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PARTIAL DIFFERENTIAL EQUATIONS

Dirichlet Problem on the Half-Line for an Abstract Euler–Poisson–Darboux Equation Containing Powers of an Unbounded Operator

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Abstract—We consider an abstract Euler–Poisson–Darboux equation containing powers of an unbounded operator that is the generator of a Bessel operator function. Sufficient conditions for the unique solvability of the Dirichlet problem on the half-line are obtained. The question concerning the convergence of the solution to zero at infinity is investigated. Examples are given.

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INTRODUCTION

The study of differential equations with unbounded operator coefficients acting in a Banach space E necessitates the development of the theory of resolving operators for the corresponding initial value problems. As a result of the study of first-order evolution equations $u'(t) = Au(t)$, semigroups of linear operators $T(t)$ arose, and studying the second-order equation (abstract wave equation) $u''(t) = Au(t)$ gave rise to operator cosine functions $C(t)$. Relaxing the requirements for resolving operators of the Cauchy problem for abstract differential equations of the first and second orders led to the concepts of an integrated semigroup and an integrated cosine operator function. For terminology and literature sources, see the monographs [1, 2] and the survey papers [3, 4].

The Bessel operator function (OBF) was introduced into consideration in the papers [5, 6] as the resolving operator of the Cauchy problem for the Euler–Poisson–Darboux (EPD) equation. However, just as in the theory of semigroups and operator cosine functions, the family of Bessel operator functions can be introduced (see [7]) independently of the EPD differential equation that it is ultimately associated with. In what follows, we recall the process of constructing a BOF.

An important role in the construction of the family is played by the generalized shift operator T_s^t depending on a parameter $k > 0$ and defined by the relation (see [8])

$$T_s^t Y(s) = \frac{\Gamma(k/2+1/2)}{\sqrt{\pi}\Gamma(k/2)} \int_0^\pi Y\left(\sqrt{s^2+t^2-2st\cos\varphi}\right)\sin^{k-1}\varphi\, d\varphi, \quad s,t \ge 0, \tag{1}$$

where $\Gamma(\cdot)$ is the Euler gamma function. The generalized shift operator depends on the parameter $k > 0$, but, following [8], we do not indicate this fact in the notation.

We also point out that in the present paper, we make do with the concept of integral of a continuous function, but if necessary, we can use the Bochner integral of a function ranging in a Banach space.

Let E be a Banach space, let $k > 0$ be a parameter, and let $Y_k(\cdot) : [0,\infty) \to B(E)$ be an operator function ranging in the space $B(E)$ of linear bounded operators.

Definition 1. A strongly continuous family $Y_k(t) : [0,\infty) \to B(E)$ of linear bounded operators depending on a parameter $k > 0$ is called a *Bessel operator function* if

$$Y_k(0) = I, \quad Y_k(t)Y_k(s) = T_s^t Y_k(s), \quad s,t \ge 0,$$

and there exist constants $\Upsilon \ge 1$ and $\omega \ge 0$ such that

$$\|Y_k(t)\| \le \Upsilon e^{\omega t}, \quad t \ge 0.$$

Associated with the BOF family is the Bessel differential operator

$$\frac{d^2}{dt^2} + \frac{k}{t}\frac{d}{dt},$$

which often occurs in differential equations with axial symmetry.

Definition 2. The *generator of the BOF* $Y_k(t)$ is the operator A whose domain $D(A)$ consists of those $x \in E$ for which the function $Y_k(t)x$ is twice differentiable at the point $t = 0$ and which is defined by the formula

$$Ax = \lim_{t\to+0}\left(\frac{d^2Y_k(t)x}{dt^2} + \frac{k}{t}\frac{dY_k(t)x}{dt}\right).$$

The following statements were proved in [7].

1. The generator A of the BOF $Y_k(t)$ is closed, and its domain $D(A)$ is dense in E; moreover, the set of elements on which all powers of the operator A are defined is dense in E.
2. For any $t, s \geq 0$ and $x \in D(A)$, one has the relations

$$Y_k(t)Y_k(s) = Y_k(s)Y_k(t),$$
$$AY_k(t)x = Y_k(t)Ax.$$

3. Let $x \in D(A)$ and $t > 0$; then $Y_k(t)x \in D(A)$ and

$$AY_k(t)x = \frac{d^2Y_k(t)x}{dt^2} + \frac{k}{t}\frac{dY_k(t)x}{dt}.$$

4. If $u_0 \in D(A)$, then the function $Y_k(t)u_0$ is a solution of the following Cauchy problem for the EPD equation:

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0, \quad u(0) = u_0, \quad u'(0) = 0;$$

in what follows, it is convenient to use the symbol $Y_0(t)$ to denote the operator cosine function $C(t)$ with generator A.

5. Let $0 \leq k < m$, and let A be the generator of the BOF $Y_k(t)$; then A is also the generator of $Y_m(t)$, where the corresponding BOF $Y_m(t)$ has the form

$$Y_m(t) = \frac{2\Gamma(m/2 + 1/2)}{\Gamma(k/2 + 1/2)\Gamma(m/2 - k/2)} \int_0^1 s^k(1 - s^2)^{(m-k)/2-1} Y_k(ts)\, ds; \tag{2}$$

relation (2) is called the *shift formula* of the BOF with respect to the parameter.

If operator A is the generator of the operator cosine function $Y_0(t) = C(t)$, then it follows from (2) for $k = 0$ that the BOF $Y_m(t)$ is the operator cosine function integrated in a special way (for more details, see the paper [9]).

1. DIRICHLET PROBLEM

In a Banach space E, on the half-line $t \geq 0$ with the parameter value $k < 1$, consider the Dirichlet problem for the Euler–Poisson–Darboux equation containing powers of an unbounded operator A,

$$u''(t) + \frac{k}{t}u'(t) = -P_m(A)u(t), \quad t > 0, \tag{3}$$

$$u(0) = u_0, \quad \sup_{t\geq 0}\|u(t)\| \leq M, \tag{4}$$

where $P_m(A)u(t) = (-1)^{m+1}B_m A^m u(t) + \sum_{n=0}^{m-1} B_n A^n u(t)$, the B_n, $n = 0, \dots, m$, are bounded operators acting in E, and A is the generator of the BOF $Y_q(t)$ for some $q \ge 0$.

We will assume that the bounded operator coefficients B_n and the generator A satisfy the following condition.

Condition 1. The domain $D(A)$ is invariant under the bounded operators B_n, $n = 0, \dots, m$, $AB_n x = B_n Ax$, $x \in D(A)$, and the spectrum $\sigma(B_m)$ of the operator B_m is located to the right of the vertical line $\operatorname{Re}\lambda = \delta > 0$ (parabolicity condition).

