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# On a Certain Functional Equation and Its Application to the Schwarz Problem

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**Abstract:** The Schwarz problem for $J$-analytic functions in an ellipse is considered. In this case, the matrix $J$ is assumed to be two-dimensional with different eigenvalues located above the real axis. The Schwarz problem is reduced to an equivalent boundary value problem for the scalar functional equation depending on the real parameter $l$. This parameter is determined by the Jordan basis of the matrix $J$. An analysis of the functional equation was performed. It is shown that for $l \in [0, 1]$, the solution of the Schwarz problem with matrix $J$ exists uniquely in the Hölder classes in an arbitrary ellipse.

**Keywords:** $J$-analytic functions; $\lambda$-holomorphic functions; matrix eigenvalue; ellipse; functional equation

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## 1. Introduction

This paper considers the Schwarz problem for $J$-analytic functions. These functions were first considered by Avron Douglis [1], who called them hyperanalytic. The theory of $J$-analytic functions was further developed by D. Pascali [2], J. Horvath [3], R. Gilbert [4], B. Bojarski [5], G. Hile [6], A.P. Soldatov [7–9], and many other authors. In particular, an analogue of the theory of analytic functions was constructed for them [5,8,10], so now, we call these functions Douglis analytic functions.

It is well known [5–7] that solutions to the Laplace equation in a simply connected domain are described as the real parts of analytic functions. Solutions of more general elliptic equations with real analytic coefficients are also expressed through analytic functions. A unified approach to the study of such representations was suggested by I.N. Vekua [10]. Later on, A.V. Bitsadze [11] obtained a representation of the general solution to elliptic systems through analytic vector-functions and their derivatives. Recently (A. P. Soldatov [9], R. Ieh [12]), it turned out that using Douglis analytic functions, the Bitsadze representation can be substantially simplified. It is possible to say that with respect to elliptic equations and systems with constant (and only higher) coefficients, these functions play the same role as the analytic functions with respect to the Laplace equation. Similar properties were identified (N.A. Zhura [13]) for second-order systems, which are elliptic according to Douglis-Nirenberg.

Thus, the study of various boundary value problems for Douglis analytic functions is relevant. The Schwarz problem considered in the paper is one of them. It should be noted that B. Bojarski [5] studied the Schwarz problem for triangular matrices in Hölder classes. He obtained significant results. However, for non-triangular matrices, this problem has been studied insufficiently.

It should be noted that A. P. Sodatov in [9] proved the Fredholm solvability of the Schwarz problem for the case when all the eigenvalues of the matrix $J$ lie in the upper half-plane. In this case, the boundary of the domain $D$ should be the Lyapunov contour. It should also be noted that the work [14] is closely related to the topic of this study.

The problem is describing a certain class of matrices $J$ and domains $D$ for which the Schwarz problem is solvable. At the same time, the solvability should be for a certain class of functions.

Let us take an arbitrary ellipse as region $D$. Let the matrix $J$ be two-dimensional, with different eigenvalues. Then, the goal of the current work is as follows. It is necessary to obtain sufficient conditions on the matrix $J$ such that the solution of the Schwarz problem exists and is unique in the Hölder classes of an arbitrary ellipse.

The authors propose the following original method. Namely, the Schwarz problem for two-dimensional matrices is reduced to an equivalent scalar functional equation. In this equation, the matrix $J$ is represented by some real parameter $l$. Then, we find a sufficient condition for the parameter $l$, under which the Schwarz problem is solvable in an arbitrary ellipse in the Hölder classes.

## 2. Statement of the Problem

Let us assume that the matrix $J \in \mathbf{C}^{\ell\times\ell}$ has no real eigenvalues. Let an $\ell$-vector-function $\phi = \phi(z) \in C^1(D)$, where $D$ is a domain in $\mathbf{R}^2$. Let us consider the following homogeneous elliptic system of first-order partial differential equations in a domain $D$:

$$\frac{\partial\phi}{\partial y} - J\cdot\frac{\partial\phi}{\partial x} = 0, \quad z\in D. \tag{1}$$

**Definition 1** ([5,7,9,15])**.** *Function $\phi = \phi(z)$, considered as a solution to system* (1)*, is called a Douglis analytic function, or a $J$-analytic function with the matrix $J$.*

Also, here is a scalar analogue of Definition 1, which will be substantially used below.

**Definition 2** ([8,9,16])**.** *Let us assume that $J = \lambda \in \mathbf{C}$, $\operatorname{Im}\lambda \neq 0$. The scalar function $f_\lambda = f_\lambda(z) \in C^1(D)$, satisfying the equation*

$$\frac{\partial f_\lambda}{\partial y} - \lambda\cdot\frac{\partial f_\lambda}{\partial x} = 0, \quad z\in D, \tag{2}$$

*is called a $\lambda$-holomorphic function.*

Examples of $J$-analytic and $\lambda$-holomorphic functions are vector polynomials of the form

$$\phi(z) = \sum_{k=0}^{m}(xE+yJ)^k\cdot c_k, \quad c_k\in\mathbf{C}^\ell, \quad \ell = 1,2,3,\ldots,$$

where $E$ is the identity matrix for $\ell\geq 2$. If $\ell = 1$, we must put $xE+yJ = x+\lambda y$.

