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VEKUA-ERDÉLYI-LOWNDES TYPE TRANSMUTATION AND APPLICATIONS

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We construct Vekua-Erdelyi-Lowndes type transmutation operators that transform solutions to the Cauchy problem for unperturbed equations into to the Cauchy problem for perturbed ones. Bibliography: 4 *titles.*

1 Introduction

In this paper, we develop the so-called transmutation method (see historical survey in [1]). Let two operators (A, B) be given. A nonzero operator T is called the *transmutation operator* if

$$T\,A = B\,T. \tag{1.1}$$

An important step of the transmutation method is to choose an appropriate function space where the equality (1.1) is valid.

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We are interested in a special class of transmutation operators that intertwines the operators $A+\lambda_1$ and $A+\lambda_2$, where $A : S_1 \to S_2$ is some operator, $\lambda_1, \lambda_2 \in \mathbb{C}$, and S_1, S_2 is a pair of function spaces. Since such operators appeared in the works of Erdélyi, Vekua, and Lowndes, it is natural that they will be referred to as Vekua–Erdélyi–Lowndes operators. We note that such operators were considered in [2] (see also the references therein).

Based on Vekua–Erdélyi–Lowndes transmutation operators, we clarify connections between the solutions to the Cauchy problems for the equations $w_{tt} = Aw$ and $w_{tt} \pm c^2 w = Aw$, where $w = w(x,t)$, $c \in \mathbb{R}$, and A is a linear operator acting by $x \in \mathbb{R}^n$. The class of equations with such operators includes, in particular, the telegraph equation and the Helmholtz equation.

2 Transmutations in the Form of Volterra Operators of the Second Kind

In this section, we construct transmutation operators $S_c^{\pm}$ with intertwining property

$$S_c^{\pm} D^2 = \left(D^2 \pm c^2\right) S_c^{\pm}.$$

Theorem 2.1. *Let $f \in C^2$. A transmutation operator satisfying the identity*

$$S_c^{\pm} D^2 f = \left(D^2 \pm c^2\right) S_c^{\pm} f, \tag{2.1}$$

where $D = d/dt$ has the form of the Volterra operator of the second kind

$$(S_c^{\pm} f)(t) = f(t) + \int_{-t}^{t} K^{\pm}(t,\tau) f(\tau)\, d\tau, \tag{2.2}$$

with the kernel

$$K^{\pm}(t,\tau) = \frac{c\sqrt{t+\tau}}{2\sqrt{t-\tau}} \begin{cases} -J_1\left(c\sqrt{t^2-\tau^2}\right), \\ I_1\left(c\sqrt{t^2-\tau^2}\right). \end{cases}$$

Proof. We are looking for a transmutation operator satisfying the identity (2.1) in the form of Volterra operator of the second kind (2.2) Here, the kernel $K^{\pm}(t,\tau)$ is smooth in both variables. Substitution into the formula (2.1) leads to the relation

$$\int_{-t}^{t} K^{\pm}(t,\tau) f''(\tau)\, dt = \frac{d^2}{dt^2} \int_{-t}^{t} K^{\pm}(t,\tau) f(\tau)\, d\tau \pm c^2 \left(f(t) + \int_{-t}^{t} K^{\pm}(t,\tau) f(\tau)\, d\tau \right).$$

Since

$$\int_{-t}^{t} K^{\pm}(t,\tau) f''(\tau)\, d\tau = K^{\pm}(t,t) f'(t) - K^{\pm}(t,-t) f'(-t) - K_\tau^{\pm}(t,\tau)\,|_{\tau=t}\, f(t)$$

$$+ K_\tau^{\pm}(t,\tau)\,|_{\tau=-t}\, f(-t) + \int_{-t}^{t} K_{\tau\tau}^{\pm}(t,\tau) f(\tau)\, d\tau$$

and

$$\frac{d^2}{dt^2}\int\limits_{-t}^{t} K^{\pm}(t,\tau)f(\tau)\,d\tau = \int\limits_{-t}^{t} K^{\pm}_{tt}(t,\tau)f(\tau)\,d\tau + K^{\pm}_{t}(t,\tau)\mid_{\tau=t} f(t) - K^{\pm}_{t}(t,\tau)\mid_{\tau=-t} f(-t)$$

$$+\frac{K^{\pm}(t,t)}{dt}f(t) + K^{\pm}(t,t)f'(t) - \frac{K^{\pm}(t,-t)}{dt}f(-t) - K^{\pm}(t,-t)f'(-t),$$

we get

$$K^{\pm}_{\tau\tau}(t,\tau) = K^{\pm}_{tt}(t,\tau) \pm c^2 K^{\pm}(t,\tau), \tag{2.3}$$

$$\frac{dK^{\pm}(t,t)}{dt} + \lim_{\tau\to t}\left(K^{\pm}_{t}(t,\tau) + K^{\pm}_{\tau}(t,\tau)\right) = \mp c^2, \tag{2.4}$$

$$\frac{dK^{\pm}(t,-t)}{dt} + \lim_{\tau\to -t}(K^{\pm}_{t}(t,\tau) + K^{\pm}_{\tau}(t,\tau)) = 0, \tag{2.5}$$

