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On the Solvability of Boundary Value Problems for an Abstract Singular Equations on a Finite Interval

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Abstract—Sufficient conditions for the unique solvability of boundary value problems for a number of abstract singular equations that are formulated in terms of the zeros of the modified function Bessel and the resolvent of the operator coefficient of the considered equations.

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1. INTRODUCTION

Let E be a Banach space and A be a closed linear operator in E whose domain $D(A) \subset E$ is not necessarily dense in E. Consider the problem of defining the function

$$u(t) \in C([0,1],E) \bigcap C^2((0,1],E) \bigcap C((0,1),D(A)),$$

satisfying the abstract Euler–Poisson–Darboux equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad 0 < t < 1, \tag{1}$$

as well as some boundary conditions. The statement of boundary conditions for the Euler–Poisson–Darboux equation, due to the singularity of the equation at the point $t = 0$, depends on the parameter $k \in \mathbb{R}$. Various types of boundary conditions at the points $t = 0$ and $t = 1$, as well as the corresponding criteria for the uniqueness of the solution boundary and nonlocal problems were established in [1, 2].

Problems of solvability of boundary value problems for a nonsingular second-order equation (the case $k = 0$ in the equation (1)) with various assumptions on the operator A can be found in ([3], Ch. 3, Sect. 2; [4]; [5], Ch. 2; [6]). Results on the solvability of boundary value problems in a half-space for the Euler–Poisson–Darboux equation in partial derivatives are given in [7, Sect. 41], and the boundary value problems on the semiaxis for abstract singular equations were studied in [8, 9]. Historical information and a detailed range of questions for equations containing the Bessel operator can be found in the introduction of monographs [10, 11].

In this paper, we present sufficient conditions for the unique solvability of the Dirichlet and Neumann boundary value problems for an abstract Euler–Poisson–Darboux equation (1) and also for a number of degenerate differential equations on a finite interval $[0, 1]$.

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2. THE $k<1$ CASE FOR THE EULER–POISSON–DARBOUX EQUATION. DIRICHLET CONDITION FOR $t=0$

For the Dirichlet problem of the form

$$u(0)=u_0, \quad u(1)=u_1 \tag{2}$$

in [1], the following criterion for the uniqueness of a solution was proved.

Theorem 1. *Let $k<1$ and A be a linear closed operator in E. We assume that the boundary problem (1) and (2) has a solution $u(t)$. For this solution to be unique, necessary and sufficient, that none of the λ_n, $n\in\mathbb{N}$ zeros of the function*

$$\Upsilon_{2-k}(\lambda)=Y_{2-k}(1;\lambda), \tag{3}$$

where

$$Y_{2-k}(t;\lambda)=\Gamma(3/2-k/2)\left(t\sqrt{\lambda}/2\right)^{k/2-1/2} I_{1/2-k/2}\left(t\sqrt{\lambda}\right),$$

$\Gamma(\cdot)$ is Euler gamma function, $I_\nu(\cdot)$ is modified Bessel function, would not be an eigenvalue of the operator A, i.e., $\lambda_n\notin\sigma_p(A)$.

Theorem 1 is established under very general conditions on the operator A, which do not ensure the solvability of the Dirichlet problem. In the following theorem, we give sufficient conditions for its unique solvability.

Theorem 2. *Let $k<1$, A be a linear closed operator in E, $u_1\in D(A^2)$, and also for all $n\in\mathbb{N}$ the zeros of λ_n defined by the equality (3) of the function $\Upsilon_{2-k}(\lambda)$, belong to the resolvent set $\rho(A)$, and the estimate*

$$\sup_{n\in\mathbb{N}}|\lambda_n|\,||(\lambda_n I-A)^{-1}||<M_0<\infty. \tag{4}$$

Then, the problem

$$u''(t)+\frac{k}{t}u'(t)=Au(t), \quad u(0)=0, \quad u(1)=u_1 \tag{5}$$

is uniquely solvable and its solution has the form

$$u(t)=-2t^{1-k}\sum_{n=1}^{\infty}\frac{Y_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}\lambda_n(\lambda_n I-A)^{-1}u_1. \tag{6}$$

Proof. As is known ([12], points 15.33–15.35), zeros λ_n of the function $\Upsilon_{2-k}(\lambda)$ simple and negative, and $\lim\limits_{n\to\infty}\frac{\lambda_n}{n^2}=-\pi^2$.