We refer to the case we are considering as *elliptic*. A *solution of the Dirichlet problem* (3), (4) is an abstract function $u(t)$ that ranges in the domain $D(A^m)$, is twice continuously differentiable for $t > 0$, is continuous for $t \ge 0$, and satisfies Eq. (3) and conditions (4).

Abstract parabolic equations with the operator $P_m(A)$ were studied earlier in [10]. In the hyperbolic case, the initial value problem for an Euler–Poisson–Darboux equation containing powers of the BOF generator was studied in the papers [11, 12].

We also point out that the papers [13–17] deal with solving elliptic problems for partial differential equations containing the Bessel operator in one or more variables and also give an extensive survey of relevant publications.

In the present paper, we consider problem (3), (4) in the elliptic case using the fundamental solution $G(t, s)$ (constructed in [10]) of the equation

$$\frac{\partial v(t,s)}{\partial t} = (-1)^{m+1} B_m \frac{\partial^{2m} v(t,s)}{\partial s^{2m}} + \sum_{n=0}^{m-1} B_n \frac{\partial^{2n} v(t,s)}{\partial s^{2n}}, \quad t > 0, \quad s \in \mathbb{R}, \tag{5}$$

which has the form

$$G(t,s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{is\sigma} Q(t,\sigma)\, d\sigma, \tag{6}$$

where

$$Q(t,\sigma) = \exp\left(-t\sigma^{2m} B_m - t\sum_{n=0}^{m-1} \sigma^{2n} B_n\right).$$

In this case, for the function $G(t, s)$ we have the convolution formula

$$\int_{-\infty}^{+\infty} G(t - t_1, s - s_1) G(t_1 - \tau, s_1 - \xi)\, ds_1 = G(t - \tau, s - \xi), \quad 0 \le \tau < t_1 < t. \tag{7}$$

Along with Eq. (5), in the domain $t > 0$, $s \in \mathbb{R}$ we consider the problem

$$\frac{\partial^2 w(t,s)}{\partial t^2} + \frac{k}{t}\frac{\partial w(t,s)}{\partial t} = (-1)^m B_m \frac{\partial^{2m} w(t,s)}{\partial s^{2m}} - \sum_{n=0}^{m-1} B_n \frac{\partial^{2n} w(t,s)}{\partial s^{2n}}, \quad w(0,s) = \delta(s), \tag{8}$$

where $\delta(s)$ is the Dirac delta function.

Applying the Fourier transform in the variable $s \in \mathbb{R}$ to problem (8) and taking into account the formula for connecting the solution of the Dirichlet problem for the EPD equation with the solution of the Cauchy problem for a parabolic equation (see [18, Theorem 3]), we introduce the following operator function, which is a solution of problem (8):

$$Z_k(t,s) = \frac{t^{1-k}}{2^k \pi \Gamma(1/2 - k/2)} \int_{-\infty}^{\infty} e^{is\sigma} \int_0^{\infty} \tau^{k/2-3/2} \exp\left(-\frac{t^2}{4\tau}\right) Q(\tau,\sigma)\, d\tau d\sigma = \int_0^{\infty} h_k(t,\tau) G(\tau,s)\, d\tau, \tag{9}$$

where the fundamental solution $G(\tau, s)$ is defined by relation (6),

$$h_k(t,\tau) = \frac{t^{1-k}\tau^{k/2-3/2}}{2^{1-k}\Gamma(1/2 - k/2)} \exp\left(-\frac{t^2}{4\tau}\right), \quad t \ge 0, \quad \tau > 0.$$

To justify the existence of a solution of the Dirichlet problem (3), (4), we need the following assertion.

Lemma 1. *Let $t \geq 0$, $b > 0$, $\beta > 0$, and $\gamma > 1$. Then for a function of the form*

$$f(t) = \int\limits_0^\infty s^{-\gamma} \exp\left(-\frac{b}{s} - \frac{t}{s^\beta}\right) ds$$

there exist constants $M_1, M_2 > 0$ such that one has the estimate

$$f(t) \leq \frac{M_1}{M_2 + t^{(\gamma-1)/\beta}}. \tag{10}$$

Proof. For $t > 0$, after a change of variables, we obtain

$$f(t) = t^{(1-\gamma)/\beta} \int\limits_0^\infty \xi^{-\gamma} \exp\left(-\frac{b}{\xi t^{1/\beta}} - \frac{1}{\xi^\beta}\right) d\xi < M_4 t^{(1-\gamma)/\beta},$$

which, together with the obvious inequality $f(t) \leq f(0) = M_3$, leads to the desired relation (10). The proof of the lemma is complete.

Depending on the type and properties of the operators A and $P(A)$, further research will be divided into two cases.

2. DIRICHLET PROBLEM IN THE CASE OF $k < 1$ WITH AN OPERATOR OF THE FORM $P_m(A)u(t) = (-1)^{m+1} \times B_m A^m u(t)$

For the fundamental solution $G(t, s)$ defined by relation (6), the paper [10] established the following estimate for this case:

$$\left\| \frac{\partial^j G(t,s)}{\partial s^j} \right\| \leq M_j t^{-(j+1)/(2m)} \exp\left(-a t^{1/(1-2m)} |s|^{2m/(2m-1)}\right), \quad a > 0, \tag{11}$$

the proof of which is carried out using methods developed in [19, Ch. 1] for the case of matrix coefficients B_j. In addition, if $Y_0(s)$ is uniformly bounded, then the analytic semigroup

$$U\big(t; P_m(A)\big)x = 2\int\limits_0^\infty G(t,s) Y_0(s) x\, ds$$

is defined with generator $P_m(A)$ whose domain is $D(A^m)$. Note that the semigroup property for $U(t; P_m(A))$ is valid due to the convolution formula (7).

Now let us estimate the derivatives of the operator function $Z_k(t, s)$.

Lemma 2. *For the operator function $Z_k(t, s)$ defined by relation (9) and its derivatives up to order $j = 0, \ldots, 2m$, the following estimate holds for $t > 0$:*

$$\left\| \frac{\partial^j Z_k(t,s)}{\partial s^j} \right\| \leq \frac{M_{k,j} t^{1-k}}{t^{1-k+(j+1)/m} + |s|^{m(1-k)+j+1}}, \quad M_{k,j} > 0. \tag{12}$$

Proof. Let us differentiate relation (9) and use the estimate (11). After a change of variables, we have

$$\left\| \frac{\partial^j Z_k(t,s)}{\partial s^j} \right\| \leq M_j \int\limits_0^\infty h_k(t,\tau) \tau^{-(j+1)/(2m)} \exp\left(-a\tau^{1/(1-2m)} |s|^{2m/(2m-1)}\right) d\tau$$

$$= \frac{M_j t^{1-k}}{2^{1-k}\Gamma(1/2-k/2)} \int\limits_0^\infty \tau^{k/2-3/2-(j+1)/(2m)} \exp\left(-\frac{t^2}{4\tau} - a\tau^{1/(1-2m)}|s|^{2m/(2m-1)}\right) d\tau$$

$$= \frac{M_j t^{-(j+1)/m}}{2^{1-k}\Gamma(1/2-k/2)} \int\limits_0^\infty \xi^{k/2-3/2-(j+1)/(2m)} \exp\left(-\frac{1}{4\xi} - \frac{a\left(|s|^m/t\right)^{2/(2m-1)}}{\xi^{1/(2m-1)}}\right) d\xi.$$

