Chapter 2
Digital Operators and Discrete Equations as Computational Tools

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2.1 Introduction

Discrete and difference equations play an important role in different branches of science, particularly in mathematical biology, signal, and image processing [1, 2]. So, for example, a lot of physical and technical processes are described by difference and discrete equations, namely, and continuous models arise as a limit transfer. According to the latter sentence, it is thought that studying of difference and discrete equations is very required.

We will deal with discrete equations related to well-known continuous mathematical objects as pseudo-differential operators and equations [3–6]. There is a series of books devoted to different aspects of the theory of discrete equations and discrete boundary value problems (see, e.g., [7–9]), but as a rule, these methods are developed for partial differential equations only. Also, some authors use projectional and algebraic methods for studying finite approximations for integral and related operator equations [10–12]. Let us remind that the theory of pseudo-differential operators was constructed to join together the theory of differential operators and certain integral ones. Starting from this point of view, we will try to construct a theory of discrete pseudo-differential operators and equations, to study such discrete equations and related discrete boundary value problems, and to verify their approximation properties.

A few years ago, such a work was started, and certain results were obtained. Some of these papers were related to discrete analogues of Calderon–Zygmund operators and corresponding integral equations [13, 14], but latter papers are devoted to studying discrete pseudo-differential equations and discrete boundary value

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problems in a discrete space and a discrete half-space [15–19]. We work with model operators with symbols non-depending on a spatial variable in special canonical domains which are cones in Euclidean space. This methodology is stipulated by a special local principle in the theory of pseudo-differential equations. We will widely use discrete and periodic analogues of classical one-dimensional singular integral operators [20, 21], methods of function theory of many complex variables [22, 23], and the key abstract result from the theory of projectional methods [24].

This paper is devoted to a new concept in the theory of discrete equations and discrete boundary value problems. We will describe here this approach and will present some key results in this direction. Most part of the paper will be related to a plane quadrant, and it is a new type of a conical domain for which we try to expand applicability of the suggested approach.

2.2 Discrete Spaces and Digital Operators

We will use the following notations. Let $\mathbf{T}$ be the segment $[-\pi, \pi]$, $h > 0$, $\hbar = h^{-1}$. We will consider all functions defined in the cube $\mathbf{T}^m$ as periodic functions in $\mathbf{R}^m$ with the same cube of periods.

If $u_d(\tilde{x})$, $\tilde{x} \in h\mathbf{Z}^m$, is a function of a discrete variable, then we call it "discrete function." For such discrete functions, one can define the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbf{Z}^m} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbf{T}^m,$$

if the latter series converges, and the function $\tilde{u}_d(\xi)$ is a periodic function on $\mathbf{R}^m$ with the basic cube of periods $\hbar\mathbf{T}^m$. This discrete Fourier transform preserves basic properties of the integral Fourier transform, particularly the inverse discrete Fourier transform is given by the formula

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in \hbar\mathbf{Z}^m.$$

The discrete Fourier transform is a one-to-one correspondence between the spaces $L_2(h\mathbf{Z}^m)$ and $L_2(\hbar\mathbf{T}^m)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x} \in h\mathbf{Z}^m} |u_d(\tilde{x})|^2 h^m \right)^{1/2}$$

and

$$||\tilde{u}_d||_2 = \left(\int\limits_{\xi \in \hbar \mathbf{T}^m} |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}.$$

Example 1 Since the definition for Sobolev–Slobodetskii spaces includes partial derivatives, we use their discrete analogue, i.e., divided difference of first order

$$\left(\Delta_k^{(1)} u_d\right)(\tilde{x}) = h^{-1}(u_d(x_1, \ldots, x_k + h, \ldots, x_m) - u_d(x_1, \ldots, x_k, \ldots, x_m)),$$

for which its discrete Fourier transform looks as follows

$$\widetilde{(\Delta_k^{(1)} u_d)}(\xi) = h^{-1}(e^{-ih\cdot\xi_k} - 1)\tilde{u}_d(\xi).$$

Further for the divided difference of second order, we have

$$\left(\Delta_k^{(2)} u_d\right)(\tilde{x}) = h^{-2}(u_d(x_1, \ldots, x_k + 2h, \ldots, x_m)$$
$$- 2u_d(x_1, \ldots, x_k + h, \ldots, x_m) + u_d(x_1, \ldots, x_k, \ldots, x_m))$$

and its discrete Fourier transform

$$\widetilde{(\Delta_k^{(2)} u_d)}(\xi) = h^{-2}(e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

Thus, for the discrete Laplacian, we have

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} (\Delta_k^{(2)} u_d)(\tilde{x}),$$

so that

$$\widetilde{(\Delta_d u_d)}(\xi) = h^{-2} \sum_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

We will use the discrete Fourier transform to introduce special discrete Sobolev–Slobodetskii spaces which are very convenient for studying discrete pseudo-differential operators and related equations.

Now we will introduce the basic space $S(h\mathbf{Z}^m)$ which consists of discrete functions with finite semi-norms

$$|u_d| = \sup_{\tilde{x} \in h\mathbf{Z}^m} (1 + |\tilde{x}|)^l |\Delta^{(\mathbf{k})} u_d(\tilde{x})|$$

for arbitrary $l \in \mathbf{N}$, $\mathbf{k} = (k_1, \ldots, k_m)$, $k_r \in \mathbf{N}$, $r = 1, \ldots, m$, where

$$\Delta^{(\mathbf{k})}u_d(\tilde{x}) = \Delta_1^{k_1}\dots,\Delta_m^{k_m}u_d(\tilde{x}).$$

In other words, the space $S(h\mathbf{Z}^m)$ is a discrete analogue of the Schwartz space $S(\mathbf{R}^m)$ of infinitely differentiable rapidly decreasing at infinity functions. Usually the space of distributions over the basic space $S(\mathbf{R}^m)$ is denoted by $S'(\mathbf{R}^m)$.

Digital distribution we call an arbitrary linear continuous functional defined on $S(h\mathbf{Z}^m)$. A set of such digital distributions we will denote by $S'(h\mathbf{Z}^m)$, and a value of the functional f_d on the basic function u_d will be denoted by (f_d, u_d).