Arrange them in descending order and using the equality

$$\lambda_n(\lambda_n I-A)^{-1}u_1=I+A(\lambda_n I-A)^{-1}u_1, \tag{7}$$

let us write (for now formally) the series (6) in the form

$$u(t)=\psi_k(t)u_1-2t^{1-k}\sum_{n=1}^{\infty}\frac{Y_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}A(\lambda_n I-A)^{-1}u_1, \tag{8}$$

where

$$\psi_k(t)=-2t^{1-k}\sum_{n=1}^{\infty}\frac{Y_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}. \tag{9}$$

In particular, it follows from the equality (7) that the multiplication of the resolvent $(\lambda_n\, I-A)^{-1}\, u_1$ by a λ_n corresponds to the application of the operator A.

Denoting $\sqrt{\lambda_n}=i\mu_n$, defined by the equality (9), the function $\psi_k(t)$ in terms of Bessel functions of the first kind $J_\nu(\cdot)$ can be rewritten in the form

$$\psi_k(t)=-2t^{1-k}\sum_{n=1}^{\infty}\frac{Y_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}=-2t^{1/2-k/2}\sum_{n=1}^{\infty}\frac{I_{1/2-k/2}(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}I'_{1/2-k/2}(\sqrt{\lambda_n})}$$

$$= -2t^{1/2-k/2}\sum_{n=1}^{\infty}\frac{J_{1/2-k/2}(t\mu_n)}{\mu_n J'_{1/2-k/2}(\mu_n)} = 2t^{1/2-k/2}\sum_{n=1}^{\infty}\frac{J_{1/2-k/2}(t\mu_n)}{\mu_n J_{3/2-k/2}(\mu_n)} = t^{1-k}; \qquad (10)$$

the well-known relation $zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$ for the derivative Bessel functions of the first kind, as well as the Fourier—Bessel expansion of a power function (see [13, 5.7.33.1])

$$\sum_{n=1}^{\infty}\frac{J_{1/2-k/2}(t\mu_n)}{\mu_n J_{3/2-k/2}(\mu_n)} = \frac{1}{2}t^{1/2-k/2}, \quad 0 \le t < 1. \qquad (11)$$

Thus, $\psi_k(t)u_1 = t^{1-k}u_1$, $t \in [0,1]$ and, as is easy to see, this function is a solution to the problem (5) for $A = 0$.

Next, we study the convergence of the series in the formula (8). Same as in formulas (10) let's write it in the form

$$\sum_{n=1}^{\infty}\frac{J_{1/2-k/2}(t\mu_n)}{\mu_n J_{3/2-k/2}(\mu_n)}A(\lambda_n I - A)^{-1}u_1, \quad \lambda_n = -\mu_n^2, \quad u_1 \in D(A). \qquad (12)$$

Using the Abel transform, one establishes (see [14, p. 306]) simultaneous convergence, moreover, to the same the sum of the next series

$$\sum_{n=1}^{\infty}a_n b_n, \quad \sum_{n=1}^{\infty}(a_1 + a_2 + \cdots + a_n)(b_n - b_{n+1})$$

provided that

$$\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n)b_n = 0. \qquad (13)$$

Let's put

$$a_n = \frac{J_{1/2-k/2}(t\mu_n)}{\mu_n J_{3/2-k/2}(\mu_n)}, \quad b_n = A(\lambda_n I - A)^{-1}u_1.$$

Then, due to (11)

$$|a_1 + a_2 + \cdots + a_n| \le M_1 t^{1/2-k/2}, \quad M_1 > 0, \qquad (14)$$

and the difference $b_n - b_{n+1}$ is estimated using the inequality (4). Get

$$||b_n - b_{n+1}|| \le M_0\left(\frac{1}{|\lambda_n|} + \frac{1}{|\lambda_{n+1}|}\right)||Au_1|| \le \frac{M_2||Au_1||}{n^2}. \qquad (15)$$

The condition (13) is obviously satisfied, and taking into account (14), (15), we have

$$\left|\left|\sum_{n=1}^{\infty}(a_1 + a_2 + \cdots + a_n)(b_n - b_{n+1})\right|\right| \le M_3 t^{1/2-k/2}\sum_{n=1}^{\infty}\frac{1}{n^2}||Au_1||,$$

therefore, the series (12) converges absolutely and uniformly in $t \in [0,1]$ and, consequently, also the series in the formulas (6), (8) also converge.

It is easy to see that the representation (8) implies the validity of the boundary conditions $u(0) = 0$, $u(1) = u_1$ of problem (5).

