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On a Difference Scheme for the Growth-Propagation Equation

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Abstract—The article presents a model written in the form of a diffusion-logistic equation, which is most often used in tasks of describing the quantitative growth and distribution of certain substances, for example, biological populations. We are interested in the finite-difference approximation of this equation. For this aim, a two-layer difference scheme with weights was used. This scheme made it possible to achieve the order of approximation $O(h^2 + \tau)$ and reduce the problem of finding a solution to a nonlinear equation to solving a system of linear algebraic equations by the run-through method.

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1. INTRODUCTION

For the first time the problem of population growth was considered in the works by the Italian scientist L. Fibonacci. The question was posed as follows: "A person grows rabbits in a closed space surrounded by a high wall. How many couples of rabbits are born in one year from one couple, if a month later a couple of rabbits gives birth to another couple, and rabbits reproduce offspring from the second month after birth". The solution to this problem is a sequence of the Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, ...$, which can be given both recursively $f_{n+2} = f_n + f_{n+1}$, $n = 1, 2, \ldots$, and by the formula of a common element

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

It means that in the absence of external restrictions, population growth is proportional to its size.

A slightly simplified version of the reasoning leading to a differential growth model can be described as follows. Consider the amount of population growth $\Delta u = (R - S)\Delta t$, where R is the number of births, and S is the number of deceased individuals during Δt. Dividing the left and right parts by Δt and going to the limit at $t \to 0$, as a result we get the differential equation $\frac{du}{dt} = R - S$. Assuming that linearization of the form $R - S = ru$ is acceptable (the net increase in the population is proportional to its current population), we get $\frac{du}{dt} = ru(t)$. Solving this equation, we come to exponential growth dynamics

$$u(t) = u_0 e^{rt}. \tag{1}$$

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Equation (1) is called the Malthusian growth model in honor of the English demographer T. Malthus [1]. In the case when the growth rate of r is negative, the population dies out, otherwise the population grows. Over time, population growth becomes faster and rushes to infinity. The population growth described by the exponential model is correctly considered only for a certain limited period of time, since the supply of resources is being exhausted. In practice, there are different scenarios for the development of events after the exhaustion of resources. One of the most common options in practical ecology is a slow deceleration of growth as it approaches a certain level. The model describing such dynamics is called logistic. The logistic equation [2] was proposed in 1838 by the Belgian mathematician P.F. Verhulst. It has the form

$$\frac{du}{dt} = ru(1 - u/K). \tag{2}$$

The main difference between model (2) and the previous exponential growth model (1) is the condition of limited resources. The variable K means the saturated capacity of the environment, i.e., the maximum possible population size consuming all available resources. At small values of x, the number increases sharply, almost exponentially, and at sufficiently large values of u it approaches the threshold of K. Using the standard method of solving the equation, we obtain the dependence of the population size on time

$$u(t) = \frac{u_0 K e^{rt}}{K - u_0 + u_0 e^{rt}}.$$

Until the beginning of the 20th century, the logistic curve had been considered the universal law of the growth of all living things. However, even then, assumptions that this mathematical model does not contain all the parameters of the real process being modeled began to appear.

A new class of mathematical models appeared after adding a term with a spatial coordinate to the logistic equation. So the basic model of the environment in which self-organization processes are possible is the reaction–diffusion equation (RDE)

$$\frac{\partial u}{\partial_t} = D\nabla^2 u + f(u).$$

In 1921, Harold Hotelling suggested this equation with $f(u) = A(s - u)u$ as a tool for researching of the population growth and migration processes. A significant weakness of the Hotelling model is that livelihood stocks are assumed to be equal to a given constant, not depending on time and population (labor force). Therefore, this model is more suitable for animal populations. The simplest RDE is given in one spatial dimension. In 1937, Fischer [4] and Soviet mathematicians Kolmogorov, Petrovsky, and Piskunov [5] developed independently a model describing the distribution of biological populations

$$u_t = u_{xx} + u(1 - u). \tag{3}$$

Equation (3) is called the Kolmogorov–Petrovsky–Piskunov equation. A large number of papers related to the propagation of waves of various nature in space, including waves of population density [6] have been devoted to this equation. In [7], exact solutions of equation (3) are found for some special cases.