Let us consider for System (1) the following boundary *Schwarz problem* [5,8,9,13,17].

Let the finite domain $D\subset\mathbf{R}^2$ be bounded by a smooth contour $\Gamma$. A $J$-analytic with the matrix $J$ in the domain $D$ function $\phi(z)\in C(\overline{D})$, which satisfies the boundary condition

$$\operatorname{Re}\phi(z)\big|_\Gamma = \psi(\omega), \quad \omega\in\Gamma, \tag{3}$$

where the real $\ell$-vector-function of $\psi(\omega)\in C(\Gamma)$ is set needs to be found.

If $\psi(\omega)\equiv 0$, then we talk about the homogeneous Schwarz problem:

$$\operatorname{Re}\phi(z)\big|_\Gamma = 0. \tag{4}$$

The obvious solutions to Problem (4) are constant functions $\phi(z)\equiv ic,\ \ c\in\mathbf{R}^\ell$, which we call *trivial solutions*. If $\ell\geq 2$, then non-constant solutions of the homogeneous Problem (4) are possible. Here is an example for $\ell = 2$.

**Example 1.** *Let*

$$J = \begin{pmatrix} 10i & -\frac{49}{3} \\ -\frac{27}{7} & -6i \end{pmatrix}, \quad \phi(z) = \begin{pmatrix} -6x(x^2+y^2-1) + (-20y^3 - 12x^2y + 18y)i \\ \frac{54}{7}\,y(x^2+y^2-1) + \big(-\frac{24}{7}\,x^3 + \frac{18}{7}\,x\big)i \end{pmatrix}. \tag{5}$$

The matrix $J$ in (5) has eigenvalues $\lambda = 3i$, $\mu = i$. The vector-polynomial of the third degree $\phi(z)$ is a $J$-analytic function with the given matrix $J$. We have $\operatorname{Re}\phi(z)\big|_\Gamma = 0$ on the unit circle $\Gamma\colon\ x^2+y^2 = 1$. Here, according to Formulas (27) and (28) below, the number $|l_2(J)| = 8$.

## 3. Transformation of the Schwarz Problem to a Boundary Value Problem for a Scalar Functional Equation

Let the matrix $J \in \mathbf{C}^{2\times 2}$ have different eigenvalues $\mu \neq \lambda$. Denote by $\mathbf{x}$ the eigenvector corresponding to $\mu$, and by $\mathbf{y}$ the eigenvector corresponding to $\lambda$. Denote also by $J_1$ and $Q$ the Jordan form and the Jordan basis of matrix $J$, respectively:

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad J_1 = \operatorname{diag}(\mu, \lambda) = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}, \quad Q = (\mathbf{x}, \mathbf{y}), \quad J = QJ_1Q^{-1}. \tag{6}$$

*Let us assume that one of the eigenvectors of the matrix $J$, for a certain vector $\mathbf{y}$, is not are real.* We expand *complex conjugation* $\overline{\mathbf{y}}$ of the vector $\mathbf{y}$ in the Jordan basis $\mathbf{x}, \mathbf{y}$ of the matrix $J$:

$$\overline{\mathbf{y}} = l_1\mathbf{x} + l_2\mathbf{y},\ \ l_1, l_2 \in \mathbf{C},\ \ l_1 = l_1(J) = \frac{\det(\overline{\mathbf{y}}, \mathbf{y})}{\det(\mathbf{x}, \mathbf{y})},\ l_2 = l_2(J) = \frac{\det(\mathbf{x}, \overline{\mathbf{y}})}{\det(\mathbf{x}, \mathbf{y})}. \tag{7}$$

In (7), Cramer formulas were applied to find the numbers $l_1, l_2$. In this article, only the number $l_2(J)$ will be used. The following statement is true.

**Lemma 1.** *The module $|l_2|$ of the number $l_2 = l_2(J)$ in (7) does not depend on the choice of the Jordan basis $Q$ of the matrix $J$. In addition, the number $l_2(J)$ itself is invariant with respect to real transformations, that is, it coincides for matrices $J$ and $BJB^{-1}$, where $B \in \mathbf{R}^{2\times 2}$.*

**Proof.** The eigenvectors of the matrix $J$ are defined with exactness up to the complex multiplier. Therefore, let $Q^* = (a\mathbf{x}, b\mathbf{y})$, $a, b \neq 0$ be another Jordan basis of the matrix $J$. Denote by $l_2^*$ the number that (7) calculated from the Jordan basis $Q^*$:

$$|l_2^*| = \left|\frac{\det(a\mathbf{x}, \overline{b\mathbf{y}})}{\det(a\mathbf{x}, b\mathbf{y})}\right| = \left|\frac{a\overline{b}\cdot\det(\mathbf{x}, \overline{\mathbf{y}})}{ab\cdot\det(\mathbf{x}, \mathbf{y})}\right| = \left|\frac{a\overline{b}}{ab}\right| \cdot \left|\frac{\det(\mathbf{x}, \overline{\mathbf{y}})}{\det(\mathbf{x}, \mathbf{y})}\right| = |l_2|,$$

which is exactly what was required.