Let us show that the series in the formula (8) can be term by term differentiated as $u_1 \in D(A^2)$. Consider a series of derivatives

$$\sum_{n=1}^{\infty}\frac{Y'_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}A(\lambda_n I - A)^{-1}u_1$$

and transform it using the equality

$$Y'_{2-k}(t;\lambda_n) = \frac{\lambda_n t}{3-k}Y_{4-k}(t;\lambda_n),$$

and the shift formula with respect to the parameter (see [15]). As a result (up to a power factor) we will have the series

$$\frac{1}{3-k}\sum_{n=1}^{\infty}\frac{\lambda_n Y_{4-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}A(\lambda_n I-A)^{-1}u_1$$

$$=\sum_{n=1}^{\infty}\frac{\lambda_n}{Y'_{2-k}(1;\lambda_n)}\int_0^1(1-s^2)s^{2-k}Y_{2-k}(ts;\lambda_n)\,ds\,A(\lambda_n I-A)^{-1}u_1$$

$$=\int_0^1(1-s^2)s^{2-k}\left(\sum_{n=1}^{\infty}\frac{Y_{2-k}(ts;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}\lambda_n(\lambda_n I-A)^{-1}Au_1\right)ds.$$

Using the equality (7), we obtain the sum of two series whose uniform convergence has already been proven earlier and which can be integrated term by term by s. Thus, it is established that the series in the formula (8) can be differentiated term by term as $u_1\in D(A)$.

Since by the condition $u_1\in D(A^2)$, the possibility of one more differentiation of the series in the formula (8) installed in the same way.

We verify by direct differentiation that the function $u(t)$ defined by the equality (8) satisfies task (5). To do this, we calculate its derivatives. We have

$$u'(t)=\psi'(t)u_1-2\sum_{n=1}^{\infty}\frac{(1-k)t^{-k}Y_{2-k}(t;\lambda_n)+t^{1-k}Y'_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}A(\lambda_n I-A)^{-1}u_1, \tag{16}$$

$$u''(t)=\psi''(t)u_1$$
$$-2\sum_{n=1}^{\infty}\frac{(1-k)(-k)t^{-k-1}Y_{2-k}(t;\lambda_n)+2(1-k)t^{-k}Y'_{2-k}(t;\lambda_n)+t^{1-k}Y''_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}$$
$$\times A(\lambda_n I-A)^{-1}u_1. \tag{17}$$

Substituting (16), (17) into the left side of the equation (1), we get

$$u''(t)+\frac{k}{t}u'(t)=-2\sum_{n=1}^{\infty}\frac{t^{1-k}\left(Y''_{2-k}(t;\lambda_n)+(2-k)t^{-1}Y'_{2-k}(t;\lambda_n)\right)}{Y'_{2-k}(1;\lambda_n)}A(\lambda_n I-A)^{-1}u_1$$

$$=-2\sum_{n=1}^{\infty}\frac{t^{1-k}Y_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}\lambda_n A(\lambda_n I-A)^{-1}u_1=Au(t),\quad 0<t<1.$$

Thus, the function $u(t)$ defined by the equality (6) is a solution to the problem (5), and thus the theorem is proved.

To conclude this section, we note that, in the language of control theory, Theorem 2 means controllability from the zero position of the system described by the conditions (5).

3. THE $k\ge 0$ CASE FOR THE EULER–POISSON–DARBOUX EQUATION. NEUMANN'S WEIGHT CONDITION AT $t=0$

For the Euler–Poisson–Darboux equation (1), consider a boundary value problem of the form

$$\lim_{t\to 0+}t^k u'(t)=u_0,\quad u(1)=u_1. \tag{18}$$

In [1], the following criterion for the uniqueness of a solution was proved.

Theorem 3. *Let $k\ge 0$ and A be a linear closed operator in E. We assume that the boundary problem* (1), (18) *has a solution $u(t)$. For this solution to be unique, necessary and sufficient, that none of the $\widehat{\lambda}_n,\ n\in\mathbb{N}$ zeros of the function*

$$\Upsilon_k(\lambda)=Y_k(1;\lambda), \tag{19}$$

where $Y_k(t;\lambda)=\Gamma(k/2+1/2)\left(t\sqrt{\lambda}/2\right)^{1/2-k/2} I_{k/2-1/2}\left(t\sqrt{\lambda}\right)$, *would not be an eigenvalue of the operator* A.