2. DIFFERENCE APPROXIMATION OF A ONE-DIMENSIONAL DIFFUSION-LOGISTIC EQUATION

Let's consider a mathematical model describing the spread of epidemics and population dynamics, which is based on the diffusion-logistic equation. This model leads to the next problem: it is necessary to find the function $u = u(x,t)$, satisfying the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Au(S - u), \tag{4}$$

at the rectangle $D = \{(x,t) : 0 < x < 1,\ 0 < t \leqslant T\}$, the initial condition $u(x,0) = u_0(x)$, $0 \leqslant x \leqslant 1$, and the boundary conditions $u(0,t) = u_1(t)$, $u(1,t) = u_2(t)$, $0 \leqslant t \leqslant T$. We divide the segments $[0,1]$ of the Ox axis and $[0,T]$ of the Ot axis into N and M parts, respectively. Thus the grid

$$\overline{w}_{h\tau} = \overline{w}_h \times \overline{w}_\tau = \{x_i = ih, i = 0,1,..,N\} \times \{t_j = j\tau, j = 0,1,..,M\}$$

with steps $h = 1/N$ and $\tau = T/M$ is introduced. To approximate the operator $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, we use a six-point pattern consisting of nodes $(x_{i\pm1}, t_{j+1})$, (x_i, t_{j+1}), $(x_{i\pm1}, t_j)$, centered at (x_i, t_{j+1}) [8]. The value of the grid function in the node (x_i, t_j) is denoted by y_i^j.

In order to make a difference approximation, we replace the derivative $\partial u/\partial t$ with the first difference derivative, and the second derivative $\partial^2 u/\partial x^2$ with respect to the spatial variable with the second difference derivative. Using the notation

$$u_{\overline{x}x} = \Lambda u = \frac{u(x+h,t) + u(x-h,t) - 2u(x,t)}{h^2}, \quad \Lambda y_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2},$$

and introducing arbitrary real parameters $\sigma, \beta \in (0,1)$, we obtain a difference scheme "with weights" [8]

$$\frac{y_i^{j+1} - y_i^j}{\tau} = \Lambda(\sigma y_i^{j+1} + (1-\sigma) y_i^j) + \phi_i^j, \quad 0 < i < N, \quad 0 \le j < M, \tag{5}$$

where ϕ_i^j is the grid approximation of the right side of the equation $f(u) = Au(S-u)$ on internal grid nodes. The initial and boundary conditions are approximated exactly

$$y_0^j = u_1(t_j), \quad y_N^j = u_2(t_j); \quad y_i^0 = y(x_i, 0) = u_0(x_i).$$

To estimate the error of the scheme under consideration, it is necessary to estimate the difference $z_i^j = y_i^j - u_i^j$ between the grid function and the continuous function $u = u(x,t)$ in the grid nodes. Let's move onto the non-index notation $y_i^j = y, y_i^{j+1} = \widehat{y}, y_t = \frac{\widehat{y} - y}{\tau}$. Then, the problem (5) can be rewritten as

$$\begin{cases} y_t = \Lambda(\sigma\widehat{y} + (1-\sigma)y) + \phi, \quad (x,t) \in w_{h\tau}, \\ y(0,t) = u_1(t), \quad y(1,t) = u_2(t),, \quad t \in w_\tau, \\ y(x,0) = u_0(x), \quad x \in \overline{w}_h. \end{cases} \tag{6}$$

Substituting $y = z + u$ in (6), we proceed to the problem for z:

$$\begin{cases} z_t = \Lambda(\sigma\widehat{z} + (1-\sigma)z) + \psi, \quad (x,t) \in w_{h\tau}, \\ z(0,t) = z(1,t) = 0, \quad t \in w_\tau, \\ z(x,0) = 0, \quad x \in \overline{w}_h. \end{cases}$$

The discrepancy of the scheme on the solutions is determined by the equation

$$\psi = \Lambda(\sigma\widehat{u} + (1-\sigma)u) - u_t + \phi,$$
$$\phi = (A-\beta)(S-u)\widehat{u} + \beta(S-\widehat{u})u.$$

