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Analytical Solution of One Non-Local Problem for Hyperbolic Equation

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Abstract—In this paper we consider small oscillations of the ideal gas near its equilibrium state inside an unbounded cylindrical tube. We investigate radial oscillations of the gas within a fixed cross-section. The gas is radially inhomogeneous. There is a power dependence of the gas density on the radial coordinate. And also the Boyle—Mariotte law is fulfilled at a constant temperature. We study the pressure in a gas, the values of which are known at the initial and final times of the experiment, and the average value of the pressure at any other time is constant. This physical problem is modeled by a non-local boundary value problem with an integral condition for a hyperbolic equation with a singular coefficient in a rectangular domain. Uniqueness and existence theorems for a solution to the problem are proved. The solution of the problem is constructed in an explicit analytical form.

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1. INTRODUCTION

Equations describing small gas oscillations are derived from the following general equations of hydrodynamics [1, pp. 349, 356]: the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \tag{1}$$

and the motion equation in the Euler form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F - \frac{1}{\rho}\operatorname{grad} p, \tag{2}$$

where v is the velocity, p is the pressure, and ρ is the density of the gas; $\mathrm{d}v/\mathrm{d}t$ is the material derivative (see [1, p. 356]), i.e., the particle velocity at the specified point; and F are the acting external forces.

The following Poisson adiabatic relation is satisfied for the pressure p and density ρ

$$p = p_0(\rho/\rho_0)^\gamma, \tag{3}$$

where $\gamma > 0$ is a constant.

Assume that the velocity v is a small value and the deviations of the pressure p and density ρ from their initial values p_0 and ρ_0, which might depend on spatial coordinates in the general case, are slight. By virtue of the smallness of acoustic oscillations, values of the second-order smallness in the equations can be disregarded; thus, the equations become linear. To linearize the equations, introduce the following

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new values: $\sigma = (\rho - \rho_0)/\rho_0$ is the relative variation of the density (the gas condensation) and $q = p/p_0$ is the relative pressure. Then, Eq. (3) takes the form

$$q = (1+\sigma)^\gamma \approx 1 + \gamma\sigma.$$

Substituting $\rho = \rho_0(1+\sigma)$ in Eq. (1), we obtain

$$\rho_0 \frac{\partial \sigma}{\partial t} + \operatorname{div}(\rho_0(1+\sigma)v) = 0.$$

Since σv is a small value, it follows that

$$\rho_0 \frac{\partial \sigma}{\partial t} + \rho_0 \operatorname{div} v + (\operatorname{grad} \rho_0, v) = 0.$$

In the last relation, pass from σ to q and take into account the relation $\sigma = (q-1)/\gamma$. We obtain the relation

$$\frac{1}{\gamma}\frac{\partial q}{\partial t} + \operatorname{div} v + \frac{1}{\rho_0}(\operatorname{grad} \rho_0, v) = 0. \tag{4}$$

Further, change $\mathrm{d}v/\mathrm{d}t$ for $\partial v/\partial t$ in Eq. (2), disregarding small values, and assume that there are no external forces. Finally, substitute $p_0 q$ instead of p in Eq. (2). We obtain

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\operatorname{grad}(p_0\, q). \tag{5}$$

In the cylindrical coordinate system (r, α, z), Eqs. (4) and (5) take the forms

$$\frac{1}{\gamma}\frac{\partial q}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\alpha}{\partial \alpha} + \frac{\partial v_z}{\partial z} + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r}v_r + \frac{1}{\rho_0\, r}\frac{\partial \rho_0}{\partial \alpha}v_\alpha + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}v_z = 0$$

and

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial r}(p_0\, q), \quad \frac{\partial v_\alpha}{\partial t} = -\frac{1}{\rho_0}\frac{1}{r}\frac{\partial}{\partial \alpha}(p_0\, q), \quad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}(p_0\, q).$$

Assume that the gas is radially heterogeneous, i.e., $\rho_0 = \rho_0(r)$. Let the density have a power law dependence on the radial coordinate: $\rho_0(r) = r^\beta$, where $\beta > 0$ is a constant. By virtue of the Boyle–Mariotte law, the relation $p_0(r) = \chi\rho_0(r)$ holds provided that the temperature is constant, where $\chi > 0$ is a constant. Let the desired functions do not depend on the coordinates z and α, i.e., we investigate radial oscillations of the gas within a fixed cross-section. Thus, the following equations are left

$$\frac{1}{\gamma}\frac{\partial q}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r}v_r = 0 \quad \text{and} \quad \frac{\partial v_r}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial r}(p_0\, q).$$

