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SYSTEM ANALYSIS AND OPERATIONS RESEARCH

Constructing Trends of Time Series Segments

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Abstract—The sub-band analysis enabling one to construct a sequence whose Fourier transform is the best approximation of a segment of Fourier transform of the original series within a given frequency interval was shown to be an efficient tool to specify the trends of segments of the nonstationary time series. Relations were established defining the matrix operator to sort out such components. A procedure for adaptive construction of the operators for trend extraction was proposed, and conditions were determined under which a wide class of sequence segments are their eigenfunctions (fixed points) corresponding to the unit eigenvalues.

Keywords: nonstationary time series, trends, sub-band analysis.

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1. INTRODUCTION AND FORMULATION OF THE PROBLEM

Let the vector $\vec{x} = (x_1, \ldots, x_N)'$, where the stroke stands for transposition, consist of real components with values fixed at observing some parameter of the process under consideration (segment of time series). The main objective of registering such vectors (empirical data) is constriction of the process behavior models. The representation

$$x_k = f_k + \varepsilon_k, \quad k = 1, \ldots, N, \tag{1.1}$$

is often used for this purpose [1–6]. There, f_k is an unknown trend reflecting the process tendency, and ε_k are the oscillations about the trend due to multiple uncontrollable factors.

Within the framework of such model, the trend distinguished for greater consistency of acts than in the deviations from it usually is of main interest.

We also note that the problem of eliminating the trend [2, 4] as the main source of nonstationarity is posed rather often in order to examine (model) the oscillations about the trend.

It is possible to note two basic current approaches to the trend construction. One of them relies on the a priori selection as the coordination model of their explicit functional dependencies on the number of reading (argument) whose parameters are then adjusted to the available empirical data (learning sample) [1, 3, 5].

Another approach is based in the idea of smoothing [1, 6] allowing one to suppress the second component in the right side of (1.1) which naturally can be called filtering. At that, some ideas about the nature of coordination of the trend values are used as well, which allows one to divide the data into the trend and oscillations about it.

We note that both approaches have certain advantages and disadvantages. The main problem of the first approach lies in the need for substantiating the adequacy of the selected functional dependence. Therefore, it is difficult to call in question the opinion formulated in [6] that the use of explicit mathematical relationships is equivalent to forcing laws to the nature.

One or another assumption about the nature of behavior of the trends and deviations from them is also often used at constructing the smoothing procedures. This is exemplified by the so-called

Spenser formulas [1] based on the assumption that the trend is a polynomial (at least locally) of one or another order.

The least rigid assumptions about the nature of trend behavior (mostly the requirement of local monotonicity) at construction of the smoothing procedures are used in [6], which shows it to the best advantage.

It also deserves noting the sufficiently well known method of exponential smoothing [7] distinguished for using the principle of reduced influence on the resulting trend estimate of the time-remote components of the processed vector of empirical data.

Of certain interest are the methods for determination of some trend parameters with the use, for example, of frequency representations [8].

The optimal methods of sub-band analysis of the time series developed in [9] are used in the present paper, which allowed one to introduce a measure of coordination of behavior of the components of the trend vector $\vec{f} = (f_1, \ldots, f_N)'$ present in all analyzed components of the vector of empirical data which is adequate to the problem at hand. At that, the a priori assumptions about the trend properties, except for certain degree of smoothness, are not used. This approach requires adaptive processing of the empirical data enabling one to acquire the desired information immediately from their particular segment.

2. GENERAL FORM OF THE SMOOTHING OPERATOR

It is only natural to represent the smoothing procedure as

$$\widehat{\vec{f}} = W(\vec{x}), \tag{2.1}$$

where W is a smoothing operator continuous over the set of real multidimensional vectors R^N.

It is easy to see that the smoothing operator must be linear in virtue of additivity of the model of right side of (1.1), that is,

$$\widehat{\vec{f}} = W(\vec{f}) + W(\vec{\varepsilon}), \tag{2.2}$$

where $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)'$.