Estimating the last integral using inequality (10) in Lemma 1 with

$$b = \frac{1}{4}, \quad \beta = \frac{1}{2m-1}, \quad \gamma = \frac{j+1-m(k-3)}{2m},$$

we obtain the desired inequality (12),

$$\left\|\frac{\partial^j Z_k(t,s)}{\partial s^j}\right\| \le \frac{M_j t^{-(j+1)/m}}{2^{1-k}\Gamma(1/2-k/2)} \frac{M_1}{M_2 + \left(1/4\left(|s|^m/t\right)^{2/(2m-1)}\right)^{(j+1+m-mk)(2m-1)/(2m)}}$$

$$\le \frac{M_{k,j} t^{1-k}}{t^{1-k+(j+1)/m} + |s|^{m(1-k)+j+1}}.$$

The proof of the lemma is complete.

In what follows, we also need estimates for weighted derivatives of the operator function $Z_k(t,s)$. To this end, we first establish the following lemma, which is proved by induction.

Lemma 3. *Let $Z(s) \in C^n(0,\infty)$, $n \in \mathbb{N}$. Then*

$$\left(\frac{1}{s}\frac{d}{ds}\right)^n Z(s) = \sum_{j=1}^n \theta_{j,n} s^{j-2n} Z^{(j)}(s), \tag{13}$$

where

$$\theta_{j,n} = \frac{(2n-j-1)!}{(-2)^{n-j}(n-j)!(j-1)!}. \tag{14}$$

Proof. Let relation (13) hold for some n. Then

$$\left(\frac{1}{s}\frac{d}{ds}\right)^{n+1} Z(s) = \sum_{j=1}^n \theta_{j,n}(j-2n) s^{j-2n-2} Z^{(j)}(s) + \sum_{j=2}^{n+1} \theta_{j-1,n} s^{j-2n-2} Z^{(j)}(s),$$

and to prove formula (13) for $n+1$ it remains to establish the relations

$$\begin{aligned} \theta_{1,n+1} &= (1-2n)\theta_{1,n}, \\ \theta_{n+1,n+1} &= \theta_{n,n} = 1, \\ \theta_{j,n+1} &= (j-2n)\theta_{j,n} + \theta_{j-1,n}, \quad 2 \le j \le n, \end{aligned}$$

which can be verified directly taking into account the definition of the numbers $\theta_{j,n}$ by formula (14). The proof of the lemma is complete.

The following lemma is a corollary of Lemmas 2 and 3.

Lemma 4. *For $t > 0$, the operator function $Z_k(t,s)$ defined by relation (9) satisfies the estimate*

$$\left\|\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s)\right\| \le \sum_{j=1}^n \frac{|\theta_{j,n}| M_{k,j} t^{1-k} s^{j-2n}}{t^{1-k+(j+1)/m} + |s|^{m(1-k)+j+1}}, \quad M_{k,j} > 0. \tag{15}$$

Assuming the uniform boundedness of the BOF $Y_q(s)$, $q \geq 0$, whose generator is the operator A, in what follows we take the smallest $n \in \mathbb{N}$ with $2n \geq q$ and introduce the operator function

$$W_k(t)x = \frac{(-1)^n \cdot 2}{(2n-1)!!} \int\limits_0^\infty \left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s) s^{2n} Y_{2n}(s) x \, ds, \quad x \in E,$$

where the BOF $Y_{2n}(s)$ is expressed via the BOF $Y_q(s)$ using formula (2).

The convergence of the integral and the possibility of its differentiation with respect to t are determined by relation (8) and the estimate (15). The operator function $W_k(t)$ bounded in the space E will be used to establish the unique solvability of the Dirichlet problem (3), (4).

Note that if the operator A is the generator of a bounded operator cosine function $C(t)$, then, as follows from [20, Ch. 9, p. 11 of the Russian translation], the operator function

$$W_0(t) = 2\int\limits_0^\infty Z_0(t,s) C(s)\, ds = \int\limits_0^\infty h_0(t,\tau) U\big(\tau; P(A)\big)\, d\tau,$$

where

$$h_0(t,\tau) = \frac{t}{2\sqrt{\pi}\tau^{3/2}} \exp\left(-\frac{t^2}{4\tau}\right), \quad t \geq 0, \quad \tau > 0,$$

is a semigroup, and the pseudodifferential operator $P_{1/2}(A) = -\sqrt{-P_m(A)}$ is the generator of this semigroup $W_0(t)$.

Theorem 1. *Assume that for some $q \geq 0$ the operator A is the generator of a uniformly bounded BOF $Y_q(s)$, $u_0 \in D(A^m)$, and Condition 1 is satisfied. Then the Dirichlet problem* (3), (4) *has a unique solution, which can be represented in the form*

$$u(t) = W_k(t)u_0 = \frac{(-1)^n \cdot 2}{(2n-1)!!} \int\limits_0^\infty \left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s) s^{2n} Y_{2n}(s) u_0 \, ds, \tag{16}$$

where $Z_k(t,s)$ is defined by relation (9), *while the BOF $Y_{2n}(s)$ is expressed via the BOF $Y_q(s)$ using formula* (2).

Proof. First, assume that $u_0 \in D(A^{m+[n/2]+2})$ and $q > 0$. Then after n-fold integration by parts we obtain

$$u(t) = W_k(t)u_0 = \frac{2}{(2n-1)!!} \int\limits_0^\infty Z_k(t,s) \left(\frac{1}{s}\frac{d}{ds}\right)^n \big(s^{2n-1} Y_{2n}(s) u_0\big) ds = 2\int\limits_0^\infty Z_k(t,s) \tilde{Y}_0(s) u_0 \, ds. \tag{17}$$

When written using the operators of shift of the solution of the EPD equation by the parameter (for more details, see [21, 22]), the function

$$\tilde{Y}_0(s)u_0 = \frac{1}{(2n-1)!!} \left(\frac{1}{s}\frac{d}{ds}\right)^n \big(s^{2n-1} Y_{2n}(s) u_0\big) \tag{18}$$

is no longer a BOF but determines the solution of the Cauchy problem

$$u''(s) = Au(s), \quad s > 0, \quad u(0) = u_0 \in D(A^{[n/2]+2}), \quad u'(0) = 0.$$