Excluding the function v_r from them, we obtain

$$\frac{1}{\gamma\chi}\frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial r^2} + \frac{2\beta+1}{r}\frac{\partial q}{\partial r} + \frac{\beta^2}{r^2}q.$$

In the last relation, pass from the relative pressure q to the pressure p as follows: $q = p/p_0 = (r^{-\beta}p)/\chi$. This yields the hyperbolic equation

$$\frac{1}{\gamma\chi}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r},$$

or the equation

$$\frac{\partial^2 p}{\partial t^2} = a^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right), \tag{6}$$

where $a^2 := \gamma\chi > 0$ is a constant.

Let us formulate the mathematical problem to investigate the gas pressure inside a cylindrical tube of radius l, restricting the consideration by radial oscillations of particles. Instead of Eq. (6), take more general equation with the Bessel differential operator

$$\frac{\partial^2 p}{\partial r^2} + \frac{k}{r}\frac{\partial p}{\partial r} = r^{-k}\frac{\partial}{\partial r}\left(r^k \frac{\partial p}{\partial r}\right),$$

where $k \in (-1, 1)$ and $k \neq 0$ is a given real number.

Let $a^2 = 1$ in Eq. (6), which does not limit the generality. Fix the beginning $t = 0$ of the experiment and its end $t = T$, assuming that the exact measuring of the investigated value is possible in no particular point of the selected radius-segment. Thus, let $D = \{(r, t) : 0 < r < l,\ 0 < t < T\}$ be a rectangular domain of the coordinate plane Ort, where $l > 0$ and $T > 0$ are given real numbers.

Formulation of the problem. It is required to find a function $p(r, t)$ satisfying the conditions

$$p(r,t) \in C(\overline{D}) \cap C^2(D), \tag{7}$$

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial r^2} + \frac{k}{r}\frac{\partial p}{\partial r}, \quad (x,t) \in D, \tag{8}$$

$$p(r,0) = \varphi(r), \quad p(r,T) = \psi(r), \quad 0 \le r \le l, \tag{9}$$

$$\lim_{r \to +0} r^k p_r(r,t) = 0, \quad 0 \le r \le T, \tag{10}$$

and

$$\int_0^l p(r,t)\, r^k\, dr = A, \quad 0 \le t \le T, \tag{11}$$

where $k \in (-1, 1)$ and $k \neq 0$ is a given real number; A is a given real number; while $\varphi(r)$ and $\psi(r)$ are given sufficiently smooth functions satisfying the coordination conditions

$$\int_0^l \varphi(r)\, r^k\, dr = \int_0^l \psi(r)\, r^k\, dr = A. \tag{12}$$

The theory of problems for equations containing the Bessel differential operator is one of the most important sections of the modern theory of partial differential equations. The importance of this class of equations is due to their use in applications to various problems of gas dynamics and acoustics, jet theory in hydrodynamics, linearized Maxwell–Einstein equations, and elasticity-plasticity theory.

A great contribution to the development of the theory of boundary value problems for equations with the Bessel operator belongs to Kipriyanov [2]. An extensive study of boundary-value problems for equations of three main classes with Bessel operator was presented in [3–8].

At the moment, problems with integral conditions are comprehensively and profoundly studied for equations of various classes. Problems with integral conditions of are studied for hyperbolic and mixed-type equations with Bessel operator (see [9–12]).

2. CONSTRUCTION OF SOLUTIONS. SOLUTION UNIQUENESS

Multiply (6) by r^k and, under a fixed $t \in (0, T)$, integrate the obtained product with respect to the variable r from ε to $l - \varepsilon$, where $\varepsilon > 0$ is sufficiently small. We obtain the relation

$$\int_\varepsilon^{l-\varepsilon} p_{tt} r^k\, dr - \int_\varepsilon^{l-\varepsilon} \frac{\partial}{\partial r}\left(r^k \frac{\partial p}{\partial r}\right) dr = 0.$$

Passing to the limit as $\varepsilon \to 0$ and taking into account conditions (10) and (11),

$$p_r(l,t) = 0, \quad 0 \le t \le T. \tag{13}$$

In the sequel, instead of problem (7)–(12), we consider problem (7)–(10), and (13).