Together with the space $S(h\mathbf{Z}^m)$, we consider the space $D(h\mathbf{Z}^m)$ consisting of discrete functions with a compact (finite) support. We say that $f_d = 0$ in the discrete domain $M_d \equiv M \cap h\mathbf{Z}^m, M \subset \mathbf{R}^m$, if $(f_d, u_d) = 0, \forall u_d \in D(M_d)$, where $D(M_d) \subset D(h\mathbf{Z}^m)$ consists of discrete functions whose supports belong to M_d. If we will denote $\widetilde{M}_d$, a union of such M_d, where $f_d = 0$ then by definition $supp\ f_d = h\mathbf{Z}^m \setminus \widetilde{M}_d$.

As usual [25], we can define some simplest operations in the space $S'(h\mathbf{Z}^m)$ excluding the differentiation (see below), and a convergence is defined as a weak convergence in the space of functionals $S'(h\mathbf{Z}^m)$.

If $f_d(\tilde{x})$ is a local summable function, then one can define the digital distribution f_d by the formula

$$(f_d, u_d) = \sum_{\tilde{x}\in h\mathbf{Z}^m} f_d(\tilde{x})u_d(\tilde{x})h^m, \quad \forall u_d \in S(h\mathbf{Z}^m).$$

Such distributions we call *regular* digital distributions. But there are so-called *singular* digital distributions like the Dirac mass-function

$$(\delta_d, u_d) = u_d(0),$$

which cannot be represented by the above formula.

Let us denote $\zeta^2 = h^{-2}\sum\limits_{k=1}^{m}(e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following

Definition 1 The space $H^s(h\mathbf{Z}^m)$ is a closure of the space $S(h\mathbf{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbf{T}^m}(1+|\zeta^2|)^s|\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}.$$

We would like to note that a lot of properties for such spaces were studied in [26].

Further, let $D \subset \mathbf{R}^m$ be a domain and $D_d = D \cap h\mathbf{Z}^m$ be a discrete domain.

Definition 2 The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbf{Z}^m)$ which supports belong to $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbf{Z}^m)$. The space $H_0^s(D_d)$ consists of discrete functions u_d with a support in D_d, and these discrete functions should admit a continuation into the whole $H^s(h\mathbf{Z}^m)$. A norm in the $H_0^s(D_d)$ is given by the formula

$$||u_d||_s^+ = \inf ||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

The Fourier image of the space $H^s(D_d)$ will be denoted by $\widetilde{H}^s(D_d)$.

Let $\widetilde{A}_d(\xi)$ be a measurable periodic function in $\mathbf{R}^m$ with the basic cube of periods $\hbar\mathbf{T}^m$. Such functions are called symbols. As usual, we will define a digital pseudo-differential operator by its symbol.

Definition 3 A digital pseudo-differential operator A_d in a discrete domain D_d is called an operator of the following kind

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbf{Z}^m} \int_{\hbar\mathbf{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d.$$

An operator A_d is called an elliptic operator if

$$ess \inf_{\xi\in\hbar\mathbf{T}^m} |\widetilde{A}_d(\xi)| > 0.$$

First as usual, we define the operator A_d on the dense set $S(h\mathbf{Z}^m)$ and then extend it on more general space.

Remark 1 One can introduce the symbol $\widetilde{A}_d(\tilde{x}, \xi)$ depending on a spatial variable $\tilde{x}$ and define a general pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbf{Z}^m} \int_{\hbar\mathbf{T}^m} \widetilde{A}_d(\tilde{x}, \xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

For studying such operators and related equations one needs to use more fine and complicated technique.

Definition 4 By definition the class E_α includes symbols satisfying the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2} \quad (2.1)$$

with universal positive constants c_1, c_2 non-depending on h and the symbol $A_d(\xi)$.

The number $\alpha \in \mathbf{R}$ is called an order of a digital pseudo-differential operator A_d.

Obviously that operator A_d satisfying (2.1) is an elliptic operator. Using the last definition one can easily get the following property.

Lemma 1 *A digital pseudo-differential operator $A_d \in E_\alpha$ is a linear bounded operator $H^s(h\mathbf{Z}^m) \to H^{s-\alpha}(h\mathbf{Z}^m)$ which norm does not depend on h.*

We study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{2.2}$$

assuming that we interested in a solution $u_d \in H^s(D_d)$ taking into account $v_d \in H_0^{s-\alpha}(D_d)$.

Main difficulty for this problem is related to a geometry of the domain D. Indeed, if $D = \mathbf{R}^m$ then the condition (2.1) guarantees the unique solvability for the Eq. (2.2). We will consider here only so-called canonical domains and simplest digital pseudo-differential operators with symbols non-depending on a spatial variable $\tilde{x}$. This fact is dictated by using in the future the local principle. The last asserts that for a Fredholm solvability of the general Eq. (2.2) with symbol $A_d(\tilde{x}, \xi)$ in an arbitrary discrete domain D_d, one needs to obtain invertibility conditions for so-called local representatives of the operator A_d, i.e., for an operator with symbol $A_d(\cdot, \xi)$ in a special canonical domain.

Earlier authors have extracted some canonical domains, namely, $D = \mathbf{R}^m, \mathbf{R}^m_+$, where $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > 0\}$.

Everywhere below we study the two-dimensional case for which a domain D is the first quadrant in a plane, $D \equiv K = \{x \in \mathbf{R}^2 : x = (x_1, x_2), x_1 > 0, x_2 > 0\}$. *Moreover, we consider homogeneous equation (2.2) for a simplicity.*

2.3 Solvability of Discrete Equations and Discrete Boundary Value Problems

Let $K_d = K \cap h\mathbf{Z}^2$. We study a solvability of the equation

$$(A_d u_d)(\tilde{x}) = 0, \quad \tilde{x} \in K_d. \tag{2.3}$$

We can describe solvability picture of the Eq. (2.3) if the symbol $\widetilde{A}_d(\xi)$ admits a special representation.

2.3.1 Periodic Wave Factorization

This concept is a periodic analogue of the wave factorization [23]. Some first preliminary considerations and results were described in [15].

We will use certain special domain in two-dimensional complex space $\mathbf{C}^2$. A domain of the type $\mathcal{T}_h(K) = \hbar\mathbf{T}^2 + iK$ is called a tube domain over the quadrant K, and we will consider analytical functions $f(x + i\tau)$ in the domain $\mathcal{T}_h(K) = \hbar\mathbf{T}^2 + iK$.

