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THE CAUCHY PROBLEM FOR THE DEGENERATE BEAM VIBRATION EQUATION

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We study the Cauchy problem for a degenerate differential equation of beam vibrations. Using the generalized Erdélyi–Kober operator, possessing the property of a transmutation operator, we reduce the original problem to a problem for a nondegenerate equation. An explicit formula for the solution is constructed. Bibliography: 18 *titles.*

Dedicated to the outstanding mathematician
Makhmud Salakhitdinovich Salakhitdinov
on the occasion of his 90th anniversary

1 Introduction

In this paper, we consider the degenerate fourth order equation

$$u_{tt} + t^p u_{xxxx} + \lambda^2 t^p u = 0, \tag{1.1}$$

where $\lambda, p \in R$, and $p \geqslant 0$. In the case $p = 0$, Equation (1.1) occurs in problems about vibrations of rods and beams, as well as the stability theory for rotation of shafts and vibration of ships.

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In the domain $\Omega = \{(x,t): x \in R, t \in R, t > 0\}$, we consider the Cauchy problem: Find a solution $u(x,t) \in M$ to Equation (1.1) satisfying the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in R, \tag{1.2}$$

where $\varphi(x)$ and $\psi(x)$ are given smooth functions, M is the class of functions that are continuously differentiable in t and twice continuously differentiable in x in the closure $\overline{\Omega} = \{(x,t): x \in R, t \in R, t \geqslant 0\}$ of Ω and also twice continuously differentiable in t and four times in x in Ω.

In the problem (1.1), (1.2), we make the change of variables $y = [2/(p+2)]t^{(p+2)/2}$. Then Equation (1.1) with the initial conditions (1.2) take the form

$$A_\lambda^\beta(u) \equiv u_{yy} + \frac{2\beta}{y}u_y + \frac{\partial^4 u}{\partial x^4} + \lambda^2 u = 0, \tag{1.3}$$

$$u(x,0) = \varphi(x), \quad \lim_{y \to +0} y^{2\beta} u_y(x,y) = \psi_0(x), \quad x \in R, \tag{1.4}$$

where $\psi_0(x) = (1-2\beta)^{2\beta}\psi(x)$, $2\beta = p/(p+2)$, and $0 < 2\beta < 1$ at $p > 0$.

We first construct a solution to Equation (1.3) satisfying the semi-homogeneous initial conditions

$$u(x,0) = \varphi(x), \quad u_y(x,0) = 0, \quad x \in R. \tag{1.5}$$

To construct a solution to the Cauchy problem (1.3), (1.5), we use the generalized Erdélyi–Kober operator of fractional order [1]. We recall some properties of this operator.

2 Generalized Erdélyi–Kober Operator

Various modifications and generalizations of the Erdélyi–Kober operators were considered, for example, in [1]–[3]. In particular, the following generalized Erdélyi–Kober operators with Bessel functions in kernels were introduced in [2]:

$$J_\lambda(\eta,\,\alpha)f(x) = 2^\alpha \lambda^{1-\alpha} x^{-2\alpha-2\eta} \int\limits_0^x t^{2\eta+1}(x^2-t^2)^{(\alpha-1)/2} J_{\alpha-1}(\lambda\sqrt{x^2-t^2}) f(t)dt, \tag{2.1}$$

where $\alpha, \eta, \lambda \in R$, $\alpha > 0$, $\eta \geqslant -(1/2)$, and $J_\nu(z)$ is the Bessel function of the first kind of order ν. The operator (2.1) coincides with the usual Erdélyi–Kober operator [1] as $\lambda \to 0$

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int\limits_0^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} f(t)dt, \tag{2.2}$$

where $\Gamma(\alpha)$ is the Euler Gamma function [4].

The inverse of the operator (2.1) with $0 < \alpha < 1$ has the form [1]

$$J_\lambda^{-1}(\eta,\alpha)f(x) = \frac{x^{-2\eta-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x \frac{\overline{I}_{-\alpha}\left(\lambda\sqrt{x^2-s^2}\right)}{(x^2-s^2)^\alpha} s^{2(\eta+\alpha)+1} f(s)ds, \tag{2.3}$$

where $\overline{I}_\nu(z) = \overline{J}_\nu(iz) = \Gamma(\nu+1)(z/2)^{-\nu} I_\nu(z)$, $I_\nu(z)$ is the Bessel function of imaginary variable. Since $\overline{I}_\nu(0) = 1$, for $\lambda = 0$ from (2.3) we obtain the inverse of the operator (2.2)

$$I_{\eta,\alpha}^{-1} g(x) = \frac{x^{-2\eta-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x (x^2-s^2)^{-\alpha} s^{2(\eta+\alpha)+1} g(s)ds \tag{2.4}$$

In a more general situation, the notion of transmutation is introduced. For given two operators (A, B) we say that a nonzero operator T is a *transmutation operator* if it possesses the following *transmutational* (or intertwining) property

$$T\,A = B\,T. \tag{2.5}$$

Usually, T is an integral operator.

