xi_1)$. We have

$$\tilde{u}(\xi_1, \xi_2) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{i\tilde{x}_1\xi_1 + i\tilde{x}_2\xi_2} u_d(\tilde{x}_1, \tilde{x}_2) h^2,$$

and then

$$\tilde{u}(\xi_1, 0) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{i\tilde{x}_1\xi_1} u_d(\tilde{x}_1, \tilde{x}_2) h^2 = \sum_{\tilde{x}_1 \in h\mathbb{Z}} e^{i\tilde{x}_1\xi_1} \left( \sum_{\tilde{x}_2 \in h\mathbb{Z}} u_d(\tilde{x}_1, \tilde{x}_2) h \right) h.$$

If we denote

$$\sum_{\tilde{x}_2 \in h\mathbb{Z}} u_d(\tilde{x}_1, \tilde{x}_2) h \equiv g_d(\tilde{x}_1), \tag{7}$$

then formula (6) implies $\tilde{c}_0(\xi_1) = A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1)$, and then

$$\begin{aligned}\tilde{u}_d(\xi) = \frac{A_{d,\neq}(\xi_1 - a_n\xi_2, 0)\tilde{g}_d(\xi_1 - a_n\xi_2) + A_{d,\neq}(\xi_1 + a_n\xi_2, 0)\tilde{g}_d(\xi_1 + a_n\xi_2)}{2A_{d,\neq}(\xi)} &+ v.p.\frac{ih}{4\pi A_{d,\neq}(\xi)} \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 - a_n\xi_2 - \eta)}{2} A_{d,\neq}(\eta, 0)\tilde{g}_d(\eta)\,d\eta \\ &- v.p.\frac{ih}{4\pi A_{d,\neq}(\xi)} \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 + a_n\xi_2 - \eta)}{2} A_{d,\neq}(\eta, 0)\tilde{g}_d(\eta)\,d\eta.\end{aligned} \tag{8}$$

The above reasoning leads to the following result.

**Theorem 2.** *Let the assumptions of Theorem 1 be satisfied, and let* $m = 1$. *Then for any right-hand side* $g \in H^{s+1/2}(h\mathbb{Z})$, *problem (2), (7) has the unique solution given by formula (8) in the space* $H^s(K_{n,d})$.

*One has the a priori estimate*

$$\|u_d\|_s \le b[g]_{s+1/2}$$

*with a constant* $b$ *independent of* $h$.

**Proof.** Taking into account the above calculations, only the a priori estimate needs proof. Indeed, according to Theorem 1,

$$\|u_d\|_s \le b[c_0]_{s_0},$$

and, since $\tilde{c}_0(\xi_1) = A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1)$, we have

$$[c_0]_{s_0} = \left[A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1)\right]_{s-æ+1/2} \le b[g]_{s+1/2}$$

by virtue of the property of a pseudodifferential operator with symbol $A_{d,\neq}(\xi_1, 0)$.

## 5. COMPARISON

Naturally, the question arises about comparing a discrete solution with its continual analog. In this section we will look at this issue.

By the continuous analog of the discrete problem (2),(7) we will mean the following problem. One is given a pseudodifferential operator $A$ with a symbol $A(\xi)$, $\xi = (\xi_1, \xi_2)$, satisfying the condition

$$c_1\big(1 + |\xi|\big)^{\alpha} \le \big|A(\xi)\big| \le c_2\big(1 + |\xi|\big)^{\alpha},$$

and the corresponding boundary value problem

$$(Au)(x) = 0, \quad x \in K_n, \tag{9}$$

$$\int_{-\infty}^{+\infty} u(x_1, x_2)\, dx_2 = g(x_1). \tag{10}$$

It is assumed that the symbol $A(\xi)$ admits wave factorization [5] with respect to $K_n$,

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

with an index æ such that $1/2 < æ - s < 3/2$.