Definition 5 A periodic wave factorization for the elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,\neq}(\xi)A_{d,=}(\xi),$$

where the factors $A_{d,\neq}(\xi)$, $A_{d,=}(\xi)$ admit analytical continuation into tube domains $\mathcal{T}_h(K)$, $\mathcal{T}_h(-K)$ respectively with estimates

$$c_1(1+|\hat{\zeta}^2|)^{\frac{æ}{2}} \leq |A_{d,\neq}(\xi+i\tau)| \leq c_1'(1+|\hat{\zeta}^2|)^{\frac{æ}{2}},$$

$$c_2(1+|\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}} \leq |A_{d,=}(\xi-i\tau)| \leq c_2'(1+|\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}},$$

and constants c_1, c_1', c_2, c_2' non-depending on h, where

$$\hat{\zeta}^2 \equiv \hbar^2\left((e^{-ih(\xi_1+i\tau_1)}-1)^2+(e^{-ih(\xi_2+i\tau_2)}-1)^2\right),$$

$$\xi=(\xi_1,\xi_2)\in\hbar\mathbf{T}^2, \quad \tau=(\tau_1,\tau_2)\in K.$$

The number $æ \in \mathbf{R}$ is called an index of periodic wave factorization.

2.3.2 Solvability Conditions

Using methods of [17, 23] one can obtain the following results on a solvability of the Eq. (2.2).

Theorem 1 *Let* $|æ - s| < 1/2$. *Then the Eq.* (2.3) *has zero solution only.*

Theorem 2 *Let* $æ - s = n + \delta, n \in \mathbf{N}$, $|\delta| < 1/2$. *Then a general solution of the Eq.* (2.3) *has the following form*

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)\left(\sum_{k=0}^{n-1}\tilde{c}_k(\xi_1)\zeta_2^k+\tilde{d}_k(\xi_2)\zeta_1^k\right), \quad \zeta_j=\hbar(e^{-i\xi_j h}-1), \quad j=1,2, \tag{$*$}$$

where $\tilde{c}_k(\xi_1), \tilde{d}_k(\xi_2), k = 0, 1, \ldots, n-1$, *are arbitrary functions from* $\widetilde{H}^{s_k}(\hbar\mathbf{T})$, $s_k = s - æ + k - 1/2$.

The a priori estimate

$$||u_d||_s \leq const \sum_{k=0}^{n-1}([c_k]_{s_k}+[d_k]_{s_k}),$$

holds, where $[\cdot]_{s_k}$ *denotes a norm in* $H^{s_k}(h\mathbf{Z})$ *and const does not depend on* h.

2.3.3 Boundary Conditions

As we see, Theorem 2 asserts that for a certain case, we have a lot of solutions. To obtain the unique solution, we need to determine uniquely all arbitrary functions in the formula (*). We consider here the case $æ - s = 1 + \delta$, $|\delta| < 1/2$ for the Eq. (2.3) and two different types of conditions.

2.3.4 Classical Variant: The Dirichlet Discrete Boundary Condition

We consider here first simple case with discrete Dirichlet boundary conditions. It follows from Theorem 2 that we have the following general solution of the Eq. (2.4)

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_2)), \tag{2.4}$$

where $c_0, d_0 \in H^{s-æ-1/2}(\hbar\mathbf{Z})$ are arbitrary functions. To determine uniquely these functions, we add the discrete Dirichlet conditions on angle sides

$$u_d|_{\tilde{x}_1=0} = f_d(\tilde{x}_2), \qquad u_d|_{\tilde{x}_2=0} = g_d(\tilde{x}_1). \tag{2.5}$$

Thus, we have the discrete Dirichlet problem (2.3), (2.5).

First, we apply the discrete Fourier transform to discrete conditions (2.5) and obtain the following form

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi_1, \xi_2)d\xi_1 = \tilde{f}_d(\xi_2), \qquad \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi_1, \xi_2)d\xi_2 = \tilde{g}_d(\xi_1).$$

Let us denote

$$\int\limits_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_1 \equiv \tilde{a}_0(\xi_2), \qquad \int\limits_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_2 \equiv \tilde{b}_0(\xi_1)$$

and suppose that $\tilde{a}_0(\xi_2), \tilde{b}_0(\xi_1) \neq 0, \forall \xi_1 \neq 0, \xi_2 \neq 0$.

Therefore, we have the following system of two linear integral equations with respect to two unknown functions $\tilde{c}_0(\xi_1), \tilde{d}_0(\xi_2)$

$$\begin{cases} \int\limits_{-\hbar\pi}^{\hbar\pi} k_1(\xi)\tilde{c}_0(\xi_1)d\xi_1 + \tilde{d}_0(\xi_2) = \tilde{F}_d(\xi_2) \\ \tilde{c}_0(\xi_1) + \int\limits_{-\hbar\pi}^{\hbar\pi} k_2(\xi)\tilde{d}_0(\xi_2)d\xi_2 = \tilde{G}_d(\xi_1), \end{cases} \tag{2.6}$$

where we have used the following notations

$$\tilde{F}_d(\xi_2) = \tilde{f}_d(\xi_2)\tilde{a}_0^{-1}(\xi_2), \quad \tilde{G}_d(\xi_1) = \tilde{g}_d(\xi_1)\tilde{b}_0^{-1}(\xi_1),$$

$$k_1(\xi) = A_{d,\neq}^{-1}(\xi)\tilde{a}_0^{-1}(\xi_2), \quad k_2(\xi) = A_{d,\neq}^{-1}(\xi)\tilde{b}_0^{-1}(\xi_1).$$

Unique solvability conditions for the system (2.6) will be equivalent to unique solvability for the discrete Dirichlet problem (2.3), (2.5).

Thus, we obtain the following result.

Theorem 3 *Let $f_d, g_d \in H^{s-1/2}(\mathbf{R}_+)$, $s > 1/2$, and*

$$\inf|\tilde{a}_0(\xi_2)| > 0, \quad \inf|\tilde{b}_0(\xi_1)| > 0.$$

Then the discrete Dirichlet problem (2.3), (2.5) is reduced to the equivalent system of linear integral equations (2.6).

2.3.5 Nonlocal Discrete Boundary Condition

Another variant of a boundary condition is the following

$$\sum_{\tilde{x}_1 \in h\mathbf{Z}_+} u_d(\tilde{x}_1, \tilde{x}_2)h = f_d(\tilde{x}_2), \qquad \sum_{\tilde{x}_2 \in h\mathbf{Z}_+} u_d(\tilde{x}_1, \tilde{x}_2)h = g_d(\tilde{x}_1),$$
$$\sum_{\tilde{x} \in h\mathbf{Z}_{++}} u_d(\tilde{x}_1, \tilde{x}_2)h^2 = 0. \tag{2.7}$$

These additional conditions will help us to determine uniquely the unknown functions c_0, d_0 in the solution (2.4).