It is namely with this condition satisfied, that separate actions on the trend and oscillations about are possible with the aim of damping the latter in order to satisfy the inequality for the Euclidean norms of the vectors

$$b = ||W(\vec{e})|| \,/\, ||W(\vec{f})|| < ||\vec{\varepsilon}|| \,/\, ||\vec{f}||. \tag{2.3}$$

The fractions in the right side of (2.3) represent the noise/signal ratio, and the inequality itself is the requirement to reduce this ratio after data processing.

It is also clear that in the ideal case—such as lack of deviations—the identity

$$W(\vec{f}) = \vec{f} \tag{2.4}$$

must be satisfied, that is, the operator used as a fixed point must have the desired trend vector which is, generally speaking, a priori is unknown.

As the basic characteristic of the time series and, first of all, trends, the present paper uses the Fourier transforms denoted below for the vectors $\vec{z} = (z_1, \ldots, z_N)'$ by the corresponding capital letters

$$Z(\omega) = \sum_{k=1}^{N} z_k \exp(-j\omega(k-1)), \quad j = (-1)^{1/2}. \tag{2.5}$$

In the general case, the inverse representation of the values of the time series

$$z_i = \int_{-\pi}^{\pi} Z(\omega)\exp(j\omega(i-1))d\omega/2\pi$$

is valid.

It is assumed within the framework of the present paper that the trends are "narrow-band" in the sense of satisfying the condition

$$f_i \approx \int_{\omega\in\Omega_R} F(\omega)\exp(j\omega(i-1))d\omega/2\pi,$$

where Ω_R in the general case denotes the union

$$\Omega_R = \bigcup_{r\in R}\Omega_r \tag{2.6}$$

of the disjoint frequency intervals symmetrical about the origin

$$\Omega_r = [-\Omega_{2r}, -\Omega_{1r}) \bigcup [\Omega_{1r}, \Omega_{2r}),$$

and satisfied are the conditions $0 \le \min\Omega_{1r}$; $\max\Omega_{2r} \le \pi$; $\forall r\ \ \Omega_{1r} \le \Omega_{2r}$, with R denoting the set of indices.

The term "narrow-band" implies that satisfied is the inequality

$$\mu_R = \sum_{r\in R}(\Omega_{2r} - \Omega_{1r})/\pi = S_R/\pi \ll 1, \tag{2.7}$$

and the smaller this relation, the more coordinated can be regarded—in distinction to the white noise—behavior of the readings of time series. Therefore, it seems natural to use the index

$$\Phi_R = 1 - \mu_R \tag{2.8}$$

as the measure of coordination.

A part of the time series energy hitting the frequency interval (2.6)

$$P_R(\vec{z}) = \int_{\omega\in\Omega_R} |Z(\omega)|^2 d\omega/2\pi \tag{2.9}$$

is important for the following constructions.

Importantly, the representation [9]

$$P_R(\vec{z}) = \vec{z}\,' A_R \vec{z} \tag{2.10}$$

is valid for this characteristic immediately in the area of the originals, where $A_R = \{a_{ik}^R\}$, $i, k = 1, \ldots, N$ is a symmetrical nonnegative definite matrix which is a sum like

$$A_R = \sum_{r\in R} A_r, \tag{2.11}$$

where $A_r = \{a_{ik}^r\}$, $i, k = 1, \ldots, N$ are the sub-band matrices with the elements

$$a_{ik}^r = (\sin(\Omega_{2r}(i-k)) - \sin(\Omega_{1r}(i-k)))/\pi(i-k), \quad a_{ii}^r = (\Omega_{2r} - \Omega_{1r})/\pi.$$

Therefore (see [10]), there exists an orthogonal $N \times N$ matrix of eigenvectors $G^R = (\vec{q}_1^R \dots \vec{q}_N^R)$ for which the equality

$$G^{R'} G^R = G^R G^{R'} = \mathrm{diag}(1, \dots, 1), \tag{2.12}$$

and the corresponding diagonal matrix of nonnegative eigenvalues

$$L^R = \mathrm{diag}(\lambda_1^R, \dots, \lambda_N^R), \tag{2.13}$$

arranged in descending order together with the eigenvectors

$$\lambda_1^R \geq \lambda_2^R \geq \dots \geq \lambda_N^{Rr} \geq 0, \tag{2.14}$$

so that satisfied is the matrix equality

$$G^R L^R = A_R G^R. \tag{2.15}$$

Thereby, representation (2.10) can be formed as

$$P_R(\vec{z}) = \sum_{k}^{N} \lambda_k^R (\beta_k^R)^2, \tag{2.16}$$

where

$$\vec{\beta}^R = (\beta_1^R, \dots, \beta_N^R)' = G^{R'} \vec{z}. \tag{2.17}$$

We introduce the notion of the sub-band distance between two vectors

$$d_R(\vec{x}, \vec{z}) = P_R(\vec{x} - \vec{z}) = \int_{\omega \in \Omega_R} |X(\omega) - Z(\omega)|^2 d\omega / 2\pi. \tag{2.18}$$

