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Asymptotic Behavior of Solution to Some Boundary Value Problem

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Abstract. We consider an elliptic pseudo-differential equation in a plane sector with additional integral condition. Using a formula for a general solution we study a limit case in which the size of a sector tends to zero. It is shown that the function in the boundary condition cannot be an arbitrary function, and it must satisfy some functional singular integral equation.

Key Words and Phrases: pseudo-differential equation, cone, boundary value problem, asymptotic behavior.

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1. Introduction

The first author studied elliptic pseudo-differential equations in domains with singular points at a boundary [14]. Using the local principle he considered the equation [15, 16, 17]

$$(Au)(x) = v(x), \quad x \in C, \tag{1}$$

where C is a convex cone in Euclidean space $\mathbb{R}^m$, A is a pseudo-differential operator

$$(Au)(x) = \int\limits_C \int\limits_{\mathbb{R}^m} A(\xi) e^{i(x-y)\cdot\xi} u(y) d\xi dy, \quad x \in C,$$

with the symbol $A(\xi)$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \leq |A(\xi)| \leq c_2(1 + |\xi|)^\alpha.$$

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There are a lot of approaches for studying this problem. We wrote many times on the methods of studying solvability of pseudo-differential equations in domains with conical points and wedges (see, for example, [1, 2, 3, 4, 5, 6, 7] and many others), but now we would like to speak on another aspect of the problem.

The main problem is obtaining conditions for unique solvability o the equation (1) in appropriate functional spaces, or invertibility conditions for the operator A. To describe such conditions a concept of the wave factorization for an elliptic symbol was introduced [14]. Unfortunately, a number of solutions depends on an index of the wave factorization [14] and to extract the unique solution one needs some additional conditions.

Each singularity (half-space, cone, wedge, etc.) corresponds to some distribution, and a convolution with the distribution describes a local representative of an initial pseudo differential operator in an appropriate point of manifold. All details can be found in [14, 17, 19]. But singularities can be of different dimensions and it is possible that singularities of lower dimensions are obtained from analogous singularities of full dimension. It means we need to find distributions for limit cases where some of parameters of singularities tend to zero. This approach was partially realized in [15, 16], and the work [17] is dedicated to multi-dimensional constructions.

Such problems have been considered from different points of view in [15, 16, 17, 18, 19, 20, 21, 22, 23]. Some results have been obtained in very special case where the size of a cone tends to zero [21, 23]. There are some interesting results concerning such cones. Here we consider simple plane case where one needs to add some condition to the equation to obtain the unique solution.

2. Preliminaries and notations

We will use standard Sobolev–Slobodetskii spaces $H^s(\mathbb{R}^m)$ [8, 14] with the norm

$$||u||_s^2 = \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi,$$

and $\sim$ over a function will denote its Fourier transform

$$(Fu)(\xi) \equiv \tilde{u}(\xi) = \int\limits_{\mathbb{R}^m} e^{ix\cdot\xi}u(x)dx.$$

We consider 2-dimensional case and the cone $C_+^a = \{x \in \mathbb{R}^2 : x_2 > a|x_1|, a > 0\}$ in the Sobolev–Slobodetskii space $H^s(C_+^a)$. The latter space consists of functions from $H^s(\mathbb{R}^m)$ with support in $\overline{C_+^a}$.

Let $\overset{*}{C^a_+}$ be a conjugate of the cone C^a_+:

$$\overset{*}{C^a_+}= \{x \in \mathbb{R}^2 : x = (x_1, x_2), ax_2 > |x_1|\},$$

$C^a_- \equiv -C^a_+$, $T(C^a_+)$ be a radial tube domain over the cone C^a_+, i.e. a domain of a two-dimensional complex space $\mathbb{C}^2$ of the type $\mathbb{R}^2 + iC^a_+$ [10, 11].

3. Wave factorization and structure of solution

Our study is based on the concept of wave factorization.

Definition 1. *The wave factorization of an elliptic symbol $A(\xi)$ is defined as its representation in the form*

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^2$, except maybe for $\{\xi \in \mathbb{R}^2 : |\xi_1|^2 = a^2\xi_2^2\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytic continuation into radial tube domains $T(\overset{*}{C^a_+}), T(\overset{*}{C^a_-})$, respectively, with the estimates

$$|A^{\pm 1}_{\neq}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A^{\pm 1}_{=}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \ \forall \tau \in \overset{*}{C^a_+} .$$

The number $æ \in \mathbb{R}$ is called an index of the wave factorization.