Indeed, using the discrete Fourier transform, we rewrite the conditions (2.7) as follows

$$\tilde{u}_d(0, \xi_2) = \tilde{f}_d(\xi_2), \quad \tilde{u}_d(\xi_1, 0) = \tilde{g}_d(\xi_1), \quad \tilde{u}_d(0, 0) = 0. \tag{2.8}$$

Now we substitute the formulas (2.8) into (2.4). The first two equality are

$$\tilde{u}_d(0, \xi_2) = A_{d,\neq}^{-1}(0, \xi_2)(\tilde{c}_0(0) + \tilde{d}_0(\xi_2)) = \tilde{f}_d(\xi_2),$$

$$\tilde{u}_d(\xi_1, 0) = A_{d,\neq}^{-1}(\xi_1, 0)(\tilde{c}_0(\xi_1) + \tilde{d}_0(0)) = \tilde{g}_d(\xi_1).$$

It implies the following relations according to the third condition that $\tilde{f}_d(0) = \tilde{g}_d(0)$, and it gives

$$\tilde{c}_0(0) + \tilde{d}_0(0) = 0, \quad \tilde{c}_0(0) = \tilde{d}_0(0) = 0.$$

So, we have at least formally the following formula

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)\left(A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1) + A_{d,\neq}(0, \xi_2)\tilde{f}_d(\xi_2)\right). \tag{2.9}$$

Theorem 4 *Let $f_d, g_d \in H^{s+1/2}(h\mathbf{Z})$. Then the discrete problem (2.3), (2.7) has unique solution which is given by the formula (2.9).*

The a priori estimate

$$||u_d||_s \leq const(||f_d||_{s+1/2} + ||g_d||_{s+1/2})$$

holds with a const non-depending on h.

2.4 Continuous Boundary Value Problems

Let A be a pseudo-differential operator with the symbol $A(\xi), \xi = (\xi_1, \xi_2)$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \leq |A(\xi)| \leq c_2(1 + |\xi|)^\alpha.$$

and admitting the wave factorization with respect to the quadrant K

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

with index æ such that $æ - s = 1 + \delta, |\delta| < 1/2$.

The continuous analogue of the discrete equation (2.3) is the following

$$(Au)(x) = 0, \quad x \in K. \tag{2.10}$$

2.4.1 The Dirichlet Condition

If we consider the Eq. (2.10), a general solution is written in the form [23]

$$\tilde{u}(\xi) = A^{-1}_{\neq}(\xi)(\tilde{C}_0(\xi_1) + \tilde{D}_0(\xi_2))$$

where arbitrary functions $\tilde{C}_0(\xi_1), \tilde{D}_0(\xi_2) \in \tilde{H}^{s-æ-1/2}(\mathbf{R})$ can be determined from the system of integral equations

$$\begin{cases} \int\limits_{-\infty}^{\infty} K_1(\xi)\tilde{C}_0(\xi_1)d\xi_1 + \tilde{D}_0(\xi_2) = \tilde{F}(\xi_2) \\ \tilde{C}_0(\xi_1) + \int\limits_{-\infty}^{\infty} K_2(\xi)\tilde{D}_0(\xi_2)d\xi_2 = \tilde{G}(\xi_1), \end{cases} \tag{2.11}$$

if we use the following boundary conditions

$$u_{|x_1=0} = f(x_2), \qquad u_{|x_2=0} = g(x_1) \tag{2.12}$$

and assume that the conditions $\inf|\tilde{A}_0(\xi_2)| \neq 0, \inf|\tilde{B}_0(\xi_1)| \neq 0$ hold. Here we have denoted

$$\int\limits_{-\infty}^{\infty} A_{\neq}^{-1}(\xi)d\xi_1 \equiv \tilde{A}_0(\xi_2), \quad \int\limits_{-\infty}^{\infty} A_{\neq}^{-1}(\xi)d\xi_2 \equiv \tilde{B}_0(\xi_1),$$

$$\tilde{F}(\xi_2) = \tilde{f}(\xi_2)\tilde{A}_0^{-1}(\xi_2), \quad \tilde{G}(\xi_1) = \tilde{g}(\xi_1)\tilde{B}_0^{-1}(\xi_1),$$

$$K_1(\xi) = A_{\neq}^{-1}(\xi)\tilde{A}_0^{-1}(\xi_2), \quad K_2(\xi) = A_{\neq}^{-1}(\xi)\tilde{B}_0^{-1}(\xi_1).$$

the following result is presented in the book [23].

Theorem 5 *If $s > 1/2$, conditions*

$$\inf|\tilde{A}_0(\xi_2)| \neq 0, \inf|\tilde{B}_0(\xi_1)| \neq 0$$

hold then the Dirichlet problem (2.10), (2.12) with data $f, g \in H^{s-1/2}(\mathbf{R}_+)$ is equivalent to the system of integral equations (2.11) with unknown functions $\tilde{C}_0, \tilde{D}_0 \in \widetilde{H}^{s_0}(\mathbf{R})$ and right hand sides $\tilde{F}, \tilde{G} \in \widetilde{H}^{s_0}(\mathbf{R})$.

2.4.2 Integral Condition

We consider the Eq. (2.10) with the following additional conditions

$$\int\limits_0^{+\infty} u(x_1, x_2)dx_1 = f(x_2), \quad \int\limits_0^{+\infty} u(x_1, x_2)dx_2 = g(x_1), \quad \int\limits_{-K} u(x)dx = 0. \tag{2.13}$$

A solution of the problem (2.10), (2.13) is sought in the space $H^s(K)$ [23], and boundary functions are taken from the space $H^{s+1/2}(\mathbf{R}_+)$. Such problem was considered in [27], and it has the solution

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\left(A_{\neq}(\xi_1, 0)\tilde{g}(\xi_1) + A_{\neq}(0, \xi_2)\tilde{f}(\xi_2)\right) \tag{2.14}$$

under condition that the symbol $A(\xi)$ admits the wave factorization with respect to the quadrant K.