Let us prove the invariance of the number $l_2$ with respect to real transformations. Since the vectors $\mathbf{x}, \mathbf{y}$ are eigenvectors for the matrix $J$, then $J\mathbf{x} = \mu\mathbf{x}$, $J\mathbf{y} = \lambda\mathbf{y}$. We write these two equations in the following equivalent form: $BJB^{-1}\cdot B\mathbf{x} = \mu B\mathbf{x}$, $BJB^{-1}\cdot B\mathbf{y} = \lambda B\mathbf{y}$.

Thus, $Q^* = (\mathbf{x}', \mathbf{y}') = (B\mathbf{x}, B\mathbf{y})$ is the Jordan basis of matrix $J^* = BJB^{-1}$. It is similar to matrix $J$, and therefore, it has the same eigenvalues $\mu, \lambda$. Let us denote, as above, by $l_2^*$ the number that (7) calculated from the Jordan basis of the matrix $Q^*$. Then,

$$l_2^* = \frac{\det(B\mathbf{x}, \overline{B\mathbf{y}})}{\det(B\mathbf{x}, B\mathbf{y})} = \frac{\det(B\mathbf{x}, B\overline{\mathbf{y}})}{\det(B\mathbf{x}, B\mathbf{y})} = \frac{\det[B\cdot(\mathbf{x}, \overline{\mathbf{y}})]}{\det[B\cdot(\mathbf{x}, \mathbf{y})]} = \frac{\det B\cdot\det(\mathbf{x}, \overline{\mathbf{y}})}{\det B\cdot\det(\mathbf{x}, \mathbf{y})} = l_2,$$

which is exactly what was required. $\square$

Thus, the real number $|l_2(J)|$ is an invariant characteristic of the matrix $J$. Taking into account (7) and the equalities $J\mathbf{x} = \mu\mathbf{x}$, $J\mathbf{y} = \lambda\mathbf{y}$, we have

$$\begin{gathered} J\overline{\mathbf{y}} = J(l_1\mathbf{x} + l_2\mathbf{y}) = \mu l_1\mathbf{x} + \lambda l_2\mathbf{y} = \mu l_1\mathbf{x} + \lambda l_2\mathbf{y} + \mu l_2\mathbf{y} - \mu l_2\mathbf{y} = \\ = \mu l_1\mathbf{x} + \mu l_2\mathbf{y} + \lambda l_2\mathbf{y} - \mu l_2\mathbf{y} = \mu(l_1\mathbf{x} + l_2\mathbf{y}) + (\lambda - \mu)l_2\mathbf{y} = \\ = \mu\overline{\mathbf{y}} + (\lambda - \mu)l_2\mathbf{y} = \mu\overline{\mathbf{y}} + (\mu - \lambda)l_2(-\mathbf{y}). \end{gathered}$$

In addition, it is obvious that $J(-\mathbf{y}) = \lambda(-\mathbf{y})$. Therefore, the matrix $J_1' = (Q')^{-1}JQ'$ of the operator $J$ in the special basis $Q' = (\overline{\mathbf{y}}, -\mathbf{y})$ has the form

$$J_1' = \begin{pmatrix} \mu & 0 \\ (\mu - \lambda)l_2 & \lambda \end{pmatrix}, \quad \mu \neq \lambda. \tag{8}$$

Let us substitute the expression $J = Q'J_1'(Q')^{-1}$ in (1) and then multiply both parts by $(Q')^{-1}$ :

$$\frac{\partial\phi}{\partial y} - Q'J_1'(Q')^{-1} \cdot \frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial}{\partial y}\left[(Q')^{-1}\phi\right] - J_1' \cdot \frac{\partial}{\partial x}\left[(Q')^{-1}\phi\right] = 0.$$

The second equality of the obtained equalities will be explained in detail, taking into account (8):

$$\frac{\partial}{\partial y}\begin{pmatrix} g \\ F \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ (\mu - \lambda)l_2 & \lambda \end{pmatrix} \cdot \frac{\partial}{\partial x}\begin{pmatrix} g \\ F \end{pmatrix} = 0, \quad (g, F)^{\mathrm{T}} = (Q')^{-1}\phi = (\overline{\mathbf{y}}, -\mathbf{y})^{-1}\phi. \tag{9}$$

From (9), according to (2), it follows that the function $g = g_\mu(z)$ will be $\mu$-holomorphic. In this case, for the function $F = F(x, y)$, there is a relation

$$\frac{\partial F}{\partial y} + l_2(\lambda - \mu)\,\frac{\partial g}{\partial x} - \lambda\,\frac{\partial F}{\partial x} = 0. \tag{10}$$