Let $K^{\pm}(t,\tau) \in C^1(\Omega)$, $\overline{\Omega} \cap \{(t,\tau) \mid t = \tau\} \neq \varnothing$. Then for $(t,x) \in \Omega$

$$\frac{d}{dt}K^{\pm}(t,t) = \lim_{\tau\to t}\left(\frac{\partial K^{\pm}(t,\tau)}{\partial t} + \frac{\partial K^{\pm}(t,\tau)}{\partial \tau}\right).$$

Therefore, (2.4) and (2.5) take the form

$$\frac{dK^{\pm}(t,t)}{dt} = \mp\frac{c^2}{2}, \tag{2.6}$$

$$K^{\pm}(t,-t) = \text{const}\,. \tag{2.7}$$

Introducing the new variables

$$u = \frac{t+\tau}{2}, \quad v = \frac{t-\tau}{2} \tag{2.8}$$

and setting $H^{\pm}(u,v) = K^{\pm}(u+v, u-v) = K^{\pm}(t,\tau)$ we obtain the problem

$$H^{\pm}_{u,v}(u,v) = \mp c^2 H^{\pm}(u,v), \tag{2.9}$$

$$H^{\pm}(u,0) = \mp\frac{c^2}{2}u. \tag{2.10}$$

To construct kernels satisfying (2.9)–(2.10), one can use the formula

$$H^{\pm}(u,v) = \mp\frac{c^2}{2}u \mp c^2\int\limits_0^u d\alpha \int\limits_0^v H^{\pm}(\alpha,\beta)\,d\beta. \tag{2.11}$$

Consider the iterations

$$H^{\pm}_0(u,v) = \mp\frac{c^2}{2}u,$$

$$H^{\pm}_{n+1}(u,v) = \mp c^2\int\limits_0^u d\alpha \int\limits_0^v H^{\pm}_n(\alpha,\beta)\,d\beta.$$

From the first iterations we get

$$H_1^{\pm}(u,v) = \frac{1}{2}\left(\mp c^2\right)^2 \frac{u^2}{2!} v,$$

$$H_2^{\pm}(u,v) = \frac{1}{2}\left(\mp c^2\right)^3 \frac{u^3}{3!}\frac{v^2}{2!},$$

$$H_n^{\pm}(u,v) = \frac{1}{2}\frac{\left(\mp c^2\right)^{n+1}}{n!(n+1)!} u^{n+1} v^n.$$

We use the formulas for the Bessel functions and modified Bessel functions of the first kind for $m \in \mathbb{N} \cup \{0\}$ (see, for example, [3])

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m} n!(m+n)!} x^{2n+m}, \quad I_m(x) = \sum_{n=0}^{\infty} \frac{1}{2^{2n+m} n!(m+n)!} x^{2n+m}.$$

Summing up the Neumann series, we get

$$H^{\pm}(u,v) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\left(\mp c^2\right)^{n+1}}{n!(n+1)!} u^{n+1} v^n = \frac{c\sqrt{u}}{2\sqrt{v}} \begin{cases} -J_1(2c\sqrt{uv}), \\ I_1(2c\sqrt{uv}). \end{cases}$$

Using the asymptotic formulas for $0 < x \ll \sqrt{\alpha+1}$

$$J_\alpha(x) \to \frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^\alpha, \quad I_\alpha(x) = i^{-\alpha} J_\alpha(ix),$$

we see that (2.10) is satisfied. Returning to x and t, we get

$$K^{\pm}(t,\tau) = \frac{c\sqrt{t+\tau}}{2\sqrt{t-\tau}} \begin{cases} -J_1\left(c\sqrt{t^2-\tau^2}\right); \\ I_1\left(c\sqrt{t^2-\tau^2}\right). \end{cases}$$

It is easy to see that $K^{\pm}(t,-t) = 0$ and the condition (2.7) is satisfied. □