2.5 Error Estimates

To construct a discrete boundary value problem which is good approximation for (2.10), (2.12), and (2.10), (2.13), we need to choose $A_d(\xi)$ and f_d, g_d in a special way. First, we introduce the operator l_h which acts as follows. For a function u defined in $\mathbf{R}$, we take its Fourier transform $\tilde{u}$ then we take its restriction on $\hbar\mathbf{T}$ and periodically extend it to $\mathbf{R}$. Finally, we take its inverse discrete Fourier transform and obtain the function of discrete variable $(l_h u)(\tilde{x}), \tilde{x} \in h\mathbf{Z}$. Thus, we put

$$f_d = l_h f, \quad g_d = l_h g.$$

Second, the symbol of digital operator A_d we construct in the same way. If we have the wave factorization for the symbol $A(\xi)$, then we take restrictions of factors on $\hbar\mathbf{T}^2$, and the periodic symbol $A_d(\xi)$ is a product of these restrictions.

2.5.1 The Discrete Dirichlet Problem

We introduce the space $\widetilde{\mathbf{H}}^s(\mathbf{R})$ of vector-functions $f = (f_1, f_2), f_j \in H^s(\widetilde{\mathbf{R}}), j = 1, 2,$

$$||f||_s \equiv ||f_1||_s + ||f_2||_s,$$

and matrix operators

$$K = \begin{pmatrix} K_1 & I \\ I & K_2 \end{pmatrix}, \qquad k = \begin{pmatrix} k_1 & I_h \\ I_h & k_2 \end{pmatrix},$$

acting in spaces $\widetilde{\mathbf{H}}^{s-æ-1/2}(\widetilde{\mathbf{R}})$ and $\widetilde{\mathbf{H}}^{s-æ-1/2}(\hbar\mathbf{T})$, respectively.

Let us remind that $s_0 = s - æ - 1/2$.

Lemma 2 *Under $s > 1, æ > 1$, the operator K is bounded in the space $\widetilde{\mathbf{H}}^{s_0}(\mathbf{R})$,*

$$K : \widetilde{\mathbf{H}}^{s_0}(\mathbf{R}) \to \widetilde{\mathbf{H}}^{s_0}(\mathbf{R}).$$

Proof It is enough to estimate $K_1 f$.

$$||K_1 f||_{s_0}^2 = \int_{-\infty}^{+\infty} (1+|\xi_2|)^{2s_0} |(K_1 f)(\xi_2)|^2 d\xi_2 =$$

$$= \int_{-\infty}^{+\infty} (1+|\xi_2|)^{2s_0} \left| \int_{-\infty}^{+\infty} K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1 \right|^2 d\xi_2 \leq$$

$$\leq \int_{-\infty}^{+\infty} (1+|\xi_2|)^{2s_0} \left(\int_{-\infty}^{+\infty} |K_1(\xi_1,\xi_2)||f(\xi_1)| d\xi_1 \right)^2 d\xi_2 \leq$$

$$\leq const \int_{-\infty}^{+\infty} (1+|\xi_2|)^{2s_0} \left(\int_{-\infty}^{+\infty} (1+|\xi_1|+|\xi_2|)^{-æ} ||f(\xi_1)| d\xi_1 \right)^2 d\xi_2$$

In the inner integral, we apply the Cauchy–Schwartz inequality using the factors $(1+|\xi_1|)^{-s_0}$ and $(1+|\xi_1|)^{s_0}$ for first and second term, respectively, and take into account that $(1+|\xi_1|)^{-s_0} \leq (1+|\xi_1|+|\xi_2|)^{-s_0}$. Thus,

$$||K_1 f||_{s_0}^2 \leq cons||f||_{s_0}^2 \int_{-\infty}^{+\infty} (1+|\xi_2|)^{2s_0} \left(\int_{-\infty}^{+\infty} (1+|\xi_1|+|\xi_2|)^{-2(s_0+æ)} d\xi_1 \right) d\xi_2 \leq$$

$$\leq const||f||_{s_0}^2 \int_{0}^{+\infty} (1+|\xi_2|)^{2s_0} \left(\int_{0}^{+\infty} (1+|\xi_1|+|\xi_2|)^{-2s+1} d\xi_1 \right) d\xi_2 \leq$$

$$\leq cons||f||_{s_0}^2 \int_{0}^{+\infty} (1+|\xi_2|)^{2s_0-2s+2} d\xi_2 \leq const||f||_{s_0}^2$$

$$\times \int_{0}^{+\infty} (1+|\xi_2|)^{-2æ+1} d\xi_2 \leq const||f||^2,$$

in view of the conditions $s > 1$, $æ > 1$. □

Let us denote by $\chi_h : H^s(\mathbf{R}) \to H^s(\hbar\mathbf{T})$ the restriction operator on the segment $\hbar\mathbf{T}$. The restriction operator on the segment $\hbar\mathbf{T}$ in the space $\mathbf{H}^s(\mathbf{R})$ will be denoted by Ξ_h so that for $f = (f_1, f_2) \in \mathbf{H}^s(\mathbf{R})$ we have

$$\Xi_h f = (\chi_h f_1, \chi_\hbar f_2).$$

We remind that h is small enough, $0 < h < 1$.

Lemma 3 *Under* $s > 1$, $æ > 1$ *the operator* K *has the following property*

$$||\Xi_h K - K\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbf{R}) \to \widetilde{\mathbf{H}}^{s_0}(\mathbf{R})} \leq \ const\ h^{s-1}.$$

Proof One can easily verify the following

$$\Xi_h K - K\Xi_h = \begin{pmatrix} \chi_h K_1 - K_1\chi_h & 0 \\ 0 & \chi_\hbar K_2 - K_2\chi_\hbar \end{pmatrix}$$

We conclude that

$$((\chi_h K_1 - K_1\chi_h)f)(\xi_2) = \begin{cases} \left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty}\right) K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1, & \xi_2 \in \hbar\mathbf{T} \\ -\int\limits_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1, & \xi_2 \notin \hbar\mathbf{T}. \end{cases}$$

Let us start from the first case and estimate one of integrals.

$$\left|\int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1\right| \leq \int\limits_{\hbar\pi}^{+\infty} |K_1(\xi)||f(\xi_1)| d\xi_1 \leq const \int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-æ} |f(\xi_1)| d\xi_1 eq$$

(we use Cauchy–Schwartz inequality)

$$\leq\ const \left(\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2æ}(1+|\xi_1|)^{-2s_0} d\xi_1\right)^{1/2}$$

$$\times \left(\int\limits_{\hbar\pi}^{+\infty} |f(\xi_1)|^2 (1+|\xi_1|)^{2s_0} d\xi_1\right)^{1/2} \leq$$

$$\leq\ const \left(\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2(æ+s_0)} d\xi_1\right)^{1/2} ||f||_{s_0},$$

Further,

$$\int_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2(æ+s_0)} d\xi_1 \sim (1+|\xi_2|+\hbar\pi)^{-2(æ+s_0)+1},$$

since $-2(æ+s_0)+1 = -2(s-1/2)+1 = -2s+2 < 0$.