Particular solutions of Eq. (6) that are different from zero in the domain D and satisfy conditions (10) and (13) are sought in the form $p(r,t) = R(r)T(t)$. Substituting this function in Eq. (6) and in conditions (10) and (13) and separating the variables, we obtain the following spectral problem with respect to the function $R(r)$:

$$R''(r) + \frac{k}{r}R'(r) + \lambda^2 R(r) = 0, \quad 0 < r < l, \tag{14}$$

$$\lim_{r\to+0} r^k R'(r) = 0, \quad R'(l) = 0, \tag{15}$$

here λ^2 is the separation constant.

For $k \in (-1, 1)$ and $k \neq 0$, the general solution of Eq. (14) is determined as follows

$$\widetilde{R}(r) = \tilde{C}_1 r^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda r) + \tilde{C}_2 r^{\frac{1-k}{2}} J_{\frac{1-k}{2}}(\lambda r),$$

where $J_\nu(\xi)$ and $J_{-\nu}(\xi)$ are the Bessel functions of the first kind of orders (respectively) $\nu = (k-1)/2$ and $-\nu = (1-k)/2$, while $\tilde{C}_1$ and $\tilde{C}_2$ are arbitrary constants.

To ensure that the found function satisfies the first condition from (15), assign $\tilde{C}_2 = 0$. Set $\tilde{C}_1 = 1$ because eigenfunctions are defined up to a constant factor. Then, the solution takes the form

$$\widetilde{R}(r) = r^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda r). \tag{16}$$

Now, substituting function (16) in the second condition from (15), we find that $\lambda_0 = 0$ and $\widetilde{R}'(l) = -l^{\frac{1-k}{2}} J_{\frac{k+1}{2}}(\lambda l)$, which implies that

$$J_{\frac{k+1}{2}}(\mu) = 0, \quad \mu = \lambda l. \tag{17}$$

From [13, p. 530], it is known that if $\nu > -1$, then the function $J_\nu(\xi)$ has a denumerable set of real roots. Then, for a given k, denoting the nth root of Eq. (17) by μ_n, find eigenvalues $\lambda_n = \mu_n/l$ of problem (14), (15). Thus, the system of eigenfunctions of problem (14), (15) has the form

$$\widetilde{R}_0(r) = 1, \quad \lambda_0 = 0,$$

$$\widetilde{R}_n(r) = r^{\frac{1-k}{2}} J_{\frac{k-1}{2}}\left(\frac{\mu_n r}{l}\right) = r^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n r), \quad n \in \mathbb{N},$$

where the eigenvalues λ_n are defined as roots of Eq. (17).

Note that the system of eigenfunctions $\widetilde{R}_0(r)$ and $\widetilde{R}_n(r)$ ($n \in \mathbb{N}$) is orthogonal and complete in the space $L_2[0,l]$ with weight r^k (see [14, p. 343]).

For further computations, we use the orthonormal system of functions

$$R_n(r) = \frac{\widetilde{R}_n(r)}{||\widetilde{R}_n(r)||}, \quad n \in \mathbb{N} \cup \{0\}, \tag{18}$$

with norm

$$||\widetilde{R}_n(r)||^2 = \int_0^l r^k\, \widetilde{R}_n^2(r)\, dr. \tag{19}$$

Introduce the functions

$$p_n(t) = \int_0^l p(r,t) r^k R_n(r)\, dx, \quad n \in \mathbb{N} \cup \{0\}, \tag{20}$$

where $R_n(r)$ are defined by means of relations (18). Basing on (20), consider auxiliary functions

$$p_{n,\varepsilon}(t) = \int\limits_{\varepsilon}^{l-\varepsilon} p(r,t) r^k R_n(r)\, dr, \quad n \in \mathbb{N}, \tag{21}$$

where $\varepsilon > 0$ is a sufficiently small number.