By analogy to (2.16), one can readily represent (2.18) in the time domain

$$d_R(\vec{x}, \vec{z}) = \sum_{k=1}^{N} \lambda_k^R (\alpha_k^R - \beta_k^R)^2, \tag{2.19}$$

$$\vec{\alpha}^R = (\alpha_1^R, \dots, \alpha_N^R)' = G^{R'} \vec{x}. \tag{2.20}$$

Since the orthonormal basis of the eigenvectors of the symmetrical matrices is complete in the space R^N (see [10]), valid are the relations (Fourier series)

$$\vec{x} = G^R \vec{\alpha}^R, \quad \vec{z} = G^R \vec{\beta}^R, \tag{2.21}$$

$$X(\omega) = \sum_{k=1}^{N} \alpha_k^R Q_k^R(\omega), \tag{2.22}$$

and as was shown in [9], the Fourier transforms of the eigenvectors $Q_k^R(\omega)$ are orthogonal not only over the entire domain $(-\pi, \pi)$, but also within the selected frequency interval (property of dual orthogonality), that is, the equalities $(i,\ k = 1, \dots, N)$,

$$\int_{\omega \in \Omega_R} Q_k^R(\omega) Q_i^R(-\omega) d\omega / 2\pi = 0, \quad i \neq k, \tag{2.23}$$

take place, and

$$P_R(\vec{q}_k^R) = \int\limits_{\omega\in\Omega_R} |Q_k^R(\omega)|^2 d\omega/2\pi = \lambda_k^R \le 1, \quad i = k. \tag{2.24}$$

Thus, the eigenvalues are equal to the part of energy of the corresponding eigenvectors hitting the selected united frequency interval, and valid is the relation

$$P_R(\vec{x}) = \sum_{k=1}^{N} \lambda_k^R (\alpha_k^R)^2, \tag{2.25}$$

defining the part of energy of the original time series hitting the same interval.

If one assumes that

$$\vec{\theta}^R = (\alpha_1^R, \ldots, \alpha_{J_R}^R)', \quad J_R < N, \tag{2.26}$$

then for the original vector the orthogonal decomposition

$$\vec{x} = \vec{y}^R + \vec{u}^R, \tag{2.27}$$

where

$$\vec{u}^R = \sum_{k=J_R+1}^{N} \alpha_k^R \vec{q}_k^R, \tag{2.28}$$

$$\vec{y}^R = GJ^R \vec{\theta}^R = GJ^R GJ^{R'} \vec{x}, \tag{2.29}$$

$$GJ^R = (\vec{q}_1^R, \ldots, \vec{q}_{J_k}^R), \tag{2.30}$$

is easily established from (2.21).

At that, with regard for assumption (2.14) one can easily determine from (2.19) a relations for the sub-band distance between the original vector and the vector (2.29):

$$d_R(\vec{x}, \vec{y}^R) = \sum_{k=J_R+1}^{N} \lambda_k^R (\alpha_k^R)^2 \le \lambda_{J_R+1}^R \sum_{k=J_{rR}+1}^{N} (\alpha_k^R)^2. \tag{2.31}$$

It is clear that the identity

$$\vec{y}^R = GJ^R GJ^{R'} \vec{y}^R \tag{2.32}$$

is satisfied in view of orthonormality of the columns of matrix (2.30).

Therefore, the vector $\vec{y}^R$ is a fixed point of the matrix operator

$$W^R = GJ^R GJ^{R'}, \tag{2.33}$$

that is, a condition like (2.4) is satisfied, and the matrix operator is linear. Therefore, it is suggested to use the vectors like (2.29) as the estimates of the time series trends.