Thus, the following inequality is obtained

$$\left| \int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right| \leq const||f||_{s_0}(1+|\xi_2|+\hbar\pi)^{-(æ+s_0)+1/2}.$$

Squaring the latter inequality, multiplying by $(1+|\xi_2|)^{2s_0}$ and integrating over $\hbar\mathbf{T}$ we obtain

$$\int_{\hbar\mathbf{T}} (1+|\xi_2|)^{2s_0} \left| \int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right|^2 d\xi_2 \leq$$

$$\leq const||f||^2_{s_0} \int_{\hbar\mathbf{T}} (1+|\xi_2|+\hbar\pi)^{-2(æ+s_0)+1} (1+|\xi_2|)^{2s_0} d\xi_2 \leq$$

$$\leq const||f||^2_{s_0}(1+\hbar\pi)^{-2s+2} \int_{\hbar\mathbf{T}} (1+|\xi_2|)^{2s_0} d\xi_2,$$

since $1+|\xi_2|+\hbar\pi \geq 1+|\hbar\pi|$, $-2(æ+s_0)+1 = -2s+2 < 0$. Let us note $2s_0 < -1$. So, we have

$$\int_{\hbar\mathbf{T}} (1+|\xi_2|)^{2s_0} \left| \int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right|^2 d\xi_2 \leq const||f||^2_{s_0} h^{2(s-1)}.$$

For the second case ($|\xi_2| > \hbar\pi$)

$$\left| \int_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1 \right| \leq const \int_{-\hbar\pi}^{+\hbar\pi} (1+|\xi|)^{-æ} |f(\xi_1)| d\xi_1 \leq$$

$$\leq const \left(\int_{-\hbar\pi}^{\hbar\pi} (1+|\xi|)^{-2æ}(1+|\xi_1|)^{-2s_0} d\xi_1 \right)^{1/2}$$

$$\times \left(\int_{-\hbar\pi}^{\hbar\pi} |f(\xi_1)|^2 (1+|\xi_1|)^{2s_0} d\xi_1 \right)^{1/2} .$$

Here we have used the Cauchy–Schwartz inequality once again. Taking into account the inequality $1+|\xi| \geq 1+|\xi_1|$ we obtain the estimate

$$\int_{-\hbar\pi}^{\hbar\pi} (1+|\xi|)^{-2æ}(1+|\xi_1|)^{-2s_0} d\xi_1 \leq const \int_{0}^{\hbar\pi} (1+\xi_1+|\xi_2|)^{-2(s_0+æ)} d\xi_1 \leq$$

$$\leq const (1+|\xi_2|)^{-2s+2} \leq const (1+\hbar\pi)^{-2s+2},$$

in view of $-2(s_0+æ) = -2(s-1/2) = -2s+1$.

Therefore, we obtain the inequality

$$\left| \int_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1 \right| \leq const ||f||_{s_0} h^{s-1}.$$

Multiplying by $(1+|\xi_2|)^{s_0}$ the latter inequality, squaring and integrating over $\mathbf{R} \setminus \hbar\mathbf{T}$ we find

$$\int_{\mathbf{R}\setminus\mathbf{T}} (1+|\xi_2|)^{2s_0} \left| \int_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right|^2 d\xi_2 \leq const ||f||^2_{s_0} h^{2(s-1)} \int_{\hbar\pi}^{+\infty} (1+\xi_2)^{2s_0} d\xi_2.$$

The latter integral converges since $2s_0 < -1$.

The same estimates are valid for K_2. □

Corollary 1 *If* $s > 1$, $æ > 1$ *and the operator* K *is invertible then for the operator* K^{-1} *the same estimate holds*

$$||\Xi_h K^{-1} - K^{-1}\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbf{R}) \to \widetilde{\mathbf{H}}^{s_0}(\mathbf{R})} \leq const\ h^{s-1}.$$

Proof Indeed, we have

$$\Xi_h K^{-1} - K^{-1}\Xi_h = K^{-1}K\Xi_h K^{-1} - K^{-1}\Xi_h K K^{-1} = K^{-1}(\Xi_h K - K\Xi_h)K^{-1},$$

and therefore

$$||\Xi_h K^{-1} - K^{-1}\Xi_h||_{\tilde{\mathbf{H}}^{s_0}(\mathbf{R})\to\tilde{\mathbf{H}}^{s_0}(\mathbf{R})} \leq ||K^{-1}||\cdot||\Xi_h K - K\Xi_h||_{\tilde{\mathbf{H}}^{s_0}(\mathbf{R})\to\tilde{\mathbf{H}}^{s_0}(\mathbf{R})}\cdot||K^{-1}||,$$

□

Lemma 4 *For* $æ > 1$ *the following estimate*

$$|K_1(\xi) - k_1(\xi)| \leq const\ (1 + |\xi|)^{-æ} h^{æ-1},\quad \xi \in \hbar\mathbf{T}^2.$$

holds.

Proof Indeed, according to our choice for $A^{-1}_{d,\neq}(\xi)$

$$\begin{aligned}|K_1(\xi) - k_1(\xi)| &= |A^{-1}_{\neq}(\xi)\tilde{A}_0^{-1}(\xi_2) - A^{-1}_{d,\neq}(\xi)\tilde{a}_0^{-1}(\xi_2)| \\ &\leq const\,(1 + |\xi|)^{-æ}|\tilde{A}_0(\xi_2) - \tilde{a}_0(\xi_2)|.\end{aligned}$$

Let us consider $|\tilde{A}_0(\xi_2) - \tilde{a}_0(\xi_2)|$. Then

$$|\tilde{A}_0(\xi_2) - \tilde{a}_0(\xi_2)| = \left|\int_{-\infty}^{\infty} A^{-1}_{\neq}(\xi)d\xi_1 - \int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_1\right| \leq$$

$$\leq const \int_{\hbar\pi}^{+\infty} (1 + |\xi|)^{-æ} d\xi_1 \leq const\ (1 + |\xi_2| + \hbar)^{-æ+1} \leq const\ h^{æ-1}$$

for enough small h. It implies the following implication $\inf|\tilde{A}_0(\xi_2)| \neq 0 \Longrightarrow \inf|\tilde{a}_0(\xi_2)| \neq 0$ for enough small h.