Some properties of the operator (2.1) were generalized in [5, 6], where the following theorem was also proved.

Let $l \in N \cup \{0\}$, $[B_\eta^x]^0 = E$, where E is the identity operator, and let $[B_\eta^x]^l = [B_\eta^x]^{l-1}[B_\eta^x]$ be the lth power of the Bessel operator

$$B_\eta^x = x^{-2\eta-1}\frac{d}{dx}x^{2\eta+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\eta+1}{x}\frac{d}{dx}.$$

Theorem 2.1. *Assume that* $\alpha > 0$, $\eta \geqslant -(1/2)$, $f(x) \in C^{2l}(0, b)$, $b > 0$, *the functions* $x^{2\eta+1}[B_\eta^x]^k f(x)$ *are integrable at zero and*

$$\lim_{x\to 0} x^{2\eta+1}\frac{d}{dx}[B_\eta^x]^k f(x) = 0, \quad k = \overline{0, l-1}.$$

Then

$$[B_{\eta+\alpha}^x + \lambda^2]^l J_\lambda(\eta,\alpha) f(x) = J_\lambda(\eta,\alpha)[B_\eta^x]^l f(x).$$

In particular, for $\lambda = 0$

$$[B_{\eta+\alpha}^x]^l I_{\eta,\alpha} f(x) = I_{\eta,\alpha}[B_\eta^x]^l f(x).$$

Owing to Theorem 2.1, it is possible to treat the operator (2.1) as a transmutation operator; more exactly, a *shift parameter operator*. This fact is useful for solving the Cauchy problem (1.3), (1.5). We note that the Erdélyi–Kober operator was used [6]–[10] to solve the Cauchy problem for partial differential equations of hyperbolic and parabolic type.

3 Application of the Erdélyi–Kober Operator

We assume that a solution to the problem (1.3), (1.5) exists. We look for it in the form of the generalized Erdélyi–Kober operator (2.1):

$$u(x, y) = J_\lambda^{(y)}(-(1/2),\, \beta)V(x, y) \tag{3.1}$$

where $V(x, y)$ is an unknown twice continuously differentiable function.

Substituting (3.1) into (1.3) and (1.5), using Theorem 2.1 with $l = 1$, $\alpha = \beta$, $\eta = -1/2$ and the inverse operator (2.3) with for $\alpha = \beta$, $\eta = -1/2$, we obtain the following problem: Find a solution $V(x, y)$ to the equation

$$\frac{\partial^2 V}{\partial y^2} + \frac{\partial^4 V}{\partial x^4} = 0 \tag{3.2}$$

satisfying the initial conditions

$$V(x, 0) = k_0\varphi(x), \quad V_y(x, 0) = 0, \quad x \in R, \tag{3.3}$$

where $k_0 = \Gamma(\beta + 1/2)/\sqrt{\pi}$.

Theorem 3.1 ([11]). *A function $V(x, y) \in M$ is a solution to the problem (3.2), (3.3) if and only if the function*

$$U(x, y) = V(x, y) + i \int_0^y V_{xx}(x, \tau) d\tau \tag{3.4}$$

is a solution to the equation

$$U_y - iU_{xx} = 0, \quad (x, y) \in \Omega, \tag{3.5}$$

satisfying the initial condition

$$U(x, 0) = k_0 \varphi(x), \quad x \in R, \tag{3.6}$$

where i is the imaginary unit.