It seems natural to call (2.29) the operator of sub-band smoothing (OSS) because its construction makes use of the eigenvectors of matrices like (2.11).

It is also easy to establish on the basis of (2.27), (2.28), and (2.12) the ratio of similitude (cosine of the angle between the vectors) between the considered vectors, the parentheses embracing the scalar product of the vectors:

$$\rho(\vec{x}, \vec{y}^R) = (\vec{x}, \vec{y}^R)/(||\vec{x}||/||\vec{y}^R||) = ||\vec{\theta}^R||/||\vec{\alpha}^R||. \tag{2.34}$$

We note that positiveness of the right side of (2.34) is an important property in terms of approximating the trend by vectors like (2.29). We also note that the parameter (2.31) defines the error of approximation of the segment of the Fourier transform of the initial series by a segment of the Fourier transform of the trend estimate. According to the theory of generalized Fourier series, the approximations with coefficients like (2.26) are the best for the given number of series terms.

The next step lies in selecting the frequency intervals containing an overwhelming part of the trend energy and estimating the least number of the eigenvectors of matrix like (2.11) used for its approximation (value of the parameter J_R).

3. ADAPTIVE CONSTRUCTION OF OSS

We note that representation (2.29) defines the set of trends satisfying the ideal condition (2.4). The real trends satisfy it only approximately, the degree of approximation being defined by the selection of the united frequency interval (2.6) and the number of the used eigenvectors of the aggregate matrix (2.11).

Since it is assumed that the desired trend is narrower than the analyzed time series, it is required to define the rule for selecting a suitable united frequency interval.

In what follows, $K + 1$ denotes the total number of the frequency intervals into which the frequency axis $[-\pi, \pi)$, $\Delta = \pi/(2K + 1)$, $\omega_r = 2r\Delta$, $r = 1, \ldots, K$ is decomposed, and the boundaries of the original frequency intervals obey the relations

$$\Omega_{10} = 0, \quad \Omega_{20} = \Delta, \quad \Omega_{1r} = \omega_r - \Delta, \quad \Omega_{2r} = \omega_r + \Delta, \quad r = 1, \ldots, K, \tag{3.1}$$

that is, the frequency intervals are disjoint and overlap the entire frequency axis.

At that, the representations of the elements of sub-band matrices from (2.11) come to the form

$$a^0_{ik} = \sin(\Delta(i - k))/\pi(i - k), \quad a^0_{ii} = 1/(2K + 1), \quad i, k = 1, \ldots, N, \tag{3.2}$$

$$a^r_{ik} = 2a^0_{ik} \cos(\omega_r(i - k)), \quad r = 1, \ldots, K. \tag{3.3}$$

Let now the analyzed vector $\vec{x}$ (1.1) consist only of noncorrelated random components (no trend) with zero expectations (symbol E), that is, the conditions

$$\mathrm{E}[x_k] = 0, \quad \mathrm{E}[x_k^2] = \sigma^2, \quad \mathrm{E}[x_i x_k] = 0, \quad i \neq k \tag{3.4}$$

be satisfied.

Then, bearing in mind notation (2.7) and (2.11), one can readily find a relation for the expectation of the characteristic (2.10)

$$\mathrm{E}[P_R(\vec{x})] = N\sigma^2\mu_R. \tag{3.5}$$

In compliance with Parseval equality [11], it is natural to use

$$B = ||\vec{x}||^2\mu_R \tag{3.6}$$

as the estimate of this characteristic.

On the basis of definition (2.5), relation (2.9) is representable as two addends

$$P_R(\vec{x}) = \int \left\{ \left(\sum_{k=1}^{N} x_k \cos(\omega(k - 1)) \right)^2 + \left(\sum_{k=1}^{N} x_k \sin(\omega(k - 1)) \right)^2 \right\} d\omega/2\pi. \tag{3.7}$$

In (3.7), it is meant integration within a frequency interval like (2.6).