Substituting $F = l_2 g_\mu + f$ in (10) after simple transformations gives the identity

$$\frac{\partial f}{\partial y} - \lambda\,\frac{\partial f}{\partial x} \equiv 0,$$

that is, $f = f_\lambda(z)$ is an arbitrary $\lambda$-holomorphic function. Thus, the general solution (9) will be the functions

$$g(z) = g_\mu(z), \quad F(z) = l_2 g_\mu(z) + f_\lambda(z). \tag{11}$$

Next, let us denote

$$g(z)\big|_\Gamma = u(x, y) + iv(x, y), \quad F(z)\big|_\Gamma = p(x, y) + iq(x, y), \tag{12}$$

where functions $u, v, p, q \in C(\Gamma)$ are real. Assuming $\phi(z) \in C(\overline{D})$ and taking into account (9) and (11), the notation in (12) is correct. Let us continue:

$$\mathbf{y} = (a_1, a_2) = (a + bi,\ c + di), \quad a, b, c, d \in \mathbf{R} \tag{13}$$

is the eigenvector of the matrix $J$, which, by condition, *is not a real vector*. Then, taking into account (9), (12), and (13), the general solution to $\phi(z)$ of Equation (1) for $\ell = 2$ and the diagonalizable matrix $J$ can be written as

$$\phi(z) = Q' \cdot (g, F)^{\mathrm{T}} = (\overline{\mathbf{y}}, -\mathbf{y}) \cdot (g, F)^{\mathrm{T}} = \begin{pmatrix} \overline{a}_1 & -a_1 \\ \overline{a}_2 & -a_2 \end{pmatrix} \cdot \begin{pmatrix} u + iv \\ p + iq \end{pmatrix}. \tag{14}$$

Let us write down the Schwarz problem (3) for $\ell = 2$ :

$$\operatorname{Re}\phi(z)\big|_{\Gamma} = \big(\psi_1(\omega), \psi_2(\omega)\big)^{\mathrm{T}}, \quad \omega \in \Gamma. \tag{15}$$

The boundary condition in (15), taking into account (14), is written as

$$\begin{cases} \operatorname{Re}[\,\overline{a}_1(u+iv) - a_1(p+iq)\,]\big|_{\Gamma} = \psi_1, \\ \operatorname{Re}[\,\overline{a}_2(u+iv) - a_2(p+iq)\,]\big|_{\Gamma} = \psi_2. \end{cases} \tag{16}$$

The following statement is true.

**Lemma 2.** *In the solution of* (16)*, which is an inhomogeneous algebraic system with respect to real functions, the variables $u, v$ can be found in the following form:*

$$u = p + r(\psi_1, \psi_2), \quad v = -q + h(\psi_1, \psi_2), \tag{17}$$

*where $r(\cdot)$, $h(\cdot)$ are linear functions of their variables.*

**Proof.** The equalities in (16) on $\Gamma$, taking into account the notation in (13), have the form

$$\begin{cases} \operatorname{Re}[\,\overline{a}_1(u+iv) - a_1(p+iq)\,] = au + bv - ap + bq = \psi_1, \\ \operatorname{Re}[\,\overline{a}_2(u+iv) - a_2(p+iq)\,] = cu + dv - cp + dq = \psi_2. \end{cases} \tag{18}$$

Obviously, after substituting (17) into (18), the variables $p, q$ are reduced. To define the variables $r, h$, we have the following system with the determinant $\Delta \neq 0$ :

$$\begin{cases} ar + bh = \psi_1, \\ cr + dh = \psi_2, \end{cases} \qquad \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. \tag{19}$$

In (19), $\Delta \neq 0$, since by condition, the vector $\mathbf{y}$ is not a multiple to a real one. From (19), by Cramer formulas, we have

$$r(\psi_1, \psi_2) = \frac{d\psi_1 - b\psi_2}{\Delta}, \quad h(\psi_1, \psi_2) = \frac{a\psi_2 - c\psi_1}{\Delta}, \quad \Delta \neq 0, \tag{20}$$

which is exactly what was required. $\square$

Further, note that the pair of equalities of real functions in (17) equals one complex functional equality:

$$(u - iv) - (p + iq) = r - ih. \tag{21}$$

Taking into account the notations in (11) and (12), Equality (21) will have the form

$$\overline{g}_{\mu} - l_2 \cdot g_{\mu} - f_{\lambda} = r - ih.$$

Multiplying both parts of this equality by $(-1)$, we have

$$f_{\lambda} - \overline{g}_{\mu} + l_2 \cdot g_{\mu} = -r + ih = \varphi, \quad l_2 \in \mathbf{C}. \tag{22}$$

The functional Equation (22) will be further considered as the problem of finding functions $f_{\lambda} = f_{\lambda}(z) \in C(\overline{D})$ and $g_{\mu} = g_{\mu}(z) \in C(\overline{D})$, which on $\Gamma = \partial D$ satisfy Equality (22). In this case, the complex boundary function $\varphi = \varphi(\omega) \in C(\overline{D})$ is set.