Since the function $Z_k(t,s)u_0$ satisfies problem (8), it can readily be verified that the function $u(t) = W_k(t)u_0$ defined by relation (17) is a solution of the Dirichlet problem (3), (4). Indeed, obviously, $u(0) = u_0$, and after integration by parts, the integrated terms vanish and we obtain

$$\begin{aligned} u''(t) + \frac{k}{t}u'(t) &= 2\int_0^\infty \left(\frac{\partial^2 Z_k(t,s)}{\partial t^2} + \frac{k}{t}\frac{\partial Z_k(t,s)}{\partial t}\right)\tilde{Y}_0(s)u_0\,ds \\ &= \int_0^\infty (-1)^m B_m \frac{\partial^{2m} Z_k(t,s)}{\partial s^{2m}}\tilde{Y}_0(s)u_0\,ds = \int_0^\infty (-1)^m B_m Z_k(t,s)\frac{d^{2m}\tilde{Y}_0(s)u_0}{ds^{2m}}\,ds \\ &= \int_0^\infty (-1)^m B_m Z_k(t,s)A^m\tilde{Y}_0(s)u_0 ds = -P_m(A)u(t). \end{aligned} \qquad (19)$$

Thus, relation (19) has been established for the elements u_0 of the set $D(A^{m+[n/2]+2})$ dense in $D(A^m)$. Owing to the boundedness of the operator function $W_k(t)$ in the space E, this relation remains valid for $u_0 \in D(A^m)$.

The case of $q = 0$ can be treated in a similar way with significant simplifications.

We will prove the uniqueness of the solution of problem (3), (4) by contradiction. Let $u_1(t)$ and $u_2(t)$ be two solutions of this problem. Consider the function

$$w(t,y) = f\Big(W_k(y)\big(u_1(t) - u_2(t)\big)\Big)$$

of two variables, where $f \in E^*$ (E^* is the dual space) and $t, y \geq 0$, which obviously satisfies the following equation and conditions:

$$\frac{\partial^2 w(t,y)}{\partial t^2} + \frac{k}{t}\frac{\partial w(t,y)}{\partial t} = \frac{\partial^2 w(t,y)}{\partial y^2} + \frac{k}{y}\frac{\partial w(t,y)}{\partial y}, \quad t, y > 0, \qquad (20)$$

$$w(0,y) = 0, \quad \sup_{t,y\geq 0}\big\|w(t,y)\big\| < M. \qquad (21)$$

We interpret the function $w(t,y)$ as a generalized function and apply the I-transform with respect to variable y. For ordinary functions decaying exponentially as $y \to +\infty$, the I-transform is defined by the relation

$$\hat{w}(t,\lambda) = \int_0^\infty \sqrt{\lambda y}\, I_p(\lambda y) w(t,y)\,dy,$$

where $p = (k-1)/2$ and $I_p(\cdot)$ is the modified Bessel function. The extension of this transform to generalized functions is presented in [23; 24, p. 63], while the space of test functions also includes functions that decay exponentially as $y \to +\infty$, on which the correct definition of the I-transform of the generalized function $w(t,y)$ is actually ensured.

From conditions (20), (21), for the transform $\hat{w}(t,\lambda)$ in the space of regular generalized functions we obtain the problem

$$\frac{\partial^2 \hat{w}(t,\lambda)}{\partial t^2} + \frac{k}{t}\frac{\partial \hat{w}(t,\lambda)}{\partial t} = \lambda^2 \hat{w}(t,\lambda), \quad t > 0, \qquad (22)$$

$$\hat{w}(0,\lambda) = 0, \quad \sup_{\substack{t\geq 0\\ \lambda\in\mathbb{R}}}\big\|\hat{w}(t,\lambda)\big\| < M. \qquad (23)$$

The general solution of Eq. (22) has the form

$$\hat{w}(t,\lambda) = t^{(1-k)/2}\big(d_1(\lambda) I_{(k-1)/2}(\lambda t) + d_2(\lambda) K_{(k-1)/2}(\lambda t)\big),$$

where $I_{(k-1)/2}(\cdot)$ is the modified Bessel function and $K_{(k-1)/2}(\cdot)$ is the Macdonald function.

The second condition in (23) implies that $d_1(\lambda) = 0$, and the first condition in (23) implies that $d_2(\lambda) = 0$; therefore $\hat{w}(t,\lambda) = w(t,y) = 0$ for any $y \geq 0$. Owing to the arbitrariness of the functional $f \in E^*$, for $y = 0$ we obtain the relation $u_1(t) \equiv u_2(t)$, and the uniqueness of the solution of the Dirichlet problem (3), (4) is thus established. The proof of the theorem is complete.

Example 1. Let $k < 1$, $B_m = I$, and $P_m(A)u(t) = (-1)^{m+1}A^m u(t)$. Then, taking into account formula (6), we obtain

$$Q(t,\sigma) = e^{-t\sigma^{2m}},$$
$$G(t,s) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{is\sigma - t\sigma^{2m}}\,d\sigma = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \cos(s\sigma)e^{-t\sigma^{2m}}\,d\sigma.$$

The last integral can be calculated in elementary functions only for $m = 1$ and determines the fundamental solution of the heat equation,

$$G(t,s)|_{m=1} = \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{s^2}{4t}\right).$$

For $m \geq 2$, the expression for the integral is very cumbersome and contains special functions. For example, for $m = 2$ it has the form

$$G(t,s) = \frac{\Gamma(5/4)}{\pi t^{1/4}}\,{}_0F_2\left(\frac{1}{2},\frac{3}{4};\frac{s^4}{256t}\right) - \frac{\Gamma(3/4)s^2}{8\pi t^{3/4}}\,{}_0F_2\left(\frac{5}{4},\frac{3}{2};\frac{s^4}{256t}\right),$$

where ${}_0F_2(\cdot)$ is the hypergeometric function.

Substituting the fundamental solution into (9) and taking into account integral 2.3.3.1 in [25], we determine

$$Z_k(t,s)|_{m=1} = \frac{t^{1-k}}{2^{2-k}\sqrt{\pi}\Gamma(1/2-k/2)}\int\limits_0^\infty \tau^{k/2-2}\exp\left(-\frac{t^2+s^2}{4\tau}\right)d\tau$$
$$= \frac{t^{1-k}}{2^{2-k}\sqrt{\pi}\Gamma(1/2-k/2)}\int\limits_0^\infty \xi^{-k/2}\exp\left(-\frac{t^2+s^2}{4}\xi\right)d\xi = \frac{\Gamma(1-k/2)t^{1-k}}{\sqrt{\pi}\Gamma(1/2-k/2)}(t^2+s^2)^{k/2-1}.$$