Assume that

$$C_K = \int_z \int_v \sum_{k=1}^{N} \sum_{i=1}^{N} x_k x_i \cos(z(k-1)) \sin(v(i-1)) dz dv/4\pi^2. \quad (3.8)$$

If the conditions (3.4) are satisfied, one can easily establish the equality for the expectation of this characteristic

$$\mathrm{E}[C_K] = \sigma^2 (D_R^+ + D_R^-)/2, \quad (3.9)$$

where

$$D_R^{\pm} = \int_{z\in\Omega_R} \int_{v\in\Omega_R} \sin(N(z \pm v)/2) \sin((N-1)(z \pm v)/2)/\sin((z \pm v)/2) dz dv/4\pi^2. \quad (3.10)$$

Obviously, here the two-dimensional domain of integration if central symmetrical about the origin, and in it the subintegral functions are odd. Therefore, the integrals (3.10) are equal to zero. Consequently, the equality $\mathrm{E}[C_K] = 0$ holds.

Thus, if the hypothesis that the components of the examined vector have a Gaussian distribution of probabilities, then characteristic (3.7) has a distribution like χ_2^2 (squared chi with two degrees of freedom).

However, we note that the assumption of the Gaussian distribution of the probabilities of deviations from the hypothesized trend is regarded as generally unrealistic. Therefore, it is advisable to use such characteristics of time series that are readily calculated and characterize rather well its behavior.

Obviously, for any $k \in \{0, \ldots, K\}$ (one of the frequency intervals) there exists the equality

$$P_k(\vec{x}) = \vec{x}' A_k \vec{x}. \quad (3.11)$$

In view of (3.2) and (3.3) (see also the definitions of the elements of sub-band matrices in (2.11)), one can easily establish the equalities

$$\sum_{r=0}^{K} A_r = I = \mathrm{diag}(1, \ldots, 1), \quad (3.12)$$

$$\sum_{r=0}^{K} P_r(\vec{x}) = ||\vec{x}||^2. \quad (3.13)$$

We notice that the latter equality is one of the forms of the Parceval equality [11]. If we introduce the notion of frequency density of time series energy

$$U = ||\vec{x}||^2/2\pi,$$

then it is possible to define the middle part of energy getting in one of the frequency intervals with boundaries (3.1):

$$h_k = ||\vec{x}||^2 (\Omega_{2k} - \Omega_{1k})/\pi, \quad k = 0, \ldots, K, \quad (3.14)$$

as well as in the combined interval like (2.6) (see also (2.7)):

$$h_R = \sum_{r\in R} h_r = ||\vec{x}||^2 \mu_R. \quad (3.15)$$

If in a frequency interval the value of the characteristic (3.11) exceeds (3.14), this can be an indication of the presence of trend energy. Therefore, to determine the frequency interval where the trend energy is concentrated, it is proposed to verify for execution the inequality

$$P_k(\vec{x}) \geq h_k(\vec{x}), \quad k = 0, \ldots, K. \tag{3.16}$$

Consequently, the set of indices of an interval like (2.6) is defined by the condition

$$\forall r \in R \Rightarrow P_r(\vec{x}) \geq h_r(\vec{x}). \tag{3.17}$$

We refer to such interval united according to (2.6) the information interval.

Consider now the problem of selecting duration of the analyzed segment. Let for definiteness the values of the time series be constant,

$$x_k = x, \quad k = 1, \ldots, N, \tag{3.18}$$

that is, the trend values are maximally consistent in a sense.

Then, the definition like (2.5) enables one to establish the relation

$$|X(\omega)|^2 = x^2 \sin^2(N\omega/2)/\sin^2(\omega/2), \quad -\pi < \omega < \pi. \tag{3.19}$$

The so-called main lobe of this energy spectrum (between the first zeros situated symmetrically about the zero frequency) containing contains the overpowering part of energy, lies within the frequency interval

$$-2\pi/N \leq \omega \leq 2\pi/N. \tag{3.20}$$

Comparison with condition (2.7) gives rise to an inequality to be satisfied for estimating the narrow-band trend:

$$N \gg 2. \tag{3.21}$$

It is suggested to select the number of frequency intervals from the condition $2\pi/N = \pi/(2K+1)$ providing

$$K = [(N-2)/4], \tag{3.22}$$

where the brackets denote the integer part.