Let the function $u(x,t)$ be sufficiently smooth. Then, using the Taylor expansion in the neighborhood of the point (x,t) and considering the operator Λ linear, we get

$$\psi = \sigma\big(\Lambda u + \tau\Lambda u_t + O(\tau^2)\big) + (1-\sigma)\Lambda u - u_t + (A-\beta)(S-u)\widehat{u} + \beta(S-\widehat{u})u. \tag{7}$$

Considering that

$$\Lambda u = u'' + \frac{h^2}{12}u^{(4)} + O(h^4) = Lu + \frac{h^2}{12}L^2 u + O(h^4), \quad Lu = \frac{\partial^2 u}{\partial x^2},$$
$$u_t = \frac{\widehat{u} - u}{\tau} = \frac{1}{\tau}\big(u + \tau\dot{u} + O(\tau^2) - u\big) = \frac{1}{\tau}\big(\tau\dot{u} + O(\tau^2)\big) = \dot{u} + O(\tau^2),$$

we can rewrite (7) as

$$\psi = \sigma\left(Lu + \frac{h^2}{12}L^2 u + \tau(L\dot{u} + \frac{h^2}{12}L^2\dot{u})\right) + (1-\sigma)\left(Lu + \frac{h^2}{12}L^2 u\right) - \dot{u} + \phi + O(h^4 + \tau^2).$$

In the final form, the approximation error is determined by the equality

$$\psi = Lu - \dot{u} + \phi + O(h^2 + \tau). \tag{8}$$

Since $\dot{u} = Lu + Au(S - u)$, then substituting in (8), we get

$$\psi = -Au(S - u) + (A - \beta)(S - u)\widehat{u} + \beta(S - \widehat{u})u + O(h^2 + \tau).$$

Let us apply the well-known expansion $\widehat{u} = u + \tau\dot{u} + O(\tau^2) = u + o(\tau)$, then

$$\psi = -Au(S - u) + (A - \beta)(S - u)u + \beta(S - u)u + O(h^2 + \tau) = O(h^2 + \tau).$$

Consequently, the considered difference equation approximates (4) with the second order of approximation error with respect to h and the first order with respect to τ.

The resulting system of equations for finding a set of values of the desired solution on the next layer is reduced to a system with a tridiagonal matrix, which is solved by the run-through method [9].

3. CONCLUSIONS

The scheme we suggested allowed to achieve the order of approximation of $O(h^2 + \tau)$ and the reduction of the nonlinear equation (4) to a system of linear algebraic equations with a tridiagonal matrix, which solution is carried out by the run-through method. When a system with a tridiagonal matrix is solved by the run-through method, the number of arithmetic operations is proportional to the number of grid nodes, which means that the time algorithm complexity is not great. Thus we constructed pretty exact and economical difference scheme.

REFERENCES

1. T. Malthus, *An Essay on the Principle of Population* (Cambridge Univ. Press, New York, 1992).
2. P. F. Verhulst, "Notice sur la loi que la population suit dans son accroissement," Corresp. Math. Phys. (Bruxelles) **10**, 113–121 (1838).
3. R. Ross, *The Prevention of Malaria,* 2nd ed. (John Murray, London, 1911).
4. R. A. Fisher, "The wave of advance of advantageous genes," Ann. Eugen. **7**, 355–369 (1937).
5. A. N. Kolmogoroff, I. G. Pretrovsky, and N. S. Piscounoff, "Ètude de l'èquation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique," Bull. Univ. Etat Moscou, Ser. Int., Sect. A **1** (6), 1–25 (1937).
6. V. G. Danilov, V. P. Maslov, and K. A. Volosov, *Mathematical Modelling of Heat and Mass Transfer Processes* (Kluwer, Dordrecht, 1995).
7. N. A. Kudryashov, "Exact solutions of a family of Fisher equations," Theor. Math. Phys. **94**, 211–218 (1993).
8. A. A. Samarskii, *Introduction to the Theory of Difference Schemes* (Nauka, Moscow, 1971) [in Russian].
9. A. A. Samarskii and A. V. Gulin, *Numerical Methods* (Nauka, Moscow, 1989) [in Russian]
10. G. E. Forsythe and C. B. Moler, *Computer Solution of Linear Algebraic Systems* (Prentice-Hall, Reading, MA, 1967).