If the symbol $A(\xi)$ admits the wave factorization [14] under the condition $1/2 < æ - s < 3/2$, where æ is the index of wave factorization, then one can show [17, 18] that a general solution of the equation (1) in Sobolev–Slobodetskii space $H^s(C^a_+)$ in Fourier image has the following form:

$$\tilde{u}(\xi) = \frac{\tilde{c}_0(\xi_1 + a\xi_2) + \tilde{c}_0(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1, \xi_2)} +$$
$$A^{-1}_{\neq}(\xi_1, \xi_2) \left(v.p. \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} - v.p. \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta)d\eta}{\xi_1 - a\xi_2 - \eta} \right), \tag{2}$$

where c_0 is an arbitrary function from $H^{s-æ+1/2}(\mathbb{R})$, and $v.p.$ denotes principal value of the integral in the Cauchy sense [9].

3.1. Proof of the formula (2).

For reader's convenience, we will describe here the scheme for obtaining a general solution of the equation (1) using the wave factorization. In other words, we will give proof for the formula (2).

Let us consider the equation (1) for the case æ $-\, s = 1 + \delta, |\delta| < 1/2$, only. A general solution can be constructed as follows. After wave factorization for the symbol with preliminary Fourier transform we get

$$A_{\neq}(\xi)\tilde{u}(\xi) + A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi) = A_{=}^{-1}(\xi)\widetilde{lv}(\xi),$$

where $u_{-}(x) = lv(x) - u(x)$, lv is an arbitrary continuation of v to the whole $\mathbb{R}^2$.

One can see that $A_{=}^{-1}(\xi)\widetilde{lf}(\xi)$ belongs to the space $\tilde{H}^{s-æ}(\mathbb{R}^2)$, and if we choose the polynomial $Q(\xi)$, satisfying the condition

$$|Q(\xi)| \sim 1 + |\xi|,$$

then $Q^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{lf}(\xi)$ will belong to the space $\tilde{H}^{-\delta}(\mathbb{R}^2)$.

Further, according to the theory of multi-dimensional Riemann problem [14], we can decompose the last function into two summands (jump problem):

$$Q^{-1}A_{=}^{-1}\widetilde{lv} = f_{+} + f_{-},$$

where $f_{+} \in \tilde{H}(C_{+}^{a}), f_{-} \in \tilde{H}(\mathbb{R}^2 \setminus C_{+}^{a})$.

So, we have

$$Q^{-1}A_{\neq}\tilde{u} + Q^{-1}A_{=}^{-1}\tilde{u}_{-} = f_{+} + f_{-},$$

or

$$Q^{-1}A_{\neq}\tilde{u} - f_{+} = f_{-} - Q^{-1}A_{=}^{-1}\tilde{u}_{-}$$

In other words,

$$A_{\neq}\tilde{u} - Qf_{+} = Qf_{-} - A_{=}^{-1}\tilde{u}_{-}.$$

The left-hand side of the last equality belongs to the space $\tilde{H}^{-1-\delta}(C_{+}^{a})$, and the right-hand side belongs to $\tilde{H}^{-1-\delta}(\mathbb{R}^2 \setminus C_{+}^{a})$, hence

$$F^{-1}(A_{\neq}\tilde{u} - Qf_{+}) = F^{-1}(Qf_{-} - A_{=}^{-1}\tilde{u}_{-}), \tag{3}$$

where the left-hand side belongs to $H^{-1-\delta}(C_{+}^{a})$, and the right-hand side belongs to $H^{-1-\delta}(\mathbb{R}^2 \setminus C_{+}^{a})$, therefore we conclude immediately that this is a distribution supported on ∂C_{+}^{a}.

We introduce $T_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the following type:

$$\begin{cases} t_1 = x_1, \\ t_2 = x_2 - a|x_1|. \end{cases}$$

(obviously, it transforms ∂C_+^a one-to-one into the hyperplane $x_2 = 0$).

To apply the Fourier transform to the formula (3), we need to obtain explicit expression for FT_a. This can be done in the following way. We calculate:

$$(FT_a u)(\xi) = \int_{-\infty}^{+\infty} e^{ia|y_1|\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1$$

$$= \int_{-\infty}^{+\infty} \chi_+(y_1) e^{iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1 + \int_{-\infty}^{+\infty} \chi_-(y_1) e^{-iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1$$

$$= \int_{-\infty}^{+\infty} \chi_+(y_1) e^{iy_1(a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2) dy_1 + \int_{-\infty}^{+\infty} \chi_-(y_1) e^{iy_1(-a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2) dy_1.$$