**Remark 1.** *Let us write the number $l_2 = l_2(J) \in \mathbf{C}$ in (7) in exponential form: $l_2 = |l_2| e^{i(\alpha+\pi)}$, that is, $\alpha = \arg(-l_2)$. Let us make the following substitutions in (22):*

$$f_{\lambda} = f^{*}_{\lambda} \cdot e^{\frac{i\alpha}{2}}, \quad g_{\mu} = -g^{*}_{\mu} \cdot e^{-\frac{i\alpha}{2}}, \quad \varphi = \varphi^{*} \cdot e^{\frac{i\alpha}{2}}. \tag{23}$$

Taking into account the identity $l_2 = |l_2|e^{i(\alpha+\pi)} = -|l_2|e^{i\alpha}$, we have

$$e^{\frac{i\alpha}{2}} f_\lambda^* + e^{\frac{i\alpha}{2}} \overline{g_\mu^*} - |l_2|e^{i\alpha}(-e^{-\frac{i\alpha}{2}})g_\mu^* = e^{\frac{i\alpha}{2}} \varphi^*, \quad \alpha = \arg(-l_2),$$

from which, after reduction by $e^{\frac{i\alpha}{2}}$, we obtain the equality

$$f_\lambda^* + \overline{g^*}_\mu + |l_2| \cdot g_\mu^* = \varphi^*. \tag{24}$$

Thus, *we study the Equation* (24). This is studied below. The transformations carried out in this paragraph are formalized in the following statement.

**Theorem 1.** *Let the matrix $J \in \mathbf{C}^{2\times 2}$ have different eigenvalues $\mu, \lambda$, and let $Q = (\mathbf{x}, \mathbf{y})$ be its Jordan basis, where the eigenvector $\mathbf{y}$ is not are real. Then, the Schwarz problem* (15) *in the class of functions $\phi(z) \in C(\overline{D})$, $\psi_1(\omega), \psi_2(\omega) \in C(\Gamma)$ is equivalent to the problem of finding the functions $f_\lambda(z)$, $g_\mu(z) \in C(\overline{D})$ as solutions to the following scalar functional equation:*

$$\begin{gathered} f_\lambda(z) + \overline{g}_\mu(z) + l \cdot g_\mu(z)\big|_\Gamma = \varphi(\omega), \quad \varphi(\omega) \in C(\Gamma), \\ l = l(J) = |l_2(J)| = \left|\frac{\det(\mathbf{x}, \overline{\mathbf{y}})}{\det(\mathbf{x}, \mathbf{y})}\right|. \end{gathered} \tag{25}$$

*In this case, the boundary function $\varphi(\omega)$ is defined by Equalities* (20)–(23).

**Proof.** Let there exist a solution $\phi(z) \in C(\overline{D})$ of Problem (15). Then, by construction and taking into account Remark 1, there exists a solution $f_\lambda(z)$, $g_\mu(z) \in C(\overline{D})$ of Equation (25) with an appropriate boundary function.

In the opposite direction, let the number $l$ be found for a given matrix $J$. Let there exist a solution $f_\lambda^* = f_\lambda^*(z)$, $f_\mu^* = f_\mu^*(z)$ of Equation (25) in the domain $D$ *for any* boundary function $\varphi^*(\omega) \in C(\Gamma)$. It must be shown that in this case, the solution to Problem (15) exists for any continuous boundary vector-function $\psi(\omega) = \big(\psi_1(\omega), \psi_2(\omega)\big)$.

Indeed, let the function $(\psi_1, \psi_2)$ and the matrix $J$. be given. Taking into account (23) and (24), we arrive at a solution to Problem (25) for the boundary function $\varphi^* = e^{-\frac{i\alpha}{2}}\varphi = e^{-\frac{i\alpha}{2}}(-r + ih)$ and the numbers $l = |l_2(J)|$. In doing so, we use (20). Recall that $\alpha = \arg(-l_2)$. Further taking into account (11), (14), (23), and the reversibility of the transformations, we have

$$\begin{gathered} \phi(z) = Q' \cdot (g, F)^{\mathrm{T}} = Q' \cdot \big(g_\mu,\ l_2 g_\mu + f_\lambda\big)^{\mathrm{T}} = Q' \cdot \big(-e^{-\frac{i\alpha}{2}} g_\mu^*,\ -l_2 \cdot e^{-\frac{i\alpha}{2}} g_\mu^* + e^{\frac{i\alpha}{2}} f_\lambda^*\big)^{\mathrm{T}}, \\ Q' = (\overline{\mathbf{y}}, -\mathbf{y}), \quad \alpha = \arg(-l_2), \end{gathered}$$

which is exactly what was required. □

Let us express the number $l = |l_2|$ in (25) in terms of the coefficients of the matrix $J$ in Notation (6). As above, let $\mathbf{x}_\mu$, $\mathbf{y}_\lambda$, the eigenvectors of the matrix $J$, correspond to its eigenvalues $\mu, \lambda$. *Let* $a_{21} \neq 0$. Using the Cayley–Hamilton theorem, it is easy to show that eigenvectors can be written as $\mathbf{x}_\mu = (a_{11} - \lambda,\ a_{21})$, $\mathbf{y}_\lambda = (a_{11} - \mu,\ a_{21})$. As a result, by Formula (7), we have

$$l_2 = \frac{\det(\mathbf{x}_\mu, \overline{\mathbf{y}}_\lambda)}{\det(\mathbf{x}_\mu, \mathbf{y}_\lambda)} = \frac{\begin{vmatrix} a_{11} - \lambda & \overline{a_{11} - \mu} \\ a_{21} & \overline{a}_{21} \end{vmatrix}}{\begin{vmatrix} a_{11} - \lambda & a_{11} - \mu \\ a_{21} & a_{21} \end{vmatrix}} =$$