Then, the inequality

$$N(\Omega_{02} - \Omega_{01})/\pi \geq 2 \quad \text{is satisfied according to (3.1).} \tag{3.23}$$

The following result generalizes the numerous experiments carried out by the present author to compute the eigenvalues of the matrices like (2.11): if (3.22) takes place and the inequalities

$$\mu_R < 0.5, \quad N \geq 8/(1 - 2\mu_R) \tag{3.24}$$

are satisfied, then for the matrices like (2.11) there exists a minimal index $IM < N$ such that the equalities

$$\lambda^R_{IM+k} = 0, \quad k = 1, \ldots, N - IM \tag{3.25}$$

are satisfied with high precision.

Consequently, for the choice of

$$J_R = IM, \tag{3.26}$$

relation (2.31) provides

$$d_R(\vec{x}, \vec{y}^R) \approx 0. \tag{3.27}$$

Stated differently, within the information frequency interval the segments of the Fourier transform of the original time series and the trend estimate obtained on the basis of (2.29) coincide actually precisely.

At the same time one must bear in mind that at making decision about the value of J_R one needs not to be guided by the requirement (3.27). In particular, If condition (3.4) is satisfied, then it makes no sense to determine the trend. Therefore, if inequality (3.16) gives rise to $\mu_R \approx 0.5$, one has make decision about the lack of trend.

Let now in representation (1.1) the deviations from the trend be zero and its spectrum finite, so that valid is the representation

$$f_i = \int\limits_{\omega \in \Omega_R} H(\omega) \exp(j\omega(i-1)) d\omega/2\pi, \tag{3.28}$$

where H is some continuous complex frequency function with even real and odd imaginary components.

Then, by substituting representation (3.28) in a definition like (2.5) and performing some simple operations, for the Fourier transform of a trend segment we can determine the convolution integral

$$F(\omega) = \int\limits_{z \in \Omega_R} H(z) \exp(-j(N-1)(\omega - z)/2) \sin(N(\omega - z)/2)/\sin((\omega - z)/2) dz.$$

This relation demonstrates the effect of expanding the domain of definition of the Fourier transform of the time series segment owing to the finiteness of its duration. In the first approximation, the variable

$$\Delta\Omega = 4/N(1 - \mu_R) \tag{3.29}$$

can be used as a relative estimate of this expansion with respect to the frequency intervals outside the information interval $\pi(1 - \mu_R))$.

At construction of trend representation (2.29), this effect is taken into consideration by using the eigenvectors corresponding to the small eigenvalues because for them the approximate equalities

$$\alpha_k^R = (\vec{q}_k^R, \vec{f}) \approx \int\limits_{z \notin \Omega_R} Q_k^R(z) F(-z) dz/2\pi \ \ \forall \lambda_k^R \ll 1 \tag{3.30}$$

take place according to the Parceval formula [11] and the property of the spectra of eigenvectors (2.24).

Consequently, projections of the trend on these eigenvectors mostly carry information about its Fourier transform outside the information frequency interval. At that, use of the rule of selection of the kind (3.16) leads to the fact that the squared projections (3.30) also are much smaller than the energy of the considered time series. It seems only natural to use the energy considerations for solving selection of the permissible eigenvectors.

In view of this property of the eigenvectors of (2.24), relation (2.25) can be rearranged in

$$P_R(\vec{x}) = \sum_{r\in R} (a_r^R)^2 P_R(\vec{q}_r^R) \tag{3.31}$$

enabling one to estimate the contributions to this characteristic of the fractions of energies of the eigenvectors with regard for the property of normalization (2.12).

The fractions of the energy of the initial time series in the information frequency interval and outside of it obey

$$\varphi_R(\bar{x}) = P_R(\vec{x})/||\vec{x}||^2 = \sum_{r\in R} P_R(\vec{q}_r^R)(\alpha_r^R)^2/||\vec{x}||^2, \tag{3.32}$$

$$\gamma_R = P_{\bar{R}}(\vec{x})/||\vec{x}||^2 = 1 - \varphi_R. \tag{3.33}$$

Here and in what follows, the symbol $\bar{R}$ stands for the set of indices of the frequency intervals complementing the set R to the initial quantity $K+1$ so that

$$\Omega_{\bar{R}} = \bigcup_{r\notin R} \Omega_r. \tag{3.34}$$

The following inequalities can be easily obtained from (3.15) and rule (3.16):