Finally, using formula (16), we write the solution of the Dirichlet problem (3), (4) for $m = 1$ in the form

$$u(t) = W_k(t)u_0 = \frac{(-1)^n\cdot 2}{(2n-1)!!}\int\limits_0^\infty\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s)s^{2n}Y_{2n}(s)u_0\,ds$$
$$= \frac{(-1)^n\cdot 2\Gamma(1-k/2)t^{1-k}}{(2n-1)!!\sqrt{\pi}\,\Gamma(1/2-k/2)}\int\limits_0^\infty\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n(t^2+s^2)^{k/2-1}s^{2n}Y_{2n}(s)u_0\,ds$$
$$= \frac{(-1)^n\cdot 2\Gamma(1-k/2)t^{1-k}}{(2n-1)!!\sqrt{\pi}\,\Gamma(1/2-k/2)}\int\limits_0^\infty(k-2)(k-4)\cdots(k-2n)(t^2+s^2)^{k/2-n-1}s^{2n}Y_{2n}(s)u_0\,ds$$
$$= \frac{2\Gamma(n+1-k/2)t^{1-k}}{\Gamma(n+1/2)\Gamma(1/2-k/2)}\int\limits_0^\infty\frac{s^{2n}Y_{2n}(s)u_0}{(t^2+s^2)^{n+1-k/2}}\,ds.$$

Below we give examples of representations of the solution of the Dirichlet problem (3), (4) for $k < 1$, $m = 1$ in specific Banach spaces.

(a) Let $E = L_p(-\infty, \infty)$, $p > 1$, let $Y_0(s)u_0(x) = (u_0(x+s) + u_0(x-s))/2$ be the operator cosine function with generator $A = d^2/dx^2$, and let $B_1 = I$. Then for $k < 1$ and $m = 1$ the solution of problem (3), (4) has the form

$$u(t,x) = \frac{\Gamma(1-k/2)t^{1-k}}{\sqrt{\pi}\Gamma(1/2-k/2)} \int\limits_0^\infty \frac{u_0(x+s) + u_0(x-s)}{(t^2+s^2)^{1-k/2}}\, ds.$$

(b) Let $E = L_p(0, \infty)$, $p > 1$, and let $B_1 = I$. If $0 < q \le 2$, then the generalized shift operator $T_x^s u_0(x)$ defined by relation (1) (after replacing the parameter k by q) is the BOF $Y_q u_0(x) = T_x^s u_0(x)$ with generator

$$A = \frac{d^2}{dx^2} + \frac{q}{x}\frac{d}{dx}.$$

Then $n = 1$, and in this case for $k < 1$ and $m = 1$ the solution of the Dirichlet problem (3), (4) has the form

$$u(t,x) = W_k(t)u_0 = \frac{-2\Gamma(1-k/2)t^{1-k}}{\sqrt{\pi}\Gamma(1/2-k/2)} \int\limits_0^\infty \frac{1}{s}\frac{\partial}{\partial s}(t^2+s^2)^{k/2-1} s^2 Y_2(s)u_0(x)\, ds$$

$$= \frac{4\Gamma(2-k/2)t^{1-k}}{\sqrt{\pi}\,\Gamma(1/2-k/2)} \int\limits_0^\infty \frac{s^2 Y_2(s)u_0(x)}{(t^2+s^2)^{2-k/2}}\, ds,$$

where for $q = 2$

$$Y_2(s)u_0(x) = \frac{1}{2}\int\limits_0^\pi u_0\left(\sqrt{x^2+s^2-2xs\cos\varphi}\right)\sin\varphi\, d\varphi, \quad x, s \ge 0,$$

while for $0 < q < 2$ the BOF $Y_q(s)$ is determined using formula (2),

$$Y_2(s)u_0(x) = \frac{\sqrt{\pi}}{\Gamma(q/2+1/2)\Gamma(1-q/2)} \int\limits_0^1 \tau^q(1-\tau^2)^{-q/2} Y_q(\tau s)u_0(x)\, d\tau,$$

$$Y_q(s)u_0(x) = \frac{\Gamma(q/2+1/2)}{\sqrt{\pi}\Gamma(q/2)} \int\limits_0^\pi u_0\left(\sqrt{x^2+s^2-2xs\cos\varphi}\right)\sin^{q-1}\varphi\, d\varphi, \quad x, s \ge 0.$$

(c) Let $E = \mathbb{R}$, $A = -A_0^2$, $A_0 > 0$, and $B_1 = 1$. Then the easiest way is to consider $n = 0$, $Y_0(s) = \cos(A_0 s)$, and for $k < 1$ and $m = 1$ the solution of the Dirichlet problem (3), (4) has the form

$$u(t) = W_k(t)u_0 = \frac{2\Gamma(1-k/2)t^{1-k}u_0}{\sqrt{\pi}\Gamma(1/2-k/2)} \int\limits_0^\infty \frac{\cos(A_0 s)}{(t^2+s^2)^{1-k/2}}\, ds.$$

Calculating the integral by formula 2.5.6.4 in [25], we obtain

$$u(t) = \frac{2^{k/2+1/2}(A_0 t)^{1/2-k/2}}{\Gamma(1/2-k/2)} K_{1/2-k/2}(A_0 t)u_0,$$

where $K_\nu(\cdot)$ is the Macdonald function.

We arrive at the same result if we take $q = 2n$,

$$Y_{2n}(s) = \Gamma(n+1/2)(A_0 s/2)^{1/2-n} J_{n-1/2}(A_0 s),$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind. Then

$$u(t) = \frac{2^{n+1/2}\Gamma(n+1-k/2)A_0^{1/2-n}t^{1-k}u_0}{\Gamma(1/2-k/2)} \int\limits_0^\infty (t^2+s^2)^{k/2-n-1}s^{n+1/2}J_{n-1/2}(A_0 s)\,ds.$$

Calculating the integral using formula 2.12.4.28 in [26], we have

$$u(t) = \frac{2^{k/2+1/2}(A_0 t)^{1/2-k/2}}{\Gamma(1/2-k/2)}K_{1/2-k/2}(A_0 t)u_0.$$

Note that due to the exponential decay of the Macdonald function as $t \to \infty$ in the last example, the solution $u(t) = W_k(t)u_0$ tends to zero as $t \to \infty$. However, in the general case, it does not follow from the estimate (15) that the solution $u(t) = W_k(t)u_0$ tends to zero as $t \to \infty$. Let us present a sufficient condition that ensures this convergence.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied, and additionally, for* $s > 0$ *let*

$$\left\| \int\limits_0^s Y_q(\tau)u_0\,d\tau \right\| < \infty. \tag{24}$$

Then $\lim\limits_{t\to\infty} W_k(t)u_0 = 0$.