$$= \frac{(a_{11} - \lambda)\overline{a}_{21} - a_{21}(\overline{a_{11} - \mu})}{a_{11}a_{21} - \lambda a_{21} - a_{21}a_{11} + \mu a_{21}} = \frac{a_{11}\overline{a}_{21} - a_{21}\overline{a}_{11} - \lambda \overline{a}_{21} + \overline{\mu} a_{21}}{-\lambda a_{21} + \mu a_{21}} =$$

$$= \frac{a_{11}\overline{a}_{21} - \overline{a_{11}\overline{a}_{21}} - \lambda\overline{a}_{21} + \overline{\mu}a_{21}}{(\mu-\lambda)a_{21}} = \frac{2i \cdot \operatorname{Im}(a_{11}\cdot\overline{a}_{21}) + \overline{\mu}a_{21} - \lambda\overline{a}_{21}}{(\mu-\lambda)\cdot a_{21}}. \tag{26}$$

Thus, by virtue of (25) and (26),

$$l = |l_2(J)| = \frac{\left|2i \cdot \operatorname{Im}(a_{11}\cdot\overline{a}_{21}) + \overline{\mu}a_{21} - \lambda\overline{a}_{21}\right|}{\left|(\mu-\lambda)\cdot a_{21}\right|}, \quad a_{21} \neq 0, \quad \mu \neq \lambda. \tag{27}$$

*Let* $a_{12} \neq 0$. Then, the eigenvectors have the form $\mathbf{x}_\mu = (a_{12}, a_{22} - \lambda)$, $\mathbf{y}_\lambda = (a_{12}, a_{22} - \mu)$. By performing transformations similar to (26), we obtain

$$l = |l_2(J)| = \frac{\left|2i \cdot \operatorname{Im}(a_{22}\cdot\overline{a}_{12}) + \overline{\mu}a_{12} - \lambda\overline{a}_{12}\right|}{\left|(\mu-\lambda)\cdot a_{12}\right|}, \quad a_{12} \neq 0, \quad \mu \neq \lambda. \tag{28}$$

Finally, if the matrix $J$ is diagonal, then both of its eigenvectors are real. Therefore, according to (7), $|l_2(J)| = 1$. Note that by Formulas (27) and (28) for matrix $J$ in (5), the number $l = |l_2(J)| = 8$.

## 4. Application of the Functional Equation (25) to the Schwarz Problem in an Ellipse

Equation (25) depends substantially on the parameter $l$ and is of interest as an independent function theory problem. For $l = 0$, it was studied in [15]. In this work, it was proved:

**Theorem 2** ([15])**.** *(A. P. Soldatov) Let* $\Gamma = \partial D$ *be Lyapunov contour, let* $\varphi(\omega) \in H^\sigma(\Gamma)$*, and let* $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\mu > 0$*. Then, the solution of the equation* $f_\lambda(z) + \overline{g}_\mu(z)\big|_\Gamma = \varphi(\omega)$ *exists and is unique up to constants in function classes* $f_\lambda, g_\mu \in H^\sigma(\overline{D})$*.*

However, for $l > 0$, the situation becomes more complicated: non-constant solutions of the homogeneous problem are possible. Let us consider the following

**Example 2.** *Let* $l = 5$, $\lambda = 2i$, $\mu = i$*. Direct calculations show that a pair of quadratic functions* $f_\lambda(z) = -3i(x+2iy)^2$ *and* $g_\mu(z) = i(x+iy)^2 - i$ *will be the solution of the homogeneous problem* $f_\lambda(z) + \overline{g}_\mu(z) + 5g_\mu(z)\big|_\Gamma = 0$ *in the ellipse* $K$ *with the* $\Gamma$ *border:* $x^2 + 8y^2 = 4$.

In this paragraph, we study Equation (25) in an ellipse. A few words about terminology. By an ellipse, we mean both the curve $\Gamma$ in the plane and the domain $K$, bounded by the curve $\Gamma$, depending on the context.

Let the real parameters $r_1 > 0$, $r_2 > 0$ be the semi-axes of the ellipse $\Gamma$, and let $\alpha \in [0, 2\pi)$ be the angle in the positive direction between the ellipse semi-axis of length $r_1$ and the positive direction of axis $Ox$. Then, the $\omega(t)$ parametrization of the $\Gamma$ ellipse with a center at the origin of coordinates can be written in the form

$$\Gamma\colon\ \omega(t) = \begin{cases} x(t) = r_1\cos\alpha\cdot\cos t - r_2\sin\alpha\cdot\sin t \\ y(t) = r_1\sin\alpha\cdot\cos t + r_2\cos\alpha\cdot\sin t, \quad t \in [-\pi, \pi). \end{cases} \tag{29}$$

Let $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\mu > 0$, $r_1 > 0$, $r_2 > 0$, $\alpha \in [0, 2\pi)$. Denote

$$\begin{aligned} a = a(\alpha,\lambda,r_1,r_2) &= \frac{r_1\cos\alpha + ir_2\sin\alpha + \lambda(r_1\sin\alpha - ir_2\cos\alpha)}{2}, \\ b = b(\alpha,\lambda,r_1,r_2) &= \frac{r_1\cos\alpha - ir_2\sin\alpha + \lambda(r_1\sin\alpha + ir_2\cos\alpha)}{2}, \\ a_1 = a_1(\alpha,\mu,r_1,r_2) &= \frac{r_1\cos\alpha + ir_2\sin\alpha + \mu(r_1\sin\alpha - ir_2\cos\alpha)}{2}, \\ b_1 = b_1(\alpha,\mu,r_1,r_2) &= \frac{r_1\cos\alpha - ir_2\sin\alpha + \mu(r_1\sin\alpha + ir_2\cos\alpha)}{2}. \end{aligned} \tag{30}$$

The numbers in (30) are determined by the parameters $r_1, r_2, \alpha$ of the ellipse $\Gamma$, as well as the complex numbers $\lambda, \mu$. In [16], it was proved.