$$\mu_R \le \varphi_R, \tag{3.35}$$

$$(1-\varphi_R)/(1-\mu_R) \le 1. \tag{3.36}$$

Since the sum of matrix eigenvalues is equal to its trace [10], from definitions (2.7) and (2.11) we get the equalities

$$\sum_{k=1}^{N} \lambda_k^R = N\mu_R, \tag{3.37}$$

$$\lambda_{sr}^R = \sum_{k=1}^{N} \lambda_k^R/N = \mu_R. \tag{3.38}$$

Therefore, the average eigenvalue is equal to the fraction of the frequency band occupied by the by the information frequency interval. Bearing in mind property (2.24) which is similar to (3.38), an equality can be also put down for the mean value of the energy fractions of the eigenvectors hitting the information frequency interval:

$$P_{sr}^R = \sum_{k=1}^{N} P_R(\vec{q}_k^R)/N = \mu_R. \tag{3.39}$$

According to (3.12), the equality is satisfied for total matrices

$$A_{\bar{R}} = I - A_R. \tag{3.40}$$

It is clear that the collections of eigenvectors of the matrices $A_{\bar{R}}$ and A_R coincide, and the following equalities hold for the corresponding eigenvalues:

$$\lambda_k^{\bar{R}} = 1 - \lambda_k^R, \quad k = 1,\ldots,N. \tag{3.41}$$

Therefore, the part of energy of the time series reaching the noninformation frequency interval (3.34) is given by

$$P_{\bar{R}}(\vec{x}) = \sum_{r\in R}(1-\lambda_r^R)(\alpha_r^R)^2. \tag{3.42}$$

Define the densities of the fractions of energies of the eigenvectors in the information frequency interval by

$$\int\limits_{z\in\Omega_R} |Q_r^R|^2 dz/S_R = P_R(\vec{q}_r^R)/\mu_R. \tag{3.43}$$

On the other hand, in the absence of deviations from the trend the left part of inequality (3.36) represents the density of its energy fraction outside the information frequency interval so that with regard for it possible expansion (3.29) the characteristic

$$\tau_R = 4(1-\varphi_R)/(1-\mu_R)^2/N \tag{3.44}$$

can be regarded as the mean density of this energy fraction in the considered neighborhood of the information frequency interval. It seems natural to demand that the right side of (3.43) be greater than the right side of (3.44), which gives rise to the inequality to be satisfied by the selected eigenvectors:

$$P_R(\vec{q}_r^R) \geq 4\mu_R(1-\varphi_R)/(1-\mu_R)^2/N. \tag{3.45}$$

With the use of (2.24), this inequality can be rearranged in the eigenvalues and conveniently used for selection of the eigenvectors

$$\lambda_r^R \geq 4\mu_R(1-\varphi_R)/(1-\mu_R)^2/N \ \ \forall r = 1,\ldots,J_R. \tag{3.46}$$

Here, the parameter J_R is equal to the maximal index of the eigenvalues satisfying this inequality.

We notice that it follows from inequality (3.36) and equalities (3.38) that the set of eigenvalues satisfying (3.46) is nonempty because there always exists an eigenvalue greater than the mean of some their quantity. Stated differently, some, if not all, number of the eigenvectors will be selected to represent the trend estimate as (2.29).

4. COMPUTER-AIDED EXPERIMENTS

To illustrate the use of the proposed adaptive OSS for extraction of trends, we give the result of some computer-aided experiments.

It is of interest to estimate through model examples the dependence of the parameter b in (2.3) on the initial noise/signal ratio

$$shs = ||\vec{\varepsilon}|| \ / \ ||\vec{f}||, \tag{4.1}$$

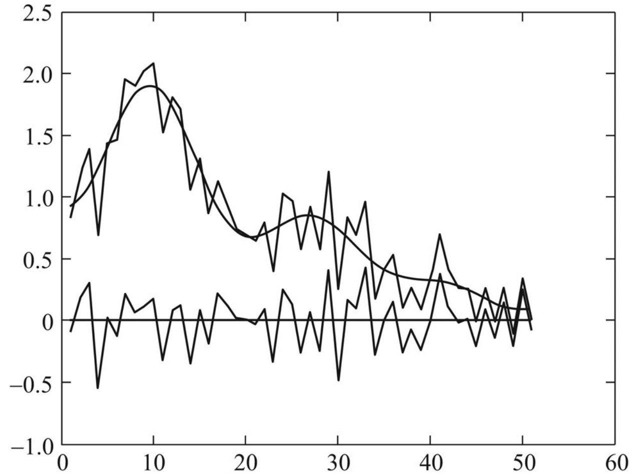
inclusive accuracy of satisfying condition (2.4).