Proof. By integrating by parts, we write the solution of the Dirichlet problem in the form

$$\begin{aligned} u(t) = W_k(t)u_0 &= \frac{(-1)^n \cdot 2}{(2n-1)!!} \int\limits_0^\infty \left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s)s^{2n}Y_{2n}(s)u_0\,ds \\ &= \frac{(-1)^n \cdot 2}{(2n-1)!!} \int\limits_0^\infty \frac{\partial}{\partial s}\left(s^{2n}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s)\right) \int\limits_0^s Y_{2n}(\tau)u_0\,d\tau ds. \end{aligned} \tag{25}$$

Taking into account relation (13), by differentiation we obtain

$$\frac{\partial}{\partial s}\left(s^{2n}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s)\right) = \sum_{j=1}^n j\theta_{j,n}s^{j-1}\frac{\partial^j Z_k(t,s)}{\partial s^j} + \sum_{j=1}^n \theta_{j,n}s^j\frac{\partial^{j+1}Z_k(t,s)}{\partial s^{j+1}};$$

applying the estimates (12) and (24) in (25), we have

$$\begin{aligned} \big\|W_k(t)u_0\big\| &\le \Upsilon_1\|u_0\|t^{1-k}\sum_{j=1}^n j|\theta_{j,n}|M_{k,j}\int\limits_0^\infty \frac{s^{j-1}\,ds}{t^{1-k+(j+1)/m}+s^{m(1-k)+j+1}} \\ &+ \Upsilon_1\|u_0\|t^{1-k}\sum_{j=1}^n |\theta_{j,n}|M_{k,j+1}\int\limits_0^\infty \frac{s^j\,ds}{t^{1-k+(j+2)/m}+s^{m(1-k)+j+2}} = \Phi(t)+\Psi(t). \end{aligned}$$

In this relation, after the changes of variables

$$s = t^\alpha\xi, \quad \alpha = \frac{1-k+(j+1)/m}{m(1-k)+j+1}, \quad \beta = \frac{-m(1-k)-j-1}{m\big(m(1-k)+j+1\big)},$$

the first term $\Phi(t)$ becomes

$$\Phi(t) = \Upsilon_1 \|u_0\| t^{\beta} \sum_{j=1}^{n} j\theta_{j,n} M_{k,j} \int_0^{\infty} \frac{\xi^{j-1}\, ds}{1+\xi^{m(1-k)+j+1}},$$

and since $\beta < 0$, we conclude that $\Phi(t) \to 0$ as $t \to \infty$.

The second term $\Psi(t)$ can be treated in a similar way with j replaced by $j+1$; the latter implies the desired assertion of the theorem. The proof of the theorem is complete.

For example, if, under the assumptions of Theorem 1, the operator A is the generator of a uniformly bounded operator cosine function $Y_0(s) = C(s)$, then, as follows from Theorem 2, in addition to this, for the solution $u(t) = W_k(t)u_0$ to tend to zero as $t \to \infty$ one should additionally require the boundedness of the operator sine function

$$S(s) = \int_0^t C(\tau)\, d\tau.$$

Here condition (24) is satisfied if, for example, $A = A_1^2$, where the operator A_1 is the generator of a uniformly bounded group $T(s; A_1)$ and in addition, the point $\lambda = 0$ is a regular point of the operator A_1, $0 \in \rho(A_1)$. Then

$$Y_0(s) = C(s) = \frac{1}{2}\big(T(s; A_1) + T(-s; A_1)\big),$$

$$\int_0^s T(\tau; A_1)u_0\, d\tau = \int_0^s AT(\tau; A_1)A^{-1}u_0\, d\tau$$

$$= \int_0^s T'(\tau; A_1)A^{-1}u_0\, d\tau = \big(T(s; A_1) - I\big)A^{-1}u_0,$$

and condition (24) is obviously satisfied, because the group $T(s; A_1)$ is uniformly bounded.

Example 2. Consider the case in which $E = H$ is a Hilbert space and $A = -A_0^2$, where A_0 is a self-adjoint operator acting in H, $0 \in \rho(A_0)$. Let E_λ be the spectral function of the operator A_0. By Stone's theorem (see, e.g., [1, Sec. 4, Theorem 4.7]), the operator A_0 is the generator of the unitary group

$$T(t; A_0)x = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE_\lambda x, \quad x \in H,$$

which satisfies inequality (24). Indeed,

$$\int_0^s T(\tau; A_0)x\, d\tau = \int_{-\infty}^{\infty} \int_0^s e^{i\lambda\tau} d\tau\, dE_\lambda x$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\lambda s} - 1}{i\lambda} dE_\lambda x = -i\big(T(s; A_0) - I\big)A_0^{-1}x;$$

hence the operator sine function is bounded, inequality (24) holds true, and the solution of the Dirichlet problem

$$u(t) = W_k(t)u_0 = 2\int_0^{\infty} Z_k(t, s)\cos(A_0 s)u_0\, ds$$

tends to zero as $t \to \infty$.

3. DIRICHLET PROBLEM IN THE CASE OF $k<1$ WITH AN OPERATOR OF THE FORM $P_m(A)u(t) = (-1)^{m+1}B_m \times A^m u(t) + \sum_{n=0}^{m-1} B_n A^n u(t)$, $\sum_{n=0}^{m-1} B_n A^n \neq 0$

Let us introduce the operator

$$B = -\mu^{2m} B_m - \sum_{n=0}^{m-1} \mu^{2n} B_n, \quad \mu \in \mathbb{R}.$$

Condition 2. If $\sum_{n=0}^{m-1} B_n A^n \neq 0$, then for any $\mu \in \mathbb{R}$ the spectrum $\sigma(B)$ of the operator B does not lie on the imaginary axis.

For the case where Condition 2 is satisfied for the fundamental solution $G(t,s)$ defined by relation (6), the paper [10] establishes the estimate

$$\left\| \frac{\partial^j G(t,s)}{\partial s^j} \right\| \le M_j t^{-(j+1)/(2m)} \exp\big(-a t^{1/(1-2m)} |s|^{2m/(2m-1)} - \delta_1 t\big), \tag{26}$$

where $M_j > 0$, $a > 0$, $0 < \delta_1 < \delta$, and the constant δ is taken from Condition 1 (parabolicity). Note that unlike the estimate (11), the estimate (26) contains the factor $\exp(-\delta_1 t)$.

Let us show that Condition 2 permits relaxing the requirement of uniform boundedness of the BOF $Y_q(s)$ when establishing the solvability of the Dirichlet problem and allows its exponential growth.

