On Boundedness of Certain Integral Operators

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Abstract—We study mapping properties of two-dimensional linear integral operators in some weighted spaces with special kernels. The considered spaces are certain variant of Sobolev–Slobodetskii spaces and their generalizations related to Banach spaces. Sufficient conditions for boundedness for such operators in these spaces are obtained.

1. INTRODUCTION

Some systems of integral equations arise under studying model pseudo-differential equations in plane corners [1, 2], for example when we consider simple boundary conditions like Dirichlet or Neumann conditions. According to this fact it seems interesting to study some mapping properties of such integral operators. These two-dimensional integral operator act in Fourier images of Sobolev–Slobodetski spaces which are weighted spaces with prescribed weight. These mapping properties play important role under studying discrete boundary value problems [4, 5] since we try to compare such integral operators with their truncated analogues, these arise under considering discrete boundary value problems. Let us note that integral operators and corresponding equations do not arise under studying model boundary value problems in a half-space, there are systems of linear algebraic equations as a result of reduction [3].

We will give here a simple example. Let $\Omega \subset \mathbb{R}^2$ be the first quadrant. We consider the following Dirichlet boundary value problem [1, 5] in Sobolev–Slobodetskii space $H^s(\Omega)$

$$\begin{cases} (Au)(x) = 0, & x \in \Omega, \\ u_{|_{x_1=0}} = f(x_2), & u_{|_{x_2=0}} = g(x_1), \end{cases} \tag{1}$$

where A is a pseudo-differential operator with the symbol $A(\xi)$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \le |A(\xi)| \le c_2(1 + |\xi|)^\alpha$$

and admitting the wave factorization [1] with respect to Ω

$$A(\xi) = A_{\ne}(\xi)A_=(\xi)$$

with the index æ such that $æ - s = 1 + \delta, |\delta| < 1/2$ then the Dirichlet problem (1) can be reduced to the system of linear integral equations

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} K_1(\xi)C(\xi_1)d\xi_1 + D(\xi_2) = F(\xi_2), \\ \displaystyle C(\xi_1) + \int_{-\infty}^{\infty} K_2(\xi)D(\xi_2)d\xi_2 = G(\xi_1) \end{cases}$$

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with respect to two unknown functions C, D. These functions should belong to Fourier image of certain H^s-space with certain $s < 0$.

2. INTEGRAL OPERATORS

We study the following integral operator

$$(Kf)(x) = \int\limits_{-\infty}^{+\infty} K(x,y)f(y)dy$$

in functional spaces with the following norm

$$||f||_s = \left(\int\limits_{\mathbb{R}} |f(x)|^2 (1+|x|)^{2s} dx \right)^{1/2}.$$

We assume that the kernel $K(x,y)$ satisfies the following condition

$$|K(x,y)| \sim (1+|x|+|y|)^{-æ}, \quad æ \in \mathbb{R}. \tag{2}$$

Theorem 1. *The operator K is a linear bounded operator $K : H^{s_1}(\mathbb{R}) \to H^{s_2}(\mathbb{R})$, where $s_1 < 0, æ > \max\{1/2 - s_1, 1 + s_2 - s_1\}$.*

Proof. Let us verify

$$||Kf||_{s_2}^2 = \int\limits_{-\infty}^{+\infty} (1+|x|)^{2s_2}|(Kf)(x)|^2 dx = \int\limits_{-\infty}^{+\infty} (1+|x|)^{2s_2} \left| \int\limits_{-\infty}^{+\infty} K(x,y)f(y)dy \right|^2 dx$$

$$\leq \int\limits_{-\infty}^{+\infty} (1+|x|)^{2s_2} \left(\int\limits_{-\infty}^{+\infty} |K(x,y)||f(y)|dy \right)^2 dx$$

$$\leq const \int\limits_{-\infty}^{+\infty} (1+|x|)^{2s_2} \left(\int\limits_{-\infty}^{+\infty} (1+|x|+|y|)^{-æ}||f(y)|dy \right)^2 dx.$$

In the inner integral, we apply the Cauchy−Schwartz inequality introducing the factors $(1+|y|)^{-s_1}$ and $(1+|y|)^{s_1}$ form the first and the second term and taking into account that $(1+|y|)^{-s_1} \leq (1+|x|+|y|)^{-s_1}$. Then, we obtain

$$||Kf||_{s_2}^2 \leq const ||f||_{s_1}^2 \int\limits_{-\infty}^{+\infty} (1+|x|)^{2s_2} \left(\int\limits_{-\infty}^{+\infty} (1+|x|+|y|)^{-2(s_1+æ)}dy \right) dx$$

$$\leq const ||f||_{s_1}^2 \int\limits_{0}^{+\infty} (1+|x|)^{2s_2} (1+|x|)^{-2(s_1+æ-1/2)} dx$$

since $-2(s_1 + æ) < -1$. Thus, we conclude

$$||Kf||_{s_2}^2 \leq const ||f||_{s_1}^2 \int\limits_{0}^{+\infty} (1+|x|)^{2(s_2-s_1-æ+1/2)} dx \leq const ||f||_{s_1}^2$$

since $2(s_2 - s_1 - æ + 1/2) < -1$. $\qquad\qquad\square$

Corollary 1. *The operator K is a linear bounded operator, $K : H^s(\mathbb{R}) \to H^s(\mathbb{R})$, where $s < 0, æ > \max\{1/2 - s, 1\}$.*