The solution of the boundary value problem (9),(10) was written in the paper [7],

$$\begin{aligned}\tilde{u}(\xi) = {} & \frac{A_{\neq}(\xi_1 - a_n\xi_2, 0)\tilde{g}(\xi_1 - a_n\xi_2) + A_{\neq}(\xi_1 + a_n\xi_2, 0)\tilde{g}(\xi_1 + a_n\xi_2)}{2A_{\neq}(\xi)} \\ & + v.p.\frac{i}{2\pi A_{\neq}(\xi)}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)\, d\eta}{\xi_1 - a_n\xi_2 - \eta} - v.p.\frac{i}{2\pi A_{\neq}(\xi)}\int_{-\infty}^{+\infty}\frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)\, d\eta}{\xi_1 + a_n\xi_2 - \eta}.\end{aligned} \tag{11}$$

Formula (11) is obtained by a reasoning similar to the above [7], with the reasoning involving an arbitrary function $\tilde{C}_0(\xi_1) \in \widetilde{H}^{s-æ+1/2}(\mathbb{R})$ in the formula for the general solution

$$\begin{aligned}\tilde{u}(\xi) = {} & \frac{\tilde{C}_0(\xi_1 - a_n\xi_2) + \tilde{C}_0(\xi_1 + a_n\xi_2)}{2A_{\neq}(\xi)} \\ & + v.p.\frac{i}{2\pi A_{\neq}(\xi)}\int_{-\infty}^{+\infty}\frac{\tilde{C}_0(\eta)\, d\eta}{\xi_1 - a_n\xi_2 - \eta} - v.p.\frac{i}{2\pi A_{\neq}(\xi)}\int_{-\infty}^{+\infty}\frac{\tilde{C}_0(\eta)\, d\eta}{\xi_1 + a_n\xi_2 - \eta},\end{aligned} \tag{12}$$

which is determined from condition (10),

$$\tilde{C}_0(\xi_1) = A_{\neq}(\xi_1, 0)\tilde{g}(\xi_1). \tag{13}$$

Being given the boundary value problem (9),(10), we select a discrete operator $A_d$ and a discrete boundary function $g_d$ so that the solvability of the boundary value problem (9),(10) implies the solvability of the discrete boundary value problem (2),(7) for sufficiently small $h$ and give a comparison of the discrete (8) and continuous (11) solutions.

Let us write formulas (8) and (12) in terms of new variables $t = (t_1, t_2)$ (in this case, the square $\hbar\mathbb{T}^2$ transforms into a certain rectangle $\Pi_h$)

$$t_1 = \xi_1 - a_n\xi_2,$$
$$t_2 = \xi_1 + a_n, \xi_2$$

by introducing the notation

$$\tilde{U}(t_1, t_2) = \tilde{u}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a_n}\right), \qquad \tilde{U}_d(t_1, t_2) = \tilde{u}_d\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a_n}\right),$$
$$a(t_1, t_2) = A_{\neq}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a_n}\right), \qquad \tilde{a}_d(t_1, t_2) = A_{d,\neq}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a_n}\right).$$

Then

$$\begin{aligned}\tilde{U}_d(t) = {} & \frac{\tilde{c}_0(t_1) + \tilde{c}_0(t_2)}{2a_d(t)} \\ & + v.p.\frac{ih}{4\pi a_d(t)}\int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(t_1 - \eta)}{2}\tilde{c}_0(\eta)\,d\eta - v.p.\frac{ih}{4\pi a_d(t)}\int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(t_2 - \eta)}{2}\tilde{c}_0(\eta)\,d\eta.\end{aligned} \tag{14}$$

$$\tilde{U}(t) = \frac{\tilde{C}_0(t_1) + \tilde{C}_0(t_2)}{2a(t)} + v.p.\frac{i}{2\pi a(t}\int\limits_{-\infty}^{+\infty} \frac{\tilde{C}_0(\eta)\,d\eta}{t_1 - \eta} - v.p.\frac{i}{2\pi a(t)}\int\limits_{-\infty}^{+\infty} \frac{\tilde{C}_0(\eta)\,d\eta}{t_2 - \eta}. \tag{15}$$

To compare $\tilde{U}$ and $\tilde{U}_d$, we need the following formula [16]:

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}, \quad x^2 < \pi^2,$$

where $B_{2n}$ are Bernoulli numbers [17, 18]. Here are the first few Bernoulli numbers