**Theorem 3.** *Let* $\operatorname{Im}\lambda > 0$; $\operatorname{Im}\mu > 0$; *and the numbers* $a, b, a_1, b_1$ *for the ellipse* $\Gamma$ (29) *be found using Formula* (30). *Let the following relations be fulfilled:*

$$\Delta_n = \Delta_n(l) = l^2 \cdot \left| \frac{b^n}{a^n} - \frac{b_1^n}{a_1^n} \right|^2 - \left| 1 - \frac{b^n}{a^n} \cdot \overline{\frac{b_1^n}{a_1^n}} \right|^2 \neq 0, \qquad l \in \mathbf{R}, \quad l \geq 0, \quad n \in \mathbf{N}. \tag{31}$$

*Then, for any boundary function* $\varphi(\omega) \in H^\sigma(\Gamma)$, *the solution of the problem* (25) *in an ellipse* $K$ *with a boundary* $\Gamma = \partial K$ *in the class of functions* $f_\lambda$, $g_\mu \in H^\sigma(\overline{K})$ *exists and is unique up to a constant.*

We obtain sufficient conditions for the parameter $l$, under which the relations in (31) are fulfilled in an arbitrary ellipse $\Gamma$. In this case, we use the following lemma [16].

**Lemma 3.** *Let* $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\mu > 0$. *Then, in* (30), $|b| < |a|$, $|b_1| < |a_1|$.

Next, taking into account (31), we formally express the parameter $l$ from the equation $\Delta_n(l) = 0$ :

$$l = \frac{\left| 1 - \frac{b^n}{a^n} \cdot \overline{\frac{b_1^n}{a_1^n}} \right|}{\left| \frac{b^n}{a^n} - \frac{b_1^n}{a_1^n} \right|}, \quad \left|\frac{b}{a}\right| < 1, \quad \left|\frac{b_1}{a_1}\right| < 1, \quad n = 1, 2, 3 \ldots \tag{32}$$

Let us prove the following statement.

**Lemma 4.** *Let* $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\mu > 0$, *and let in* (32),

$$\frac{b^n}{a^n} - \frac{b_1^n}{a_1^n} \neq 0, \quad n \in \mathbf{N}' \subseteq \mathbf{N}.$$

*Then, if* $\Delta_n(l) = 0$ *for* $n \in \mathbf{N}'$, *then the number* $l > 1$.

**Proof.** We denote the following, taking into account Lemma 3:

$$\frac{b^n}{a^n} = \xi_1 e^{it_1}, \quad \frac{b_1^n}{a_1^n} = \xi_2 e^{it_2}, \quad 0 \leq \xi_1 < 1, \; 0 \leq \xi_2 < 1, \quad n \in \mathbf{N}'. \tag{33}$$

For $n \in \mathbf{N}'$, Equality (32) is equivalent to the equality $\Delta_n(l) = 0$. We transform (32), taking into account the notations in (33):

$$\begin{aligned} l &= \frac{\left| 1 - \frac{b^n}{a^n} \cdot \overline{\frac{b_1^n}{a_1^n}} \right|}{\left| \frac{b^n}{a^n} - \frac{b_1^n}{a_1^n} \right|} = \frac{\left|1 - \xi_1\xi_2 e^{i(t_1 - t_2)}\right|}{\left|\xi_1 e^{it_1} - \xi_2 e^{it_2}\right|} = \\ &= \frac{\left|1 - \xi_1\xi_2 e^{i(t_1 - t_2)}\right|}{\left|e^{it_1}\right| \cdot \left|\xi_1 - \xi_2\, e^{i(t_2 - t_1)}\right|} = \frac{\left|1 - \xi_1\xi_2 e^{i(t_1 - t_2)}\right|}{\left|\xi_1 - \xi_2\, e^{i(t_2 - t_1)}\right|}. \end{aligned} \tag{34}$$

We denote $t = t_1 - t_2$ and write the inequality $l^2 > 1$, taking into account the last fraction in (34):

$$\left|1 - \xi_1\xi_2 e^{it}\right|^2 > |\xi_1 - \xi_2 e^{-it}|^2. \tag{35}$$

Let us rewrite (35) taking into account the Euler formula $e^{it} = \cos t + i \sin t$ :

$$(1 - \xi_1\xi_2 \cos t)^2 + \xi_1^2\xi_2^2 \sin^2 t > (\xi_1 - \xi_2 \cos t)^2 + \xi_2^2 \sin^2 t. \qquad (36)$$