The results obtained in one of the experiments are compiled in Table 1. Here, a mix of cosines was used as the model trend

$$f_k = \cos(8\pi k/N + \phi_1) + \cos(15\pi k/N + \phi_2), \quad k = 1,\ldots,N, \tag{4.2}$$

where $\phi_1, \phi_2 \in (-\pi,\pi)$ are the random phases distributed uniformly over the aforementioned interval and N is the duration of the processed segment of the time series.

It is easy to understand that in model (4.2) a variation in N implies a variation in the sampling frequency because it defines the number of points of indication on the corresponding periods of the cosines.



Dependence (smooth curve) of the estimated trend, initial empirical data (broken line), and the differences of these curves (lower broken line with zero level) on the number of the reading (abscissa).

The additive distortions and random phases were reproduced by pseudo-random Gaussian numbers. The values of b were calculated from the results of extracting the trend at each act of modeling the time series segment on the basis of a trend like (4.2) and averaged over the number of distortion acts $M = 100$. Table 1 compiles the average values of the parameter b.

It is easy to understand the accuracy of trend restoration grows with the discretization frequency (approximately according to the law of proportionality $N^{-1/2}$).

We also note that the numbers in the first column of Table 1 characterize the degree of satisfying condition (2.4), that is, the effect of processing in the lack of trend distortions.

To illustrate the effect of duration of the processed segment on the accuracy of trend estimation, instead of model (4.2) the sampling frequency was fixed and the duration was varied so that the model is given by

$$f_k = \cos(8\pi k/64 + \phi_1) + \cos(15\pi k/64 + \phi_2), \quad k = 1, \ldots, N.$$

The experimental method and notation were the same as above. The results of its application are compiled in Table 2. We note that the first experiment (Table 1) can also be used to compare the results under different durations.

The tables are indicative of the fact that both with increase in the sampling frequency and in the duration of the processed segment under the same sampling frequency the resulting noise/signal ratio (2.3) decreases relative to the initial ratio.

Table 1. Impact of the discretization frequency on the accuracy of trend restoration

shs		0	0.1	0.2	0.3	0.4	0.5
$N = 64$	b	0.0020	0.052	0.105	0.159	0.204	0.264
$N = 128$	b	0.0010	0.039	0.076	0.116	0.154	0.202
$N = 512$	b	0.0001	0.024	0.047	0.070	0.087	0.117

Table 2. Impact of segment duration on the accuracy of trend restoration

shs		0	0.1	0.2	0.3	0.4	0.5
$N = 128$	b	0.0020	0.043	0.086	0.132	0.175	0.224
$N = 512$	b	0.0002	0.029	0.058	0.086	0.119	0.144

The next group of computer-aided experiments concerns processing of the actual time series obtained by finding the logarithm of the width of annual rings of the sawn down trees. The figure depicts a typical example of such processing. We notice that the mean value of difference between the initial data and trend estimate is 0.00004, that is, practically zero. Similar values were obtained by processing other dendroseries.

5. CONCLUSIONS

The present paper developed a method to extract in the segments of the time series the trends with energies concentrated in a small part of the frequency domain (narrow band). A measure of differences of the Fourier transforms of the time series within the given frequency intervals was proposed. It was shown that with the use of the apparatus of the so-called special sub-band matrices this measure can be determined directly in the time domain. The eigenvectors of these matrices make up an orthonormal basis adequate to the variational problem of the best approximation within the given frequency interval of the segment of Fourier transform of the initial time series by a segment of the transform of the narrow-band time series which is taken as the trend estimate.

An adaptive procedure for calculation of the trend estimates was proposed. Computer-aided experiments illustrating sufficiently high efficiency of the proposed method of trend extraction were carried out.

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