Collecting the obtained estimates we complete the proof. □

Let us introduce the operator $\Xi_h K \Xi_h$. Lemma 3 implies that for enough small h an invertibility of the operator $\Xi_h K \Xi_h$ in the space $\tilde{\mathbf{H}}^{s-æ-1/2}(\hbar\mathbf{T})$ follows from an invertibility of the operator K in the space $\tilde{\mathbf{H}}^{s-æ-1/2}(\mathbf{R})$ [24]. Moreover,

$$||(\Xi_h K \Xi_h)^{-1}||_{\tilde{\mathbf{H}}^{s_0}(\hbar\mathbf{T})\to\tilde{\mathbf{H}}^{s_0}(\hbar\mathbf{T})} \leq const$$

for enough small h.

Lemma 5 *If* $æ > 1$ *then a comparison for norms of operators* $\Xi_h K \Xi_h$ *and* k *is given by the estimate*

$$||\Xi_h K \Xi_h - k||_{\tilde{\mathbf{H}}^{s_0}(\hbar\mathbf{T})\to\tilde{\mathbf{H}}^{s_0}(\hbar\mathbf{T})} \leq const\ h^{æ-1}.$$

Proof The difference of the operators looks as follows

$$\Xi_h K \Xi_h - k = \begin{pmatrix} \chi_h K_1 \chi_h - k_1 & 0 \\ 0 & \chi_\hbar K_2 \chi_\hbar - k_2 \end{pmatrix},$$

and we need to estimate the norm $\chi_h K_j \chi_h - k_j$, $j = 1, 2$. Let us estimate K_1 using Lemma 3. So, we obtain

$$||\chi_\hbar K_1 \chi_\hbar f - k_1 f||^2_{s_0} = \int\limits_{\hbar\mathbf{T}} (1 + |\xi_2|)^{2s_0} \left| \int\limits_{\hbar\mathbf{T}} [K_1(\xi) - k_1(\xi)] f(\xi_1) d\xi_1 \right|^2 d\xi_2 \leq$$

$$\leq \int\limits_{\hbar\mathbf{T}} (1 + |\xi_2|)^{2s_0} \left(\int\limits_{\hbar\mathbf{T}} |K_1(\xi) - k_1(\xi)||f(\xi_1)| d\xi_1 \right)^2 d\xi_2 \leq$$

$$\leq \; const \; h^{2æ-2} \int\limits_{\hbar\mathbf{T}} (1 + |\xi_2|)^{2s_0} \left(\int\limits_{\hbar\mathbf{T}} (1 + |\xi|)^{-æ} |f(\xi_1)| d\xi_1 \right)^2 d\xi_2.$$

In the inner integral, we apply the Cauchy–Schwartz inequality with the factor $(1 + |\xi_1|)^{s_0}$

$$\int\limits_{\hbar\mathbf{T}} (1 + |\xi|)^{-æ} |f(\xi_1)| d\xi_1 \leq ||f||_{s_0} \left(\int\limits_{\hbar\mathbf{T}} (1 + |\xi|)^{-2æ} (1 + |\xi_1|)^{-2s_0} d\xi_1 \right)^{1/2} \leq$$

$$\leq ||f||_{s_0} \left(\int\limits_0^{+\infty} (1 + |\xi|)^{-2(æ+s_0)} d\xi_1 \right)^{1/2} \leq ||f||_{s_0} (1 + |\xi_2|)^{-(æ+s_0)+1/2}$$

$$= ||f||_{s_0} (1 + |\xi_2|)^{-s+1},$$

since $s_0 = s - æ - 1/2$. We have

$$||\chi_\hbar K_1 \chi_\hbar f - k_1 f||^2_{s_0} \leq \; const \; h^{2æ-2} ||f||^2_{s_0} \int\limits_{-\hbar\pi}^{\hbar\pi} (1 + |\xi_2|)^{2s_0-2s+2} d\xi_2 \leq$$

$$\leq const\ h^{2æ-2}||f||^2_{s_0} \int_0^{+\infty} (1+|\xi_2|)^{-2æ+1} d\xi_2 \leq const\ h^{2æ-2}||f||^2,$$

in view of $s_0 + 1 - s = -æ + 1/2$. Taking a square root, we obtain the required assertion. □

At this time, we are able to compare discrete and continuous solutions.

Theorem 6 *Let the conditions of Theorem 5 hold and $s > 1, æ > 1$. A comparison for solutions of problems (2.10), (2.12) (2.3), (2.5) for enough small h is given by the estimate*

$$||u - u_d||_{\tilde{H}^s(\hbar\mathbf{T}^2)} \leq const\ h^{s-1}(||f||_{s-1/2} + ||g||_{s-1/2})$$

where $const$ does not depend on h.

Proof We start from a comparison of solutions of systems (2.6) and (2.11).

We have the continuous solution

$$\tilde{u}(\xi) = A^{-1}_{\neq}(\xi)(\tilde{C}_0(\xi_1) + \tilde{D}_0(\xi_2))$$

and the discrete one

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_2)).$$

Taking into account that $\xi \in \hbar\pi$, we obtain the conclusions below.

Let us denote by $\tilde{\Phi}_d$ and $\tilde{\Phi}$ vectors with components $(\tilde{F}_d, \tilde{G}_d)^T$ and $(\tilde{F}, \tilde{G})^T$, $\tilde{C}$ and $\tilde{c}$ are vectors with components– $(\tilde{C}_0, \tilde{D}_0)^T$ $(\tilde{c}_0, \tilde{d}_0)^T$, respectively. Then we write

$$\tilde{C} = K^{-1}\tilde{\Phi}, \qquad \tilde{c} = k^{-1}\tilde{\Phi}_d,$$

where C_1, C_2 and c_1, c_2, j-th coordinates of vectors $\tilde{C}, \tilde{c}$, $j = 1, 2$. Therefore,

$$(\chi_h\tilde{u})(\xi) - \tilde{u}_d(\xi) = \chi_h A^{-1}_{\neq}(\xi)\left((\tilde{C}_0(\xi_1) - \tilde{c}_0(\xi_1)) + (\tilde{D}_0(\xi_2) - \tilde{d}_0(\xi_2)\right) =$$

$$= \chi_h A^{-1}_{\neq}(\xi)\left((K^{-1}\tilde{\Phi})_1(\xi_1) - (k^{-1}\tilde{\Phi}_d)_1(\xi_1) + (K^{-1}\tilde{\Phi})_2(\xi_2) - (k^{-1}\tilde{\Phi}_d)_2(\xi_2)\right).$$