3. SOME GENERALIZATIONS: $H^{s,p}$-SPACES

3.1. First Variant

This section is devoted to a generalization of the above result to Banach spaces $H^{s,p}(\mathbb{R}), 1 < p < \infty$. Let us write that for $f \in H^{s,p}(\mathbb{R})$

$$||f||_{s,p} = \left(\int\limits_{\mathbb{R}} |f(x)|^p (1 + |x|)^{ps} dx \right)^{1/p}.$$

Obviously, for $p = 2$ we have $H^{s,2}(\mathbb{R}) \equiv H^s(\mathbb{R})$. Let us remind also that the pair (p, q) is so that $1/p + 1/q = 1$.

Theorem 2. *Let the kernel $K(x, y)$ satisfies the condition (2). Then, the operator K is a linear bounded operator*

$$K : H^{s_1, p_1}(\mathbb{R}) \to H^{s_2, p_2}(\mathbb{R}),$$

where $s_1, s_2, æ, p_1, p_2$ such that $s_1 < 0, æ > \max\{1/q_1 - s_1, 1/p_2 + 1/q_1 + s_2 - s_1\}$.

Proof. We estimate

$$||Kf||_{s_2, p_2}^{p_2} = \int\limits_{-\infty}^{+\infty} (1 + |x|)^{p_2 s_2} |(Kf)(x)|^{p_2} dx = \int\limits_{-\infty}^{+\infty} (1 + |x|)^{p_2 s_2} \left| \int\limits_{-\infty}^{+\infty} K(x, y) f(y) dy \right|^{p_2} dx$$

$$\leq \int\limits_{-\infty}^{+\infty} (1 + |x|)^{p_2 s_2} \left(\int\limits_{-\infty}^{+\infty} |K(x, y)| |f(y)| dy \right)^{p_2} dx$$

$$\leq const \int\limits_{0}^{+\infty} (1 + |x|)^{p_2 s_2} \left(\int\limits_{-\infty}^{+\infty} (1 + |x| + |y|)^{-æ} |f(y)| dy \right)^{p_2} dx.$$

Let us consider the inner integral

$$\int\limits_{-\infty}^{+\infty} (1 + |x| + |y|)^{-æ} |f(y)| dy = \int\limits_{-\infty}^{+\infty} (1 + |y|)^{-s_1} (1 + |x| + |y|)^{-æ} |f(y)| (1 + |y|)^{s_1} dy.$$

Applying the Hölder inequality and taking into account the inequality $(1 + |y|)^{-s_1} \leq (1 + |x| + |y|)^{-s_1}$, we have

$$\int\limits_{-\infty}^{+\infty} (1 + |x| + |y|)^{-æ} ||f(y)| dy \leq ||f||_{s_1, p_1} \left(\int\limits_{0}^{+\infty} (1 + |x| + |y|)^{-q_1(s_1 + æ)} dy \right)^{\frac{1}{q_1}}$$

$$\leq const ||f||_{s_1, p_1} (1 + |x|)^{\frac{-q_1(s_1 + æ) + 1}{q_1}},$$

since $-q_1(s_1 + æ) < -1 \Longleftrightarrow æ > 1/q_1 - s_1$.

If so then we have

$$||Kf||_{s_2, p_2}^{p_2} \leq const ||f||_{s_1, p_1}^{p_2} \int\limits_{0}^{+\infty} (1 + |x|)^{p_2(1/q_1 - s_1 - æ + s_2)} dx \leq const ||f||_{s_1, p_1}^{p_2},$$

since $p_2(1/q_1 - s_1 - æ + s_2) < -1 \Longleftrightarrow æ > 1/p_2 + 1/q_1 + s_2 - s_1$. $\square$

Corollary 2. *Let the kernel $K(x, y)$ satisfies the condition (2). Then, the operator K is a linear bounded operator $K : H^{s,p}(\mathbb{R}) \to H^{s,p}(\mathbb{R})$, where $s, æ, p$ such that $s < 0, æ > \max\{1/q - s, 1\}$.*

3.2. Second Variant

This section is devoted to another generalization of the above result to Banach spaces $H^{s,p}(\mathbb{R}), 1 < p < \infty$. We say that for $f \in H^{p,s}(\mathbb{R})$

$$||f||_{p,s} = \left(\int\limits_{\mathbb{R}} |f(x)|^p (1 + |x|)^{2s} dx \right)^{1/p}.$$

Obviously, for $p = 2$ we have $H^{s,2}(\mathbb{R}) \equiv H^s(\mathbb{R})$. Let us remind also that the pair (p, q) is so that $1/p + 1/q = 1$.