Corollary 3.1. *Let $U(x, y) \in M$ be a solution to the problem (3.5), (3.6), where $\varphi(x)$ is a real-valued function. Then $V(x, y) = \operatorname{Re} U(x, y)$ is a solution to the problem (3.2), (3.3), where $\operatorname{Re} U$ denotes the real part of $U(x, y)$.*

To solve the problem (3.2), (3.3), we apply Corollary 3.1. Then Equation (3.5) becomes the one-dimensional Schrödinger equation

$$\frac{\partial U}{\partial y} - i \frac{\partial^2 U}{\partial x^2} = 0.$$

The solution to the problem (3.5), (3.6) in this case takes the form [12]

$$U(x, y) = \int_{-\infty}^{+\infty} \varphi(\xi) G(x, \xi, y) d\xi,$$

where

$$G(x, \xi, y) = \frac{1}{2\sqrt{\pi y}} \exp\Big[i\Big(\frac{(x - \xi)^2}{4y} - \frac{\pi}{4}\Big)\Big].$$

By Corollary 3.1, the solution to the problem (3.2), (3.3) takes the form

$$V(x, y) = k_0 \int_{-\infty}^{+\infty} \varphi(\xi) G_1(x, y, \xi) d\xi, \tag{3.7}$$

where

$$G_1(x, y, \xi) = \frac{1}{2\sqrt{\pi y}} \cos\Big[\frac{(\xi - x)^2}{4y} - \frac{\pi}{4}\Big].$$

Substituting (3.7) into (3.1) and changing the integration order, we get

$$u(x, y) = \frac{k_0 y^{1-2\beta}}{\sqrt{2\pi}\Gamma(\beta)} \int_{-\infty}^{+\infty} \varphi(\xi) G_2(x, y, \xi) d\xi, \tag{3.8}$$

where

$$G_2(x,y,\xi) = \int_0^y (y^2-\eta^2)^{\beta-1}\overline{J}(\lambda\sqrt{y^2-\eta^2})G_1(x,\xi,\eta)d\eta. \tag{3.9}$$

Substituting $G_1(x,\xi,\eta)$ into (3.9), replacing the integration variable, using the series expansion of the Bessel–Clifford (or normalized Bessel) function, and applying formula (2.5.8.3) in [13], we find

$$u(x,y) = \frac{k_0}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi(x+2\xi\sqrt{y})G_3(y,\xi;\beta,\lambda)d\xi, \tag{3.10}$$

where

$$\begin{aligned} G_3(y,\xi;\beta,\lambda) = {} & \frac{\Gamma(1/4)}{\Gamma(\beta+(1/4))}K_1\Big(\beta+\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4},\quad -\frac{1}{4}\lambda^2y^2\Big) \\ & + \frac{\Gamma(-1/4)}{\Gamma(\beta-(1/4))}\xi^2K_1\Big(\beta-\frac{1}{4};\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4},\,-\frac{1}{4}\lambda^2y^2\Big), \end{aligned} \tag{3.11}$$

where

$$K_1(a,b,c;\,x,y) = \sum_{m=0}^{\infty}\frac{y^m}{(a)_m m!}\,{}_1F_2(1-a-m;\,b,\,c;\,x)$$

and ${}_1F_2(a;\,b,\,c;\,z)$ is the generalized hypergeometric function [4].

We note that for $\beta=0$ and $\lambda\neq 0$ Equation (1.3) takes the form

$$A^0_\lambda(u) \equiv \frac{\partial^2u}{\partial y^2}+\frac{\partial^4u}{\partial x^4}+\lambda^2u=0 \tag{3.12}$$

and the solution to the problem (1.3), (1.5) has the form (3.10) with $k_0=\Gamma(1/2)/\sqrt{\pi}=1$,

$$G_3(y,\xi;0,\lambda) = K_1\Big(\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4},-\frac{1}{4}\lambda^2y^2\Big)+\xi^2K_1\Big(-\frac{1}{4};\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4},\,-\frac{1}{4}\lambda^2y^2\Big).$$

Assume that $\beta\neq 0$ and $\lambda=0$. Then Equation (1.3) takes the form

$$A^\beta_0(u) \equiv u_{yy}+\frac{2\beta}{y}u_y+\frac{\partial^4u}{\partial x^4}=0 \tag{3.13}$$

and the solution to the problem (1.3), (1.5) has the form (3.10) with $k_0=\Gamma(\beta+1/2)/\sqrt{\pi}$,

$$\begin{aligned} G_3(y,\xi;\beta,0) = {} & \frac{\Gamma(1/4)}{\Gamma(\beta+(1/4))}\,{}_1F_2\Big(\frac{3}{4}-\beta;\,\frac{3}{4},\,\frac{1}{2};-\frac{\xi^4}{4}\Big) \\ & + \frac{\Gamma(-1/4)}{\Gamma(\beta-(1/4))}\xi^2\,{}_1F_2\Big(\frac{5}{4}-\beta;\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4}\Big). \end{aligned}$$