Twice differentiate relation (21) with respect to t for $0 < t < T$ and take into account Eq. (6). We obtain

$$p''_{n,\varepsilon}(t) = \int\limits_{\varepsilon}^{l-\varepsilon} p_{tt}(r,t) r^k R_n(r)\, dr = \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\partial}{\partial r}(r^k p_r) R_n(r)\, dr. \tag{22}$$

By virtue of Eq. (14), derive the following relation from (21)

$$p_{n,\varepsilon}(t) = -\frac{1}{\lambda_n^2} \int\limits_{\varepsilon}^{l-\varepsilon} p(r,t) \frac{d}{dr}\left(r^k R'_n(r)\right)\, dr.$$

Substituting this expression in (22). In the last relation, pass to the limit as $\varepsilon \to 0$. Then, taking into account (7) and the boundary-value conditions given by (13) and (15), we obtain the following ordinary differential equation for determining the functions $p_n(t)$

$$p''_n(t) + \lambda_n^2 p_n(t) = 0, \quad t \in (0,T).$$

Its general solution has the form

$$p_n(t) = a_n \cos \lambda_n t + b_n \sin \lambda_n t, \tag{23}$$

where a_n and b_n are arbitrary constants. To determine a_n and b_n, substitute functions (20) in the initial-value conditions given by (9):

$$p_n(0) = \int\limits_0^l \varphi(r) r^k R_n(r)\, dr = \varphi_n,$$

$$p_n(T) = a_n \cos \lambda_n T + b_n \sin \lambda_n T = \psi_n. \tag{24}$$

From (23) and (24), we conclude that

$$a_n = \varphi_n, \quad b_n = \frac{\psi_n - \varphi_n \cos \lambda_n T}{\sin \lambda_n T}.$$

Substituting the found values of aa_n and b_n in (23), we obtain the final form of the functions

$$p_n(t) = \varphi_n \cos \lambda_n t - \frac{\psi_n - \varphi_n \cos \lambda_n T}{\sin \lambda_n T} \sin \lambda_n t \tag{25}$$

defined for

$$\sin \lambda_n T = \sin \mu_n \alpha \neq 0 \quad \text{and} \quad \alpha = T/l, \quad n \in \mathbb{N}, \tag{26}$$

where μ_n are roots of Eq. (17). Then, condition (26) is satisfied for all $k \in (-1,1)$ and $k \neq 0$ provided that $\alpha = T/l$ is an irrational number (see [10]).

In the same way, we find

$$p_0(t) = \varphi_0 + \psi_0 t, \tag{27}$$

$$p_0(0) = l^{-\frac{k+1}{2}} \sqrt{k+1} \int\limits_0^l \varphi(r) r^k dr = \varphi_0,$$

$$p_0(T) = l^{-\frac{k+1}{2}}\sqrt{k+1}\int\limits_0^l \psi(r)r^k dr = \psi_0. \tag{28}$$

Let $\varphi(r) = \psi(r) \equiv 0$ and condition (26) be satisfied. Then, (24) and (28) imply that $\varphi_n = \psi_n \equiv 0$ and (25) and (27) imply that $p_n(t) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Then, (20) implies that $\int_0^l p(r,t)r^k R_n(r)\,dr = 0$ for each $t \in [0,T]$. By virtue of the completeness of system (18) in the space $L_2[0,l]$ with weight r^k, this implies that $p(r,t) = 0$, a.e., on the segment $r \in [0,l]$ provided that $t \in [0,T]$. Since $p(r,t) \in C(\overline{D})$ due to (7), it follows that $p(r,t) \equiv 0$ in $\overline{D}$. In the same way (see [10]), it is easy to show that, $\varphi(r) = \psi(r) \equiv 0$ and condition (26) is broken, i.e., $\alpha = T/l$ is a rational number, then the homogeneous problem ($\varphi(r) = \psi(r) \equiv 0$) has nontrivial solutions. Thus, the following assertion is proved.

Theorem 1. *Let there exist a solution of problem (7)–(10) and (13) and the fraction $T/l = \alpha$ of the sides of the rectangular domain D be an irrational number. Then, this solution is unique.*

Let condition (26) be satisfied for all $n \in \mathbb{N}$. Basing on the found particular solutions, represent a solution of problem (7)–(10) and (13) by the Fourier–Bessel series

$$p(r,t) = p_0(t)R_0(r) + \sum_{n=1}^{\infty} p_n(t)R_n(r), \tag{29}$$

where the functions $R_n(r)$, $n \in \mathbb{N} \cup \{0\}$, are defined by relation (18), the functions $p_n(t)$ are defined by relation (25), and the function $p_0(t)$ is defined by relation (27).