The last two summands are the Fourier transforms of the functions

$$\chi_+(y_1) e^{iy_1(a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2), \quad \chi_-(y_1) e^{iy_1(-a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2)$$

with respect to the first variable y_1, respectively, so we can use the following properties ([7], Plemelj–Sokhotskii formulas [12, 13], and we write them for one variable):

$$\int_{-\infty}^{+\infty} \chi_+(x) e^{ix\xi} u(x) dx = \frac{1}{2} \tilde{u}(\xi) + v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta) d\eta}{\xi - \eta},$$

$$\int_{-\infty}^{+\infty} \chi_-(x) e^{ix\xi} u(x) dx = \frac{1}{2} \tilde{u}(\xi) - v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta) d\eta}{\xi - \eta}.$$

Taking into account these properties, we have

$$(FT_a u)(\xi) = \frac{\tilde{u}(a\xi_2 + \xi_1, \xi_2) + \tilde{u}(-a\xi_2 + \xi_1, \xi_2)}{2}$$

$$+ v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{a\xi_2 + \xi_1 - \eta} - v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{-a\xi_2 + \xi_1 - \eta}.$$

Taking into account a general form of a distribution from $S'(\mathbb{R}^2)$ supported on the straight line $x_2 = 0$ [8, 10]

$$\sum_{k=0}^{n} c_k(x_1) x_2^k,$$

we conclude there is only one summand $c_0(x_1)$ in the latter formula, because $æ - s = 1 + \delta$. Further, applying the operator FT_a to the left-hand side of the formula (3) we obtain the formula (2).

4. Limit analysis of factors

4.1. First limit case

It is about a half-space. Indeed, if $a \to 0$, this case corresponds to a half-space case, and we have

$$\tilde{u}(\xi) = \frac{\tilde{c}_0(\xi_1)}{\lim\limits_{a\to 0} A_{\neq}(\xi_1, \xi_2)}.$$

Example 1. *For the Laplacian $\xi_1^2 + \xi_2^2$, we have [14]*

$$A_{\neq}(\xi_1, \xi_2) = \sqrt{a^2+1}\xi_2 + \sqrt{a^2\xi_2^2 - \xi_1^2},$$

so as $a \to 0$ we obtain well known factorization factor from [8]

$$\lim_{a\to 0} A_{\neq}(\xi_1, \xi_2) = \xi_2 + i|\xi_1|.$$

4.2. Second limit case

This case corresponds to the situation when the size of the cone becomes very small, in other words, $a \to \infty$. Our main result is related to this case.

Example 2. *If we are interested in second limit case $a \to +\infty$ for the same symbol*

$$A_{\neq}(\xi_1, \xi_2) = \sqrt{a^2+1}\xi_2 + \sqrt{a^2\xi_2^2 - \xi_1^2},$$

looks like it will be infinite. But if we apply change of variables

$$\begin{cases} t_1 = \xi_1 + a\xi_2, \\ t_2 = \xi_1 - a\xi_2, \end{cases}$$

we will obtain

$$a_{\neq}(t_1, t_2) = A_{\neq}\left(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a}\right),$$

or after simple calculations

$$a_{\neq}(t_1, t_2) = \sqrt{\frac{a^2+1}{4a^2}}(t_1 - t_2) + \sqrt{-t_1t_2},$$

so that

$$\lim_{a\to+\infty} a_{\neq}(t_1, t_2) = \frac{1}{2}(t_1 - t_2) + \sqrt{-t_1t_2}.$$

Remark 1. *In fact,*

$$\frac{1}{2}(t_1 - t_2) + \sqrt{-t_1 t_2} = \lim_{a\to+\infty} a_{\neq}(t_1, t_2) \neq A_{\neq}\left(\frac{t_1 + t_2}{2}, 0\right) = \frac{i}{2}|t_1 + t_2|,$$

because the parameter a affects the change of variables. Nevertheless, these values coincide for $t_1 = t_2$.

5. The main result

We add to the equation (1) the following integral condition:

$$\int_{-\infty}^{+\infty} u(x_1, x_2)dx_2 \equiv g(x_1), \tag{4}$$

and study a solvability of the boundary value problem (1),(3).

Given wave factorization

$$A(\xi_1, \xi_2) = A_{\neq}(\xi_1, \xi_2) \cdot A_{=}(\xi_1, \xi_2),$$

let us denote $A_{\neq}(t, 0)\tilde{g}(t) \equiv G(t)$ and $\lim\limits_{a\to+\infty} a_{\neq}(t_1, t_2) \equiv h(t_1, t_2)$ assuming that latter limit exists. Further, we introduce the following one-dimensional functional integral equation with respect to the function $\tilde{g}$:

$$2h(t_1, t_2)\tilde{g}\left(\frac{t_1 + t_2}{2}\right) = G(t_1) + G(t_2) + (SG)(t_1) - (SG)(t_2), \tag{5}$$

where

$$(SG)(t) = v.p.\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{G(\eta)d\eta}{t - \eta}.$$

Now we are ready to formulate and prove our main result.