Assume that $\beta=0$ and $\lambda=0$, Then Equation (1.3) takes the form

$$A^0_0(u) \equiv \frac{\partial^2u}{\partial y^2}+\frac{\partial^4u}{\partial x^4}=0 \tag{3.14}$$

and the solution to the problem (1.3), (1.5) has the form (3.10) with $k_0 = \Gamma(1/2)/\sqrt{\pi} = 1$,

$$G_3(y, \xi; 0, 0) = {}_1F_2\Big(\frac{3}{4}; \frac{3}{4}, \frac{1}{2}; -\frac{\xi^4}{4}\Big) + \xi^2 {}_1F_2\Big(\frac{5}{4}; \frac{5}{4}, \frac{3}{2}; -\frac{\xi^4}{4}\Big). \tag{3.15}$$

By the formula

$${}_0F_1(b; -z) = \Gamma(b) z^{\frac{1-b}{2}} J_{b-1}(2\sqrt{z}),$$

where

$$J_{-1/2}\big(z\big) = \sqrt{\frac{2}{\pi z}} \cos z, \quad J_{1/2}\big(z\big) = \sqrt{\frac{2}{\pi z}} \sin z,$$

we have

$${}_1F_2\Big(\frac{3}{4}; \frac{3}{4}, \frac{1}{2}; -\frac{\xi^4}{4}\Big) = {}_0F_1\Big(\frac{1}{2}; -\frac{\xi^4}{4}\Big) = \Gamma\Big(\frac{1}{2}\Big)\frac{\xi}{\sqrt{2}} J_{-1/2}(\xi^2) = \cos(\xi^2),$$

$${}_1F_2\Big(\frac{5}{4}; \frac{5}{4}, \frac{3}{2}; -\frac{\xi^4}{4}\Big) = {}_0F_1\Big(\frac{3}{2}; -\frac{\xi^4}{4}\Big) = \Gamma\Big(\frac{3}{2}\Big)\frac{\sqrt{2}}{\xi} J_{1/2}(\xi^2) = \frac{\sin(\xi^2)}{\xi^2}.$$

Substituting the last equalities into (3.15), we find

$$G_3(y, \xi; 0, 0) = \cos(\xi^2) + \sin(\xi^2) = \sqrt{2}\cos\Big(\xi^2 - \frac{\pi}{4}\Big).$$

The last expression coincides with the results of [14] obtained by other methods.

Now, we study the problem of finding a solution to Equation (1.3) satisfying the conditions

$$u(x, 0) = 0, \quad \lim_{y \to +0} y^{2\beta} u_y(x, y) = \psi_0(x), \quad x \in R. \tag{3.16}$$

We apply the following property of this equation.

Proposition 3.1. *If $u(x, y; 1-\beta)$ is a solution to the equation $A_\lambda^{1-\beta}(u) = 0$ satisfying the conditions* (1.5), *then the function $w(x, y; \beta) = y^{1-2\beta} u(x, y; 1-\beta)$ is a solution to the equation $A_\lambda^{\beta}(w) = 0$ satisfying the conditions*

$$w(x, 0) = 0, \quad \lim_{y \to +0} y^{2\beta} w_y(x, y) = (1-2\beta)\varphi(x), \quad x \in R.$$

Proposition 3.1 is proved by a direct computation. Taking into account Proposition 3.1 and replacing $(1-2\beta)\varphi(x)$ by $\psi_0(x)$ on the basis of the solution to the equation $A_\lambda^{\beta}(u) = 0$ satisfying (1.5), we can construct a solution to the equation $A_\lambda^{\beta}(w) = 0$ satisfying the conditions (3.16)

$$w(x, y) = k_1 y^{1-2\beta} \int\limits_{-\infty}^{+\infty} \psi_0(x + 2\xi\sqrt{y}) G_3(y, \xi; 1-\beta, \lambda) d\xi, \tag{3.17}$$

where $k_1 = \Gamma(1/2 - \beta)/(2\sqrt{\pi})$,

$$G_3(y, \xi; 1-\beta, \lambda) = \frac{\Gamma(1/4)}{\Gamma((5/4-\beta))} K_1\Big(\frac{5}{4} - \beta; \frac{3}{4}, \frac{1}{2}; -\frac{\xi^4}{4}, \quad -\frac{1}{4}\lambda^2 y^2\Big)$$
$$+ \frac{\Gamma(-1/4)}{\Gamma((3/4)-\beta)} \xi^2 K_1\Big(\frac{3}{4} - \beta; \frac{5}{4}, \frac{3}{2}; -\frac{\xi^4}{4}, -\frac{1}{4}\lambda^2 y^2\Big).$$