3. SOLUTION EXISTENCE

Apart from series (29), consider the series

$$p_t(r,t) = \psi_0 R_0(r) + \sum_{n=1}^{\infty} p_n'(t)R_n(r), \quad p_r(r,t) = \sum_{n=1}^{\infty} p_n(t)R_n'(r), \tag{30}$$

$$p_{tt}(r,t) = \sum_{n=1}^{\infty} p_n''(t)R_n(r), \quad p_{rr}(r,t) = \sum_{n=1}^{\infty} p_n(t)R_n''(r). \tag{31}$$

It is known from [10] that, if $\alpha = T/l > 0$ is an irrational number of power $m \geq 2$, then there exists $C_0 > 0$ such that the estimates

$$|\sin \mu_n \alpha| \geq \frac{C_0}{n} \quad (m = 2), \quad |\sin \mu_n \alpha| \geq \frac{C_0}{n^{1+\varepsilon}} \quad (m > 2, \quad \varepsilon > 0) \tag{32}$$

are satisfied.

Lemma 1. *If $\alpha = T/l > 0$ is an irrational number of power $m \geq 2$, then the estimates*

$$\begin{gathered} |p_n(t)| \leq C_1 n \left(|\varphi_n| + |\psi_n|\right), \\ |p_n'(t)| \leq C_2 n^2 \left(|\varphi_n| + |\psi_n|\right), \quad m = 2, \\ |p_n''(t)| \leq C_3 n^3 \left(|\varphi_n| + |\psi_n|\right), \end{gathered}$$

$$\begin{gathered} |p_n(t)| \leq C_1 n^{1+\varepsilon} \left(|\varphi_n| + |\psi_n|\right), \\ |p_n'(t)| \leq C_2 n^{2+\varepsilon} \left(|\varphi_n| + |\psi_n|\right), \quad m > 2, \\ |p_n''(t)| \leq C_3 n^{3+\varepsilon} \left(|\varphi_n| + |\psi_n|\right), \end{gathered}$$

where C_i are positive constants (here and after), are satisfied for sufficiently large n and each $t \in [0,T]$.

The proof follows from relation (25), estimate (32), and the asymptotic relation

$$\mu_n = \lambda_n l = \pi n + \frac{\pi}{4}k + O\left(n^{-1}\right)$$

for roots of Eq. (17) and large values n (see [15, p. 17]).

Lemma 2. *If n is sufficiently large and $r \in [0, l]$, then*

$$|R_n(r)| \leq C_4, \quad |R_n'(r)| \leq C_5 n, \quad |R_n''(r)| \leq C_6 n^2. \tag{33}$$

Proof. It is known that, for large ξ,

$$J_\nu(\xi) = O\left(\xi^{-1/2}\right). \tag{34}$$

From (19) we obtain

$$||\widetilde{R}_n|| = \frac{l}{\sqrt{2}} |J_{\frac{k+1}{2}}(\mu_n)|. \tag{35}$$

Then, it follows from Eqs. (34) and (35) that

$$||\widetilde{R}_n|| = O\left(n^{-1/2}\right) \quad \text{as} \quad n \to +\infty. \tag{36}$$

With regard to Eq. (35), formula (18) becomes

$$R_n(r) = \frac{\sqrt{2}\, r^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n r)}{l\, |J_{\frac{k+1}{2}}(\mu_n)|}. \tag{37}$$

Then, the first estimate in Eq. (33) follows from relations (34), (36), and (37).

Now we calculate

$$\widetilde{R}_n'(r) = -\lambda_n r^{\frac{1-k}{2}} J_{\frac{k+1}{2}}(\lambda_n r). \tag{38}$$

Then, the second estimate in Eq. (33) follows from Eqs. (34), (36), and (38).

Equation (14) implies the relation

$$\widetilde{R}_n''(r) = -\frac{k}{r} \widetilde{R}_n'(r) - \lambda_n^2 \widetilde{R}_n(r).$$

By virtue of the first two estimates, this implies the third estimate in Eq. (33). □

Due to Lemmas 1 and 2, for each $(r, t) \in \overline{D}$, series (29) is estimated by the series

$$C_7 \sum_{n=1}^{+\infty} n^{1+\varepsilon} \left(|\varphi_n| + |\psi_n|\right) \tag{39}$$

and series (30) and (31) are estimated by the series

$$C_8 \sum_{n=1}^{+\infty} n^{2+\varepsilon} \left(|\varphi_n| + |\psi_n|\right), \quad C_9 \sum_{n=1}^{+\infty} n^{3+\varepsilon} \left(|\varphi_n| + |\psi_n|\right), \tag{40}$$

respectively.