It implies that it is enough to estimate the norm $||\Xi_h K^{-1}\Phi - k^{-1}\Phi_d||_{\mathbf{H}^{s_0}(\hbar\mathbf{T})}$. We write

$$\Xi_h K^{-1}\Phi - k^{-1}\Phi_d = (\Xi_h K^{-1}\Phi - K^{-1}\Xi_h\Phi) + (K^{-1}\Xi_h\Phi - k^{-1}\Phi_d).$$

We use Corollary 1 for an estimate of first summand. We obtain

$$||\Xi_h K^{-1}\Phi - K^{-1}\Xi_h\Phi||_{s_0} \leq const\ h^{s-1}||\Phi||_{s_0}$$

and then the second summand, we represent as the sum

$$K^{-1}\Xi_h\Phi - k^{-1}\Phi_d = (K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi) + (k^{-1}\Xi_h\Phi - k^{-1}\Phi_d),$$

each summand we will estimate separately.

Let us consider $k^{-1}\Xi_h\Phi - k^{-1}\Phi_d$. Since norm of the operator k^{-1} is bounded by a constant non-depending on h, we obtain

$$\begin{aligned} ||k^{-1}\Xi_h\Phi - k^{-1}\Phi_d||_{s_0} &\leq const||\Xi_h\Phi - \Phi_d||_{s_0} \\ &\leq const(||\chi_h F - F_d||_{s_0} + ||\chi_h G - G_d||_{s_0}). \end{aligned}$$

Last step is to estimate, for example, $||\chi_h F - F_d||_{s_0}$. We have

$$||\chi_h F - F_d||_{s_0}^2 = \int_{-\hbar\pi}^{\hbar\pi} |\tilde{f}(\xi_2)A_0^{-1}(\xi_2) - \tilde{f}_d(\xi_2)a_0^{-1}(\xi_2)|^2(1+|\xi_2|)^{2s_0}d\xi_2 \leq$$

$$\leq const\ h^{2æ-2}\int_{-\hbar\pi}^{\hbar\pi} |\tilde{f}(\xi_2)|^2(1+|\xi_2|)^{2s_0}d\xi_2 \leq const\ h^{2æ-2}||f||_{s_0}^2$$

according to coincidence for f_d and f on $\hbar\mathbf{T}$ and the estimate of Lemma 4.

Left summands can be estimated by the following operator identity

$$K^{-1} - k^{-1} = K^{-1}(k-K)k^{-1}.$$

(Let us remind that an invertibility of the operator k follows from an invertibility of the operator K.) Therefore, comparing over $\hbar\mathbf{T}$

$$K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi = \Xi_h(K^{-1} - k^{-1})\Xi_h\Phi = \Xi_h K^{-1}(k-K)k^{-1}\Xi_h\Phi,$$

and taking into account Lemma 5, we have the estimate

$$||K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi||_{s_0} \leq const\ h^{æ-1}||\Phi||_{s_0} \leq const\ h^{æ-1}(||f||_{s_0} + ||g||_{s_0}).$$

Summing obtained estimates, we complete the proof, taking into account mapping properties of operators which admit to obtain H^s-norm. □

2.5.2 Nonlocal Discrete Boundary Value Problem

For such f_d, g_d and the symbol $A_d(\xi)$, we obtain the following result.

Theorem 7 *Let $f, g \in S(\mathbf{R})$, æ > 1. Then we have the following estimate for solutions u and u_d of the continuous problem (2.10), (2.13) and the discrete one (2.3), (2.7)*

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq C(f, g)h^{\beta},$$

where the const $C(f, g)$ depends on functions f, g, $\beta > 0$ can be an arbitrary number.

Proof First, let us note that solvability conditions for the problem (2.10), (2.13) guarantee satisfying solvability conditions of the problem (2.3), (2.7) for enough small h.

Further, we need to compare two functions (2.9) and (2.14), more exactly their inverse discrete Fourier transform and inverse Fourier transform at points $\tilde{x} \in K_d$. We have

$$u_d(\tilde{x}) - u(\tilde{x}) = \frac{1}{4\pi^2}\left(\int\limits_{\hbar\mathbf{T}^2} e^{i\tilde{x}\cdot\xi}\tilde{u}_d(\xi)d\xi - \int\limits_{\mathbf{R}^2} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi\right) =$$

$$= \frac{1}{4\pi^2}\int\limits_{\mathbf{R}^2\setminus\hbar\mathbf{T}^2} e^{i\tilde{x}\cdot\xi}A_{\neq}^{-1}(\xi)\left(A_{\neq}(\xi_1, 0)\tilde{g}(\xi_1) + A_{\neq}(0, \xi_2)\tilde{f}(\xi_2)\right)d\xi,$$

since according to our choice for A_d, f_d, g_d the functions $\tilde{u}_d$ and $\tilde{u}$ coincide in points $\xi \in \hbar\mathbf{T}^2$.

We will estimate one summand.

$$\left|\frac{1}{4\pi^2}\int\limits_{\mathbf{R}^2\setminus\hbar\mathbf{T}^2} e^{i\tilde{x}\cdot\xi}A_{\neq}^{-1}(\xi)A_{\neq}(\xi_1, 0)\tilde{g}(\xi_1)d\xi\right|$$

$$\leq C\int\limits_{\hbar\pi}^{+\infty}\frac{d\xi_2}{(1 + |\xi_1| + |\xi_2|)^{æ}}\int\limits_{\hbar\pi}^{+\infty}|\xi_1|^{-\gamma}d\xi_1,$$

since $\tilde{g} \in S(\mathbf{R})$. It implies the required estimate. □

2.6 Conclusion

We have considered very simple variants of discrete boundary value problems for digital operators. Particularly, our considerations are based on Theorem 2 which gives a general solution of our discrete equation. There are a lot of different situations in this studying, for example, the case $n > 1$ which permits to use more complicated boundary conditions, or the case $n \in \mathbf{N}, n < 0$ which admits to introduce more unknowns in Eq. (2.2) and potential like discrete operators similar considered ones in [18]. We work in this direction and present these studies in forthcoming publications.

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