Lemma 5. *Under Condition* 2, *for the operator function* $Z_k(t,s)$ *defined by relation* (9) *and its derivatives up to order* $j = 0, \dots, 2m$ *there exist constants* $M_{k,j}$, Ω, $\Omega_1 > 0$ *such that for* $t > 0$ *one has the estimate*

$$\left\| \frac{\partial^j Z_k(t,s)}{\partial s^j} \right\| \le \frac{M_{k,j} t^{1-k}}{t^{1-k+(j+1)/m} + |s|^{m(1-k)+j+1}} e^{-\Omega_1 t - \Omega |s|}. \tag{27}$$

Proof. Just as in the proof of Lemma 2, let us differentiate relation (9) and use the estimate (26). Denoting $a = a_1 + a_2$, $0 < a_1 < a$, $a_2 = a - a_1$, $1 = b_1 + b_2$, $0 < b_1 < 1$, $b_2 = 1 - b_1$, $\delta_1 = \delta_2 + \delta_3$, $0 < \delta_2 < \delta_1$, and $\delta_3 = \delta_1 - \delta_2$, we obtain

$$\begin{aligned} \left\| \frac{\partial^j Z_k(t,s)}{\partial s^j} \right\| &\le M_j \int_0^\infty h_k(t,\tau) \tau^{-(j+1)/(2m)} \exp\big(-a\tau^{1/(1-2m)} |s|^{2m/(2m-1)} - \delta_1 \tau\big)\, d\tau \\ &= \frac{M_j t^{1-k}}{2^{k-1}\Gamma(1/2 - k/2)} \int_0^\infty \tau^{k/2 - 3/2 - (j+1)/(2m)} \\ &\quad \times \exp\left(-\frac{(b_1+b_2)t^2}{4\tau} - (a_1+a_2)\tau^{1/(1-2m)}|s|^{2m/(2m-1)} - (\delta_2+\delta_3)\tau\right) d\tau. \end{aligned} \tag{28}$$

Further, let us show that for $s \in \mathbb{R}$ and $t, \tau > 0$ one has the inequalities

$$\exp\big(-a_2 \tau^{1/(1-2m)} |s|^{2m/(2m-1)} - \delta_3 \tau\big) \le e^{-\Omega|s|}, \tag{29}$$

$$\exp(-b_2 \tau^{-1} t^2 - \delta_2 \tau) \le e^{-\Omega_1 t}, \tag{30}$$

where

$$\begin{aligned} \Omega &= a_2^{(2m-1)/(2m)} \big(\delta_3(2m-1)\big)^{1/(2m)} + \delta_3^{1/(2m)} \big(a_2/(2m-1)\big)^{(2m-1)/(2m)}, \\ \Omega_1 &= 2\sqrt{b_2 \delta_2}. \end{aligned} \tag{31}$$

Obviously, inequality (29) is satisfied for $s = 0$. Now let $s \neq 0$. Let us prove the relation

$$a_2 \tau^{1/(1-2m)} |s|^{2m/(2m-1)} + \delta_3 \tau \ge \Omega |s|$$

or the one equivalent to it,

$$a_2\left(\frac{|s|}{\tau}\right)^{1/(2m-1)}+\delta_3\frac{\tau}{|s|}\geq\Omega.$$

The least value of the function $\varphi(t)=a_2t^{1/(2m-1)}+\delta_3/t$ for $t>0$ is equal to Ω, as proved by the estimate (29).

Inequality (30) is obtained from (29) by replacing a_2 with b_2, m with 1, s with t, and δ_3 with δ_2.

Taking into account inequalities (29) and (30), from (28) we obtain

$$\left\|\frac{\partial^j Z_k(t,s)}{\partial s^j}\right\|\leq\frac{M_jt^{1-k}}{2^{k-1}\Gamma(1/2-k/2)}e^{-\Omega_1t-\Omega|s|}\times\int\limits_0^\infty\tau^{k/2-3/2-(j+1)/(2m)}\exp\left(-\frac{b_1t^2}{4\tau}-a_1\tau^{1/(1-2m)}|s|^{2m/(2m-1)}\right)d\tau. \tag{32}$$

The estimate of the integral in inequality (32) was actually carried out earlier in Lemma 2. Applying this estimate, we arrive at the desired inequality (27). The proof of the lemma is complete.

The next lemma is a straightforward consequence of Lemmas 3 and 5.

Lemma 6. *Under Condition* 2, *for the operator function* $Z_k(t,s)$ *defined by relation* (9) *with* $t>0$ *one has the estimate*

$$\left\|\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^nZ_k(t,s)\right\|\leq t^{1-k}e^{-\Omega_1t-\Omega|s|}\sum_{j=1}^n\frac{|\theta_{j,n}|M_{k,j}s^{j-2n}}{t^{1-k+(j+1)/m}+|s|^{m(1-k)+j+1}},\quad M_{k,j}>0. \tag{33}$$

Theorem 3. *Let Conditions* 1 *and* 2 *be satisfied, let* $u_0\in D(A^m)$, *and let for some* $q\geq0$ *the operator* A *be the generator of a BOF* $Y_q(s)$ *satisfying the estimate*

$$\left\|Y_q(s)\right\|\leq\Upsilon e^{\omega s},\quad s\geq0,\quad\Upsilon\geq1,\quad0\leq\omega<\Omega, \tag{34}$$

where Ω *is the constant in* (31). *Then the Dirichlet problem* (3), (4) *has a unique solution tending to zero as* $t\to\infty$, *which can be represented in the form*

$$u(t)=W_k(t)u_0=\frac{(-1)^n\cdot2}{(2n-1)!!}\int\limits_0^\infty\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^nZ_k(t,s)s^{2n}Y_{2n}(s)u_0\,ds,$$

where $Z_k(t,s)$ *is defined by relation* (9) *and the BOF* $Y_{2n}(s)$ *is expressed via the BOF* $Y_q(s)$ *by formula* (2).

Proof. The convergence of the integral defining the function $u(t)=W_k(t)u_0$ and the possibility of differentiation of this integral are determined by the estimate (33). Let us verify that this function is a solution of problem (3), (4).

Just as in the proof of Theorem 1, let us first assume that $u_0\in D(A^{m+[n/2]+2})$, $q>0$, and $\tilde{Y}_0(s)u_0$ is defined by relation (18). After integration by parts, the integrated terms vanish, and we obtain

$$u''(t)+\frac{k}{t}u'(t)=2\int\limits_0^\infty\left(\frac{\partial^2Z_k(t,s)}{\partial t^2}+\frac{k}{t}\frac{\partial Z_k(t,s)}{\partial t}\right)\tilde{Y}_0(s)u_0\,ds$$
$$=2\int\limits_0^\infty\left((-1)^mB_m\frac{\partial^{2m}Z_k(t,s)}{\partial s^{2m}}-\sum_{n=0}^{m-1}B_n\frac{\partial^{2n}Z_k(t,s)}{\partial s^{2n}}\right)\tilde{Y}_0(s)u_0\,ds$$

$$= 2\int\limits_0^\infty Z_k(t,s)\left((-1)^m B_{2m}\frac{d^{2m}\tilde{Y}_0(s)u_0}{ds^{2m}} - \sum_{n=0}^{m-1} B_n \frac{d^{2n}\tilde{Y}_0(s)u_0}{ds^{2n}}\right)ds$$

$$= 2\int\limits_0^\infty Z_k(t,s)\left((-1)^m B_m A^m \tilde{Y}_0(s)u_0 - \sum_{n=0}^{m-1} B_n A^n \tilde{Y}_0(s)u_0\right)ds$$

$$= -2P_m(A)\int\limits_0^\infty Z_k(t,s)\tilde{Y}_0(s)u_0\,ds$$

$$= \frac{(-1)^n\cdot 2}{(2n-1)!!}P_m(A)\int\limits_0^\infty \left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_k(t,s)s^{2n}Y_{2n}(s)u_0\,ds = -P_m(A)u(t).$$

Thus, the function $W_k(t)u_0$ is a solution of Eq. (3) on the set $D(A^{m+[n/2]+2})$, dense in $D(A^m)$, of elements u_0. Owing to the boundedness of the operator function $W_k(t)$ in the space E, the statement also holds true for $u_0 \in D(A^m)$.