Similarly, for $\beta = 0$ and $\lambda \neq 0$ Equation (1.3) takes the form (3.12) and the solution to the problem (1.3), (3.16) has the form

$$w(x,y) = \frac{y}{2}\int_{-\infty}^{+\infty} \psi_0(x + 2\xi\sqrt{y})G_3(y,\xi;1,\,\lambda)d\xi,$$

where

$$G_3(y,\xi;1,\lambda) = 4K_1\Big(\frac{5}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4},\quad -\frac{1}{4}\lambda^2y^2\Big)$$
$$- 4\xi^2K_1\Big(\frac{3}{4};\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4},\,-\frac{1}{4}\lambda^2y^2\Big).$$

For $\beta \neq 0$ and $\lambda = 0$ Equation (1.3) takes the form (3.13) and the solution to the problem (1.3), (3.16) has the form (3.17) with $k_1 = \Gamma(1/2-\beta)/(2\sqrt{\pi})$,

$$G_3(y,\xi;1-\beta,0) = \frac{\Gamma(1/4)}{\Gamma((5/4)-\beta)}\quad {}_1F_2\Big(\beta-\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4}\Big)$$
$$+\frac{\Gamma(-1/4)}{\Gamma((3/4)-\beta)}\xi^2{}_1F_2\Big(\frac{1}{4}+\beta;\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4}\Big).$$

For $\beta = 0$ and $\lambda = 0$ Equation (1.3) takes the form (3.13) and the solution to the problem (1.3), (3.16) has the form (3.17) with $k_1 = \Gamma(1/2)/2\sqrt{\pi} = 1/2$,

$$G_3(y,\xi;1,0) = 4\,{}_1F_2\Big(-\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4}\Big) - 4\xi^2{}_1F_2\Big(\frac{1}{4};\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4}\Big). \tag{3.18}$$

By the formulas

$${}_1F_2\Big(-\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-x^2\Big) = \cos(2x) + 2\sqrt{\pi x}S(2x),$$
$${}_1F_2\Big(\frac{1}{4};\,\frac{5}{4},\,\frac{3}{2};\,-x^2\Big) = \sqrt{\frac{\pi}{x}}C(2x) - \frac{\sin(2x)}{2x},$$

we have

$${}_1F_2\Big(-\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4}\Big) = \cos(\xi^2) + 2\xi\sqrt{\frac{\pi}{2}}S(\xi^2),$$

$${}_1F_2\Big(\frac{1}{4};\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4}\Big) = \sqrt{2\pi}\frac{1}{\xi}C(\xi^2) - \frac{\sin\left(\xi^2\right)}{\xi^2}$$

where

$$S(z) = \int_0^z \frac{\sin t}{\sqrt{t}}dt, \quad C(z) = \int_0^z \frac{\cos t}{\sqrt{t}}dt$$

are the Fresnel sine and cosine integrals.

Substituting the last equalities into (3.18), we find

$$G_3(y,\xi;1,0) = 4\,{}_1F_2\Big(-\frac{1}{4};\,\frac{3}{4},\,\frac{1}{2};\,-\frac{\xi^4}{4}\Big) - 4\xi^2{}_1F_2\Big(\frac{1}{4};\,\frac{5}{4},\,\frac{3}{2};\,-\frac{\xi^4}{4}\Big)$$

$$= 4\Big[\cos(\xi^2) + 2\xi\sqrt{\frac{\pi}{2}}S(\xi^2)\Big] - 4\xi^2\Big[\sqrt{2\pi}\frac{1}{\xi}C(\xi^2) - \frac{\sin(\xi^2)}{\xi^2}\Big]$$

$$= 4[\cos(\xi^2) + \sin(\xi^2)] + 4\sqrt{2\pi}\xi[S(\xi^2) - C(\xi^2)].$$

The latter coincides with the results of [14] obtained by a different method.

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

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