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_3 = 0,$$
$$B_4 = -\frac{1}{30}, \qquad B_5 = 0, \qquad B_6 = \frac{1}{42}, \qquad \ldots$$

and their asymptotics

$$|B_{2n}| \sim \frac{2\cdot(2n)!}{(2\pi)^{2n}}, \quad n \to \infty. \tag{16}$$

To obtain a convergent series for $\cot\frac{h(t_1-\eta)}{2}$, we estimate

$$\left|\frac{h(t_1 - \eta)}{2}\right| \le \frac{h\big(|t_1| + |\eta|\big)}{2} \le \frac{\pi}{2}$$

under the condition that $|t_1| \leq \frac{\hbar\pi}{2}$, $|\eta| \leq \frac{\hbar\pi}{2}$. For such $t_1, \eta (t_1 \neq \eta)$ we have

$$\cot \frac{h(t_1-\eta)}{2} = \frac{2\hbar}{t_1-\eta} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} \frac{h^{2n-1}(t_1-\eta)^{2n-1}}{2^{2n-1}}$$
$$= \frac{2\hbar}{t_1-\eta} - \sum_{n=1}^{\infty} \frac{2|B_{2n}|h^{2n-1}(t_1-\eta)^{2n-1}}{(2n)!}.$$

Then

$$v.p.\frac{ih}{4\pi a_d(t)} \int_{-\hbar\pi/2}^{\hbar\pi/2} \cot \frac{h(t_1-\eta)}{2} \tilde{c}_0(\eta)\, d\eta = v.p.\frac{i}{2\pi a_d(t)} \int_{-\hbar\pi/2}^{\hbar\pi/2} \frac{\tilde{c}_0(\eta)\, d\eta}{t_1-\eta}$$
$$- \frac{ih}{4\pi a_d(t)} \int_{-\hbar\pi/2}^{\hbar\pi/2} \sum_{n=1}^{\infty} \frac{2|B_{2n}|h^{2n-1}(t_1-\eta)^{2n-1}}{(2n)!} \tilde{c}_0(\eta)\, d\eta.$$

Let us estimate

$$\left| \sum_{n=1}^{\infty} \frac{2|B_{2n}|h^{2n-1}(t_1-\eta)^{2n-1}}{(2n)!} \right| \leq \sum_{n=1}^{\infty} \frac{2|B_{2n}|h^{2n-1}\big(|t_1|+|\eta|\big)^{2n-1}}{(2n)!} \leq \sum_{n=1}^{\infty} \frac{2|B_{2n}|\pi^{2n-1}}{(2n)!},$$

and the convergence of the last series easily follows from (16). Thus,

$$\left| \int_{-\hbar\pi/2}^{\hbar\pi/2} \sum_{n=1}^{\infty} \frac{2|B_{2n}|h^{2n-1}(t_1-\eta)^{2n-1}}{(2n)!} \tilde{c}_0(\eta)\, d\eta \right| \leq c \int_{-\hbar\pi/2}^{\hbar\pi/2} \big|\tilde{c}_0(\eta)\big|\, d\eta,$$

and the constant $c$ is independent of $h$.

Note also that if $t \in \frac{\hbar}{2}\mathbb{T}^2$, then $\xi \in \Pi_h = [-\frac{\hbar}{2}, \frac{\hbar}{2}] \times [-\frac{\hbar}{2a_n}, \frac{\hbar}{2a_n}]$.

To construct a suitable discrete boundary value problem for (9),(10), it is necessary to choose $A_d(\xi)$ and $g_d$ in a special way. Let us introduce the operator $l_h$ that acts as follows. For a function $u$ defined on the line $\mathbb{R}$, we take its Fourier transform $\tilde{u}$, then its restriction to $\hbar\mathbb{T}$ and extend it periodically to $\mathbb{R}$. Next, its inverse discrete Fourier transform is taken and the function of the discrete variable $(l_h u)(\tilde{x}), \tilde{x} \in h\mathbb{Z}$ is obtained. Thus $g_d = l_h g$.

The symbol of the discrete operator $A_d$ is constructed in a similar way. Using the existing wave factorization of the symbol $A(\xi)$, we take restrictions of the factors to $\hbar\mathbb{T}^2$ with subsequent periodic continuation to $\mathbb{R}^2$; the periodic symbol $A_d(\xi)$ is the product of these periodic extensions. Below, we consider only such $A_d$, $g_d$ in problem (2),(7).

Now we are ready to formulate the main result about the comparison of discrete and continuous solutions.