Theorem 3. *Let the kernel $K(x, y)$ satisfies the condition (2). Then, the operator K is a linear bounded operator $K : H^{p_1,s_1}(\mathbb{R}) \to H^{p_2,s_2}(\mathbb{R})$, where s_1, s_2, æ, p_1, p_2 such that $s_1 < 0$, æ $>$ $\max\{1/q_1 - 2s_1/p_2, 1/p_2 + 1/q_1 + 2s_2/p_2 - 2s_1/p_1\}$.*

Proof. Let us estimate the norm

$$||Kf||_{p_2,s_2}^{p_2} = \int\limits_{-\infty}^{+\infty} (1 + |x|)^{2s_2} |(Kf)(x)|^{p_2} dx = \int\limits_{-\infty}^{+\infty} (1 + |x|)^{2s_2} \left| \int\limits_{-\infty}^{+\infty} K(x,y)f(y)dy \right|^{p_2} dx$$

$$\leq \int\limits_{-\infty}^{+\infty} (1 + |x|)^{2s_2} \left(\int\limits_{-\infty}^{+\infty} |K(x,y)||f(y)|dy \right)^{p_2} dx$$

$$\leq const \int\limits_{0}^{+\infty} (1 + |x|)^{2s_2} \left(\int\limits_{-\infty}^{+\infty} (1 + |x| + |y|)^{-æ}|f(y)|dy \right)^{p_2} dx.$$

Let us consider the inner integral

$$\int\limits_{-\infty}^{+\infty} (1 + |x| + |y|)^{-æ}|f(y)|dy = \int\limits_{-\infty}^{+\infty} (1 + |y|)^{-2s_1/p_1}(1 + |x| + |y|)^{-æ}|f(y)|(1 + |y|)^{2s_1/p_1}dy.$$

Applying the Hölder inequality and taking into account the inequality $(1 + |y|)^{-2s_1/p-1} \leq (1 + |x| + |y|)^{-2s_1/p-1}$, we have

$$\int\limits_{-\infty}^{+\infty} (1 + |x| + |y|)^{-æ}||f(y)|dy \leq ||f||_{p_1,s_1} \left(\int\limits_{0}^{+\infty} (1 + |x| + |y|)^{-q_1(2s_1/p_1+æ)}dy \right)^{\frac{1}{q_1}}$$

$$\leq const \, ||f||_{p_1,s_1}(1 + |x|)^{\frac{-q_1(2s_1/p_1+æ)+1}{q_1}},$$

since $-q_1(2s_1/p_1 + æ) < -1 \Longleftrightarrow æ > 1/q_1 - 2s_1/p_2$.

If so then we have

$$||Kf||_{p_2,s_2}^{p_2} \leq const \, ||f||_{p_1,s_1}^{p_2} \int\limits_{0}^{+\infty} (1 + |x|)^{p_2(1/q_1-2s_1/p_1-æ)+2s_2}dx \leq const \, ||f||_{p_1,s_1}^{p_2},$$

since $p_2(1/q_1 - 2s_1/p_1 - æ) + 2s_2 < -1 \Longleftrightarrow æ > 1/p_2 + 1/q_1 + 2s_2/p_2 - 2s_1/p_1$.   □

Corollary 3. *Let the kernel $K(x, y)$ satisfies the condition (2). Then, the operator K is a linear bounded operator $K : H^{p,s}(\mathbb{R}) \to H^{p,s}(\mathbb{R})$, where s, æ, p such that $s < 0$, æ $> \max\{1/q - 2s/p, 1\}$.*

4. CONCLUSIONS

The considered spaces $H^{s,p}(\mathbb{R})$ and $H^{p,s}(\mathbb{R})$ like almost the analogous spaces considered in [6, 7]. But we work in Fourier images, and the introduced spaces are usual weighted spaces with power weights.

REFERENCES

1. V. B. Vasil'ev, *Wave Factorization of Elliptic Symbols: Theory and Applications. Introduction to the Theory of Boundary Value Problems in Non-Smooth Domains* (Kluwer Academic, Dordrecht, 2000).
2. A. V. Vasilyev and V. B. Vasilyev, "Difference equations and boundary value problems," in *Differential and Difference Equations and Applications,* Ed. by S. Pinelas, Z. Dǒslá, O. Dǒslý, and P. Kloeden, Springer Proc. Math. Stat. **164**, 421−432 (2016).
3. G. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations* (AMS, Providence, RI, 1981).
4. A. A. Mashinets, A. V. Vasilyev, and V. B. Vasilyev, "On discrete Neumann problem in a quadrant," Lobachevskii J. Math. **44**, 1011−1021 (2023).
5. A. V. Vasilyev, V. B. Vasilyev, and A. A. Khodyreva, "The discrete Dirichlet problem. Solvability and approximation properties," J. Math. Sci. **270**, 654−664 (2023).
6. H. Triebel, *Theory of Function Spaces* (Birkhäuser, Basel, 1983).
7. M. W. Wong, *An Introduction to Pseudo-Differential Operators* (World Scientific, Singapore, 1991).