Let us remove the parentheses in (36). After performing simple transformations and using the identity $\cos^2 t + \sin^2 t = 1$, we have the following inequalities:

$$\begin{gathered} 1 - 2\xi_1\xi_2 \cos t + \xi_1^2\xi_2^2 \cos^2 t + \xi_1^2\xi_2^2 \sin^2 t > \xi_1^2 - 2\xi_1\xi_2 \cos t + \xi_2^2 \cos^2 t + \xi_2^2 \sin^2 t, \\ 1 + \xi_1^2\xi_2^2 > \xi_1^2 + \xi_2^2, \quad 1 - \xi_1^2 - \xi_2^2 + \xi_1^2\xi_2^2 > 0, \\ 1 - \xi_1^2 - \xi_2^2(1 - \xi_1^2) > 0, \quad (1 - \xi_1^2)(1 - \xi_2^2) > 0, \quad \xi_1, \xi_2 \in [0, 1). \end{gathered} \qquad (37)$$

The inequality $(1 - \xi_1^2)(1 - \xi_2^2) > 0$ in (37) is independent of the parameter $t$ and holds for all $\xi_1 < 1$, $\xi_2 < 1$. Therefore, by virtue of (34) and (35) and the reversibility of the transformations performed, we have $l > 1$, which was required. Lemma 4 is proved. □

From Lemma 4, we have the following:

**Lemma 5.** *Let* $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\mu > 0$. *Then, for* $l \in [0, 1]$, *the relations are valid as* $\Delta_n(l) \neq 0$, $n \in \mathbf{N}$ (31) *for any ellipse* $\Gamma$.

**Proof.** If $n \in \mathbf{N}'$, then, by virtue of Lemma 4, the equality $\Delta_n(l) = 0$ can be fulfilled only if $l > 1$. Let $n \notin \mathbf{N}'$. Then,

$$\frac{b^n}{a^n} - \frac{b_1^n}{a_1^n} = 0.$$

Then, by virtue of (31), Lemma 3, and the two inequalities in (7), we have

$$\Delta_n(l) \equiv -\left|1 - \frac{b^n}{a^n} \cdot \frac{\overline{b_1^n}}{a_1^n}\right|^2 \neq 0, \quad \left|\frac{b}{a}\right| < 1, \quad \left|\frac{b_1}{a_1}\right| < 1,$$

which is exactly what was required. □

The obvious consequence of Lemma 5 is as follows:

**Lemma 6.** *When* $l \in [0, 1]$, *the statement of Theorem* 3 *is valid for any ellipse* $K$ *with boundary* $\Gamma = \partial K$.

In deriving Equation (25), we used the condition that the matrix $J$ has a complex eigenvector. However, in [17], we have the following:

**Theorem 4.** *Let the matrix* $J \in \mathbf{C}^{2\times 2}$ *have at least one real eigenvector. Let the domain* $D$ *be bounded by a Lyapunov contour* $\Gamma$. *Then, for any boundary function* $\big(\psi_1(\omega), \psi_2(\omega)\big) \in H^\sigma(\Gamma)$, *the solution to the Schwarz problem in* (15) *exists and is unique up to a vector-constant in the classes of functions* $\phi(z) \in H^\sigma(\overline{D})$.

Let us note in connection with this theorem that if the matrix $J$ has a real eigenvector, then according to (25), the number $l(J) = 1$. Thus, by virtue of Theorems 1, 3, and 4 and Lemma 6, the following basic statement is proved.

**Theorem 5.** *Let the matrix* $J \in \mathbf{C}^{2\times 2}$ *have different eigenvalues* $\lambda, \mu$, *where* $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\mu > 0$. *Let* $l = l(J)$ (25) *fulfill the condition* $l \in [0, 1]$.

*Then, for any boundary function* $\big(\psi_1(\omega), \psi_2(\omega)\big) \in H^\sigma(\Gamma)$, *the solution of the Schwarz problem in* (15) *in an arbitrary ellipse* $K$ *with boundary* $\Gamma$ *in the classes of functions* $\phi(z) \in H^\sigma(\overline{K})$ *exists and is unique up to a vector-constant.*

In conclusion, we note that the number $l(J) = 1$ is the same for matrices $J$ with a Jordan basis of the form $Q = (\mathbf{x}, \overline{\mathbf{x}})$.

## 5. Conclusions

Thus, instead of the Schwarz problem for two-dimensional matrices with different eigenvalues, we can study the functional Equation (25), where $l$ is an arbitrary real parameter. In this paper, it is shown that for the ellipse, this approach is quite promising. Moreover, Equation (25) is also interesting as an independent problem of the function theory. Further, it is necessary to consider the case when the parameter $l > 1$. The most interesting values are those $l > 1$ at which the solution of Equation (25) in Hölder classes exists and is unique precisely in an arbitrary ellipse with the boundary $\Gamma$. But finding such values of $l$ will require a more in-depth analysis of the relations (31), which is beyond the scope of the paper.

**Remark 2.** *We suggest using the theory of reproducing kernels [14]. It may permit simplification of the proofs. We will try to verify this opportunity in the near future.*

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