**Theorem 3.** *Let the symbol $A(\xi)$ admit wave factorization with respect to $K_n$ with an index* æ *such that $1/2 <$ æ $- s < 3/2$, and let $g \in H^{s+1/2}(\mathbb{R})$. Then problems (9), (10) and (2), (7) have unique solutions $u \in H^s(K_n)$ and $u_d \in H^s(K_{n,d})$, respectively. If we additionally assume that the Fourier transform $\tilde{g}$ is compactly supported, then the following estimate holds for sufficiently small $h$:*

$$\big|\tilde{u}(\xi) - \tilde{u}_d(\xi)\big| \leq ch, \quad \xi \in \frac{\hbar}{4b_n}\mathbb{T}^2, \quad b_n = \max\{1, a_n\}, \tag{17}$$

*with a constant $c$ independent of $h$.*

**Proof.** The unique solvability of problem (9),(10) was proven in [7], and the unique solvability of problem (2),(7) follows from Theorem 2 due to the choice of the discrete symbol $A_d(\xi)$ and boundary function $g_d(\tilde{x}$. It remains to verify the validity of estimate (17).

Note that due to formulas (6),(13) and the choice of function $g_d$, the functions $\tilde{c}_0$ and $\tilde{C}_0$ coincide on $\hbar\mathbb{T}$. Let us compare the Fourier images of discrete and continuous solutions in variables $t=(t_1,t_2)$. We write out the difference

$$\tilde{U}_d(t)-\tilde{U}(t)=\underbrace{\frac{\tilde{c}_0(t_1)+\tilde{c}_0(t_2)}{2a_d(t)}-\frac{\tilde{C}_0(t_1)+\tilde{C}_0(t_2)}{2a(t)}}_{I_1}$$

$$\underbrace{+\,v.p.\frac{ih}{4\pi a_d(t)}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(t_1-\eta)}{2}\tilde{c}_0(\eta)\,d\eta-v.p.\frac{i}{2\pi a(t)}\int\limits_{-\infty}^{+\infty}\frac{\tilde{C}_0(\eta)\,d\eta}{t_1-\eta}}_{I_2}$$

$$\underbrace{-v.p.\frac{ih}{4\pi a_d(t)}\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(t_2-\eta)}{2}\tilde{c}_0(\eta)\,d\eta+v.p.\frac{i}{2\pi a(t)}\int\limits_{-\infty}^{+\infty}\frac{\tilde{C}_0(\eta)\,d\eta}{t_2-\eta}}_{I_3}.$$

As noted above, the term $I_1$ is equal to zero, and the terms $I_2$ and $I_3$ look alike. Since some estimates were given in this section for $I_2$, we will focus on this term.

Since $\tilde{C}_0(\eta)=A_{\neq}(\eta,0)\tilde{g}(\eta)$ and the function $\tilde{g}$ is finite, we can choose an $h$ so that $\operatorname{supp}\tilde{C}_0\subset\frac{\hbar}{2}\mathbb{T}$, and then

$$I_2=v.p.\frac{ih}{4\pi a_d(t)}\int\limits_{-\hbar\pi/2}^{\hbar\pi/2}\cot\frac{h(t_1-\eta)}{2}\tilde{c}_0(\eta)\,d\eta-v.p.\frac{i}{2\pi a(t)}\int\limits_{-\hbar\pi/2}^{+\hbar\pi/2}\frac{\tilde{C}_0(\eta)\,d\eta}{t_1-\eta}$$

$$=-\frac{ih}{4\pi a_d(t)}\int\limits_{-\hbar\pi/2}^{\hbar\pi/2}\sum_{n=1}^{\infty}\frac{2|B_{2n}|h^{2n-1}(t_1-\eta)^{2n-1}}{(2n)!}\tilde{c}_0(\eta)\,d\eta.$$

The estimate for the last expression was obtained above, therefore we can conclude that

$$|I_2|\le bh\int\limits_{-\hbar\pi/2}^{\hbar\pi/2}\left|\tilde{c}_0(\eta)\right|d\eta\le bh\int\limits_{\operatorname{supp}\tilde{g}}\left|A_{\neq}(\eta,0)\tilde{g}(\eta)\right|d\eta$$

with a constant $b$ independent of $h$, since $|a_d(t)|\ge\text{const}$

## CONCLUSIONS

The considered discrete boundary value problem with a nonlocal boundary condition, of course, does not exclude the possibility of considering problems with other boundary conditions. Some cases under similar conditions on the symbol were considered in [14, 15]. It is expected that the methods presented in this paper may be applicable in some multidimensional cases.

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## CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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