Theorem 1. *If the symbol $A(\xi_1, \xi_2)$ admits the wave factorization with respect to C_+^a for sufficiently large a, then the boundary value problem (1),(4) is solvable if and only if the right-hand side g satisfies the equation (5) as $a \to +\infty$.*

Proof. Let us denote

$$\begin{aligned} v.p.\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} &\equiv \tilde{d}_0(\xi_1 + a\xi_2), \\ v.p.\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta)d\eta}{\xi_1 - a\xi_2 - \eta} &\equiv \tilde{d}_0(\xi_1 - a\xi_2). \end{aligned} \tag{6}$$

Then we have

$$\tilde{u}(\xi_1,\xi_2) = \frac{\tilde{c}_0(\xi_1+a\xi_2)+\tilde{c}_0(\xi_1-a\xi_2)+\tilde{d}_0(\xi_1+a\xi_2)-\tilde{d}_0(\xi_1-a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)} \equiv$$

$$\frac{\tilde{c}(\xi_1+a\xi_2)+\tilde{d}(\xi_1-a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)}, \tag{7}$$

where $\tilde{c}(\xi_1+a\xi_2) \equiv \tilde{c}_0(\xi_1+a\xi_2)+\tilde{d}_0(\xi_1+a\xi_2)$, $\tilde{d}(\xi_1-a\xi_2) \equiv \tilde{c}_0(\xi_1-a\xi_2)-\tilde{d}_0(\xi_1-a\xi_2)$.

For the Fourier images, the condition (4) means the following:

$$\tilde{u}(\xi_1,0) = \tilde{g}(\xi).$$

Then, according to the formula (2), we have

$$\frac{\tilde{c}_0(\xi_1)}{A_{\neq}(\xi_1,0)} = \tilde{g}(\xi_1).$$

Thus, at least formally we can find the function

$$\tilde{c}_0(\xi_1) = A_{\neq}(\xi_1,0)\tilde{g}(\xi_1)$$

and then using formulas (6) we find $\tilde{d}_0(\xi_1)$. Hence, the formula (7) gives the solution of the equation (1). Finally, the solution of the equation (1) under the condition (4) takes the following form:

$$\tilde{u}(\xi_1,\xi_2) = \frac{A_{\neq}(\xi_1+a\xi_2,0)\tilde{g}(\xi_1+a\xi_2)+A_{\neq}(\xi_1-a\xi_2,0)\tilde{g}(\xi_1-a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)}+$$

$$\frac{1}{2A_{\neq}(\xi_1,\xi_2)} v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{\xi_1+a\xi_2-\eta} - \frac{1}{2A_{\neq}(\xi_1,\xi_2)} v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{\xi_1-a\xi_2-\eta}.$$

Let us apply the change of variables

$$\begin{cases} t_1 = \xi_1 + a\xi_2, \\ t_2 = \xi_1 - a\xi_2, \end{cases}$$

and denote

$$a_{\neq}(t_1,t_2) \equiv A_{\neq}\left(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a}\right).$$

Then, denoting $\tilde{U}(t_1,t_2) \equiv \tilde{u}\left(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a}\right)$, we can write

$$\tilde{U}(t_1,t_2)=\frac{A_{\neq}(t_1,0)\tilde{g}(t_1)+A_{\neq}(t_2,0)\tilde{g}(t_2)}{2a_{\neq}(t_1,t_2)}+$$

$$\frac{1}{2a_{\neq}(t_1,t_2)}v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_1-\eta}-\frac{1}{2a_{\neq}(t_1,t_2)}v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_2-\eta}.$$

Therefore, we have the following relation as $a\to+\infty$:

$$\tilde{u}\left(\frac{t_1+t_2}{2},0\right)=\tilde{U}(t_1,t_2)=\frac{A_{\neq}(t_1,0)\tilde{g}(t_1)+A_{\neq}(t_2,0)\tilde{g}(t_2)}{2a_{\neq}(t_1,t_2)}+ \\ \frac{1}{2a(t_1,t_2)}v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_1-\eta} \\ -\frac{1}{2a_{\neq}(t_1,t_2)}v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)d\eta}{t_2-\eta}. \tag{8}$$

Taking into account our notations, we see that the relation (8) coincides with the equation (5). This completes the proof. ◀

Conclusion

Some plane problems have been considered in this work. The approach used here can be also applied in a multidimensional space. Such situations will be studied in our forthcoming papers.

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