Investigate the convergence of series (39) and (40).

Lemma 3. *If $\varphi(r), \psi(r) \in C^{2+\delta}[0, l]$, $\varepsilon < \delta < 1$, there exist derivatives $\varphi'''(r)$ and $\psi'''(r)$ such that their variations on $[0, l]$ are finite and*

$$\varphi'(0) = \varphi''(0) = \psi'(0) = \psi''(0) = \varphi'(l) = \psi'(l) = 0,$$

then the estimates

$$|\varphi_n| \leq \frac{C_{10}}{n^{4+\delta}}, \quad |\psi_n| \leq \frac{C_{11}}{n^{4+\delta}}$$

are satisfied.

The proof is similar to the proof of the corresponding lemma in [9].

Due to Lemma 3, series (39) and (40) are majorized by the converging number series $C_{12} \sum_{n=1}^{\infty} n^{-1-(\delta-\varepsilon)}$. Due to the Weierstrass criterion, series (29)–(31) uniformly converge in the domain $\overline{D}$. Thus, the function $u(x,t)$ defined by series (29) satisfies all conditions of problem (7)–(10) and (13) and belongs to the class $C^2(\overline{D})$. Thus, the following assertion is proved.

Theorem 2. *If $\alpha = T/l$ is a positive irrational algebraic number of power $m \geq 2$ and functions $\varphi(r)$ and $\psi(r)$ satisfy assumptions of Lemma 3, then there exists a unique solution $p(r,t)$ of problem (7)–(10) and (13), defined by series (29). This solution belongs to $C^2(\overline{D})$.*

Now, let us prove that the function $p(r,t)$ defined by series (29) is a solution of problem (7)–(12).

Theorem 3. *If $\alpha = T/l$ is a positive irrational algebraic number of power $m \geq 2$ and functions $\varphi(r)$ and $\psi(r)$ satisfy assumptions of Lemma 3 and conditions (12), then there exists a unique solution of problem (7)–(12), defined by series (29).*

Proof. Let $p(r,t)$ be a solution of problem (7)–(10) and (13) and functions $\varphi(r)$ and $\psi(r)$ satisfy the assumptions of the theorem. Then, Eq. (8) is satisfied everywhere in the domain D. Multiply Eq. (8) by r^k and, for a fixed $t \in (0,T)$, integrate the obtained product with respect to the variable r from ε to $l-\varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. We obtain

$$\int\limits_{\varepsilon}^{l-\varepsilon} p_{tt}(r,t) r^k\, dr - \left(r^k \frac{\partial p}{\partial r} \right)\Bigg|_{\varepsilon}^{l-\varepsilon} = 0.$$

In the last relation, pass to the limit as $\varepsilon \to 0$. Taking into account conditions (10) and (13), we conclude that $\int_0^l p_{tt}(r,t) r^k\, dr = 0$. Now, integrating this relation with respect to the variable t twice, we arrive at the relation

$$\int\limits_0^l p(r,t) r^k\, dr = K_1 t + K_2, \quad A,\, B = \text{const}. \tag{41}$$

Assigning $t = 0$ in (41) and taking into account conditions (9) and (12), we find that $\int_0^l \varphi(r) r^k\, dr = K_2 = A$. In (41), assign $t = T$ and take into account conditions (9) and (12) and the obtained value $K_2 = A$. We obtain $\int_0^l p(r,T) r^k\, dr = \int_0^l \psi(r) r^k\, dr = K_1 T + A = A$ and, therefore, $K_1 = 0$.

Substituting the found values of the constants $K_1 = 0$ and $K_2 = A$ in relation (41), we arrive at the integral condition given by (11). The inverse reasoning is provided in Section 1. Thus, it is proved that conditions (11) and (13) are equivalent to each other provided that the coordination conditions given by (12) are satisfied. Hence, problem (7)–(12) and problem (7)–(10) and (13) are equivalent to each other as well, which completes the proof of the theorem. □

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