3 Application of Vekua–Erdélyi–Lowndess Transmutations to the Cauchy Problem

Theorem 3.1. *Let A be a linear operator, and let w be a solution to the problem*

$$w_{tt} = Aw, \quad w = w(x,t), \tag{3.1}$$

$$w(x,0) = f(x), \quad w_t(x,0) = g(x). \tag{3.2}$$

Then the function $w^c = S_+ w$, where

$$(S_+)_t w(x,t) = w(x,t) - \frac{c}{2}\int_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1(c\sqrt{t^2-\tau^2}) w(x,\tau)\, d\tau,$$

is a solution to the problem

$$w^c_{tt} + c^2 w^c = Aw^c, \quad w^c = w^c(x,t), \tag{3.3}$$

$$w^c(x,0) = f(x), \quad w^c_t(x,0) = g(x). \tag{3.4}$$

Proof. It is easy to see that $w^c(x,0) = w(x,0)$. Therefore, if w satisfies the first condition in (3.2), then w^c satisfies the first condition in (3.4). The converse is also valid. We have

$$w_t^c(x,t) = w_t(x,t) - \frac{c}{2}\frac{\partial}{\partial t}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1(c\sqrt{t^2-\tau^2}) w(x,\tau)\, d\tau$$

$$= w_t(x,t) - \frac{c}{2}\Bigg(\lim_{\tau\to t}\Big(\frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1(c\sqrt{t^2-\tau^2}) w(x,\tau)\Big) - \lim_{\tau\to -t}\Big(\frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1(c\sqrt{t^2-\tau^2}) w(x,\tau)\Big) +$$

$$+ \int\limits_{-t}^{t} \frac{\partial}{\partial t}\frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1(c\sqrt{t^2-\tau^2}) w(x,\tau)\, d\tau \Bigg)$$

$$= w_t(x,t) - \frac{c^2}{2}\Bigg(t w(x,t) + \int\limits_{-t}^{t} \Big(\frac{t J_0(c\sqrt{t^2-\tau^2})}{t-\tau} - \frac{\sqrt{t+\tau}}{c(t-\tau)^{3/2}} J_1(c\sqrt{t^2-\tau^2}) \Big) w(x,\tau)\, d\tau \Bigg).$$

Letting $t \to 0$, we get $w_t^c(x,0) = w_t(x,0)$.

We show that, if w satisfies Equation (3.1), then w^c satisfies Equation (3.3). We have

$$(D_t^2 + c^2) w^c = (D_t^2 + c^2) S_+ w = S_+ D_t^2 w = S_+ A w = A S w = A w^c.$$

Therefore, $(D_t^2 + c^2) w^c = A w^c$ and w^c satisfies Equation (3.3). □

The following result is proved in the same way as Theorem 3.1.

Theorem 3.2. *Let A be a linear operator, and let w be a solution to the problem*

$$w_{tt} = Aw, \quad w = w(x,t),$$
$$w(x,0) = f(x), \quad w_t(x,0) = g(x).$$

Then the function $w^c = S_- w$, where

$$(S_-)_t w(x,t) = w(x,t) + \frac{c}{2}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} I_1\left(c\sqrt{t^2-\tau^2}\right) w(x,\tau)\, d\tau$$

is a solution to the problem

$$w_{tt}^c - c^2 w^c = A w^c, \quad w^c = w^c(x,t),$$
$$w^c(x,0) = f(x), \quad w_t^c(x,0) = g(x).$$

Example 3.1. Consider the wave equation in the one-dimensional case with the initial conditions

$$w_{tt} = a^2 w_{xx}, \quad w(x,0) = f(x), \quad w_t(x,0) = g(x).$$

Recall that the solution is given by the d'Alembert formula

$$w(x,t) = \frac{f(x-at) + f(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} g(s) ds.$$

By Theorem 3.1,

$$w^c(x,t) = (S_+)_t w(x,t) = w(x,t) - \frac{c}{2}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right) w(x,\tau)\,d\tau$$

$$= \frac{f(x-at)+f(x+at)}{2} - \frac{c}{4}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left(f(x-a\tau)+f(x+a\tau)\right)d\tau$$

$$+ \frac{1}{2a}\int\limits_{x-at}^{x+at} g(s)ds - \frac{c}{4a}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left(\int\limits_{x-a\tau}^{x+a\tau} g(s)ds\right) d\tau \tag{3.5}$$

is a solution to the telegraph equation

$$w^c_{tt} = a^2 w^c_{xx} - c^2 w^c, \quad w^c(x,0) = f(x), \quad w^c_t(x,0) = g(x).$$