The case of $q = 0$ can be treated in a similar way with significant simplifications.

Since the estimate (33) contains the factor $\exp(-\Omega_1 t)$, it obviously follows that the solution $u(t)$ tends to zero as $t \to \infty$.

The remaining statements of Theorem 3 in fact have already been established in Theorem 1. The proof of the theorem is complete.

Remark. If Condition 2 is not satisfied in Theorem 3, then, just as in Theorem 1, the BOF $Y_q(s)$ must be uniformly bounded. In this case, the indicated solution of the Dirichlet problem (3), (4), generally speaking, does not have to tend to zero as $t \to \infty$. For the solution to tend to zero, just as in Theorem 2, the BOF $Y_q(s)$ should additionally be required to satisfy inequality (24).

Example 3. Let $E = \mathbb{R}$, $m = 1$, $P_1(A) = B_1A + B_0$, where $B_1 > 0$ (the parabolicity condition), $A = -A_0^2$, $A_0 \in \mathbb{R}$, and then $Y_0(s) = \cos(A_0 s)$. Assume also that $B_0 < 0$ (the condition of ellipticity of the polynomial $P_1(A)$) with the operator $B = -\mu^2 B_1 - B_0$ satisfying Condition 2 for $\mu \in \mathbb{R}$. Then, taking into account the results in Example 1, we obtain

$$Q(t,\sigma) = \exp(-tB_1\sigma^2 + tB_0),$$

$$G(t,s) = \frac{1}{2\sqrt{\pi t B_1}}\exp\left(-\frac{s^2}{4tB_1} + tB_0\right),$$

$$Z_k(t,s) = \frac{t^{1-k}}{2^{2-k}\Gamma(1/2-k/2)\sqrt{\pi B_1}}\int\limits_0^\infty \tau^{k/2-2}\exp\left(-\frac{t^2}{4\tau} - \frac{s^2}{4\tau B_1} + \tau B_0\right)d\tau$$

$$= \frac{t^{1-k}}{2^{2-k}\Gamma(1/2-k/2)\sqrt{\pi B_1}}\int\limits_0^\infty \tau^{k/2-2}\exp\left(-\frac{t^2B_1 + s^2}{4\tau B_1} + \tau B_0\right)d\tau$$

$$= \frac{2^{k/2}t^{1-k}(t^2B_1+s^2)^{k/4-1/2}}{\Gamma(1/2-k/2)\sqrt{\pi B_1}(-B_0B_1)^{k/4-1/2}}K_{1-k/2}\left(\frac{\sqrt{-B_0B_1(t^2B_1+s^2)}}{B_1}\right);$$

here we have used integral 2.3.16.1 in [25].

Using formula (16), we write the solution of the Dirichlet problem (3), (4) for $m = 1$ decaying as $t \to \infty$ in the form

$$u(t) = W_k(t)u_0 = 2\int\limits_0^\infty Z_k(t,s)\cos(A_0 s)u_0\,ds.$$

4. WEIGHTED BOUNDARY VALUE PROBLEMS IN THE CASE OF $k \geq 1$

A straightforward verification establishes the following assertion.

Lemma 7. *If for $k < 1$ the function $v_k(t)$ satisfies the Dirichlet problem* (3), (4), *then for $k > 1$ the function $u(t) = t^{1-k}v_{2-k}(t)$ is a solution of Eq.* (3) *bounded at infinity and satisfying the condition*

$$\lim_{t\to 0} t^{k-1}u(t) = u_0. \tag{35}$$

Lemma 7 and Theorem 3 imply the following statement.

Theorem 4. *Let $k > 1$, let $u_0 \in D(A^m)$, let Conditions* 1 *and* 2 *be satisfied, and let the operator A for some $q \geq 0$ be the a generator of a BOF $Y_q(s)$ satisfying the estimate* (34). *Then the weighted Dirichlet problem* (3), (35) *has a unique solution tending to zero as $t \to \infty$, which can be represented in the form*

$$u(t) = \frac{(-1)^n \cdot 2t^{1-k}}{(2n-1)!!} \int_0^\infty \left(\frac{1}{s}\frac{\partial}{\partial s}\right)^n Z_{2-k}(t,s)s^{2n}Y_{2n}(s)u_0\,ds,$$

where $Z_{2-k}(t,s)$ is defined by relation (9), *while the BOF $Y_{2n}(s)$, $2n \geq q$, is expressed via the BOF $Y_q(s)$ using formula* (2).

As was already established earlier, if Condition 2 is satisfied and the operator A is the generator of the operator cosine function $Y_0(s)$, then the semigroup $U(t; P_m(A))$ defined by relation (11) is a contraction; therefore, by Theorem 5.6 in [1, Ch. 1, Sec. 5], the operator

$$P_{1/2}(A) = -\sqrt{-P_m(A)}$$

is the generator of the contraction semigroup $U_1(t; P_{1/2}(A))$ given by

$$U_1\big(t; P_{1/2}(A)\big) = \frac{1}{\pi}\int_0^\infty \sin\left(t\sqrt{\tau}\right)\left(\tau I + P_{1/2}(A)\right)^{-1} d\tau.$$

The following holds by virtue of Theorem 4.1 in [27].

Theorem 5. *Let $k \geq 1$, let $u_0 \in D(A^{2m})$, let Conditions* 1 *and* 2 *be satisfied, and let the operator A be the generator of an operator cosine function $Y_0(s)$ satisfying the estimate* (34). *Then the function*

$$u(t) = -\frac{1}{\Gamma(k)}\int_1^\infty (\xi^2-1)^{k/2-1}U_1\big(t\xi; P_{1/2}(A)\big)\big(-P_{1/2}(A)\big)^{k-1}u_1\,d\xi$$

is the unique solution of Eq. (3) *with the conditions*

$$\lim_{t\to 0} t^k u'(t) = u_1, \qquad \lim_{t\to\infty} u(t) = 0. \tag{36}$$

Theorem 5, in comparison with problem (3), (35), contains the solution of another weighted boundary value problem for $k > 1$ as well as the solution of problem (3), (36) for $k = 1$, which has the form

$$u(t) = -\int_1^\infty (\xi^2-1)^{-1/2}U_1\left(t\xi; -\sqrt{-P_m(A)}\right)u_1\,d\xi.$$

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