We transform the second term on the right-hand side of (3.5) as follows:

$$\frac{c}{4}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left(f(x-a\tau)+f(x+a\tau)\right)d\tau$$

$$= \frac{c}{4}\int\limits_{-t}^{t} \frac{t+\tau}{\sqrt{t^2-\tau^2}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left[f(x-a\tau)+f(x+a\tau)\right]d\tau$$

$$= \frac{ct}{2}\int\limits_{-t}^{t} \frac{J_1\left(c\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}} f(x-a\tau)\,d\tau = \{x+a\tau=s\} = \frac{ct}{2a}\int\limits_{x-at}^{x+at} \frac{J_1\left(c\sqrt{t^2-\left(\frac{x-s}{a}\right)^2}\right)}{\sqrt{t^2-\left(\frac{x-s}{a}\right)^2}} f(s)\,ds.$$

Taking account that

$$\int \tau \frac{J_1\left(c\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}} d\tau = \frac{1}{c}\left(J_0(c\sqrt{t^2-\tau^2}) - 1\right) + C,$$

we transform the third and fourth terms on the right-hand side of (3.5) as follows:

$$(S_+)_t \frac{1}{2a}\int\limits_{x-at}^{x+at} g(s)ds = \frac{1}{2a}\int\limits_{x-at}^{x+at} g(s)\mathrm{d}s - \frac{c}{4a}\int\limits_{-t}^{t} \frac{t+\tau}{\sqrt{t^2-\tau^2}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left(\int\limits_{x-a\tau}^{x+a\tau} g(s)ds\right) d\tau$$

$$= \frac{1}{2a}\int\limits_{x-at}^{x+at} J_0\left(c\sqrt{t^2-\frac{(x-s)^2}{a^2}}\right) g(s)ds.$$

As a result, we get the known formula (see, for example, [4])

$$w^c(x,t) = \frac{f(x-at)+f(x+at)}{2} - \frac{ct}{2a}\int\limits_{x-at}^{x+at} \frac{J_1\left(c\sqrt{t^2-\frac{(x-s)^2}{a^2}}\right)}{\sqrt{t^2-\frac{(x-s)^2}{a^2}}} f(s)\mathrm{d}s$$

$$+\frac{1}{2a}\int\limits_{x-at}^{x+at} J_0\left(c\sqrt{t^2-\frac{(x-s)^2}{a^2}}\right) g(s)ds.$$

Example 3.2. The following problem describes transverse vibrations of elastic rods:

$$w_{tt} = -a^2 w_{xxxx}, \quad w(x,0) = f(x), \quad w_t(x,t) = ag''(x) \tag{3.6}$$

and has the Boussinesq solution (see, for example, [4])

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x-2p\sqrt{at})(\cos(p^2)+\sin(p^2))dp$$

$$+\frac{1}{a\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(x-2p\sqrt{at})(\cos(p^2)-\sin(p^2))dp.$$

By Theorem 3.1,

$$w^c(x,t) = (S_+)_t w(x,t) = w(x,t) - \frac{c}{2}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right) w(x,\tau)\, d\tau$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x-2p\sqrt{at})(\cos(p^2)+\sin(p^2))dp + \frac{1}{a\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(x-2p\sqrt{at})(\cos(p^2)-\sin(p^2))dp$$

$$-\frac{c}{2\sqrt{2\pi}}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left(\int\limits_{-\infty}^{\infty} f(x-2p\sqrt{a\tau})(\cos(p^2)+\sin(p^2))dp\right) d\tau$$

$$-\frac{c}{2a\sqrt{2\pi}}\int\limits_{-t}^{t} \frac{\sqrt{t+\tau}}{\sqrt{t-\tau}} J_1\left(c\sqrt{t^2-\tau^2}\right)\left(\int\limits_{-\infty}^{\infty} g(x-2p\sqrt{a\tau})(\cos(p^2)-\sin(p^2))dp\right) d\tau$$

is a solution to the Cauchy problem for the perturbed Boussinesq type equation with an additional parameter

$$w^c_{tt} = -a^2 w^c_{xxxx} - c^2 w^c, \quad w^c(x,0) = f(x), \quad w^c_t(x,0) = ag''(x).$$

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

References

1. V. V. Katrakhov and S. M. Sitnik, "The transmutation method and boundary-value problems for singular elliptic equations" [in Russian], *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya* **64**, No. 2, 211–426 (2018).

2. E. L. Shishkina and S. M. Sitnik, *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics*, Elsevier, Amsterdam (2020).

3. M. Abramowitz (Ed.) and I. Stegun (Ed.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, John Wiley and Sons, New York etc. (1972).

4. A. D. Polyanin and V. E. Nazaikinskii, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, CRC Press, Boca Raton, FL (2002).

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