Theorem 4. *Let* $k\ge 0$, A *be a linear closed operator in* E, $u_1\in D(A^2)$, *and also for all* $n\in\mathbb{N}$ *the zeros of* $\widehat{\lambda}_n$ *defined by the equality (19) of the function* $\Upsilon_k(\lambda)$, *belong to the resolvent set* $\rho(A)$, *and the estimate*

$$\sup_{n\in\mathbb{N}}|\widehat{\lambda}_n|\,||(\widehat{\lambda}_n I-A)^{-1}||<M_0<\infty.$$

Then, the problem

$$u''(t)+\frac{k}{t}u'(t)=Au(t),\quad \lim_{t\to 0+}t^k u'(t)=0,\quad u(1)=u_1 \tag{20}$$

is uniquely solvable and its solution has the form

$$u(t)=-2\sum_{n=1}^{\infty}\frac{Y_k(t;\widehat{\lambda}_n)}{Y_k'(1;\widehat{\lambda}_n)}\widehat{\lambda}_n(\widehat{\lambda}_n I-A)^{-1}u_1.$$

The proof of Theorem 4 is basically similar to the proof of Theorem 2.

4. BOUNDARY VALUE PROBLEMS FOR DEGENERATE DIFFERENTIAL EQUATIONS WITH POWER DEGENERACY

As applications of Theorems 2 and 4 in the Banach space E, consider the equation that degenerates with respect to the variable t

$$t^{\gamma}v''(t)+bt^{\gamma-1}v'(t)=Av(t),\quad 0<t<T. \tag{21}$$

Let $0<\gamma<2,\ b\in\mathbb{R}$. The value of the parameter $\gamma,\ 0<\gamma<2$ means a weak degeneration of the equation (21), in contrast to the case of strong degeneracy $\gamma>2$, which will also be considered further in the paper. For $\gamma=2$, the Euler equation is obtained, which, as is well known, reduces to a non-degenerate equation.

The setting of boundary conditions at the degeneracy point $t=0$ depends on the coefficients b and $\gamma>0$ of the equation and these boundary conditions will be given below.

For $b<1$, consider the problem of determining the function $v(t)\in C([0,T],E)\bigcap C^2((0,T],E)$, belonging to $D(A)$ for $t\in(0,T)$, satisfying the equation (21) and the Dirichlet conditions

$$v(0)=0,\quad v(T)=v_1. \tag{22}$$

Change of independent variable and unknown function

$$t=\left(\frac{\tau}{\delta}\right)^{\delta},\quad \delta=\frac{2}{2-\gamma},\quad v(t)=v\left(\left(\frac{\tau}{\delta}\right)^{\delta}\right)=w(\tau),$$

taking into account the equalities

$$v'(t)=\left(\frac{\tau}{\delta}\right)^{1-\delta}w'(\tau),\quad v''(t)=\left(\frac{\tau}{\delta}\right)^{2(1-\delta)}\left(w''(\tau)+\frac{1-\delta}{\tau}w'(\tau)\right),$$

reduces the weakly degenerate equation (21) to the Euler–Poisson–Darboux equation of the form

$$w''(\tau)+\frac{k}{\tau}w'(\tau)=Aw(\tau),\quad \tau\in[0,l], \tag{23}$$

where $k=b\delta-\delta+1,\ \delta=\dfrac{2}{2-\gamma},\ l=\delta T^{1/\delta}$. To simplify the notation, we will further assume that T is chosen so that $l=1$. At the same time, the conditions (22) are converted into conditions respectively

$$w(0)=0,\quad w(1)=v_1. \tag{24}$$

The resulting problem (23) and (24) has already been studied by us in paragraph 2. Returning to the original Dirichlet problem (21) and (22) for a weakly degenerate equation, using Theorem 2, we formulate the following conditions unambiguous resolution.

Theorem 5. *Let* $0 < \gamma < 2$, $b < 1$, $k = \frac{2(b-1)}{2-\gamma} + 1$, $\delta = \frac{2}{2-\gamma}$, $T = \frac{1}{\delta^\delta}$, *A is a linear closed operator in E,* $v_1 \in D(A^2)$, *and also for all* $n \in \mathbb{N}$ *the zeros of* λ_n *defined by the equality (3) of the function* $\Upsilon_{2-k}(\lambda)$, *belong to the resolvent set* $\rho(A)$, *and the estimate*

$$\sup_{n\in\mathbb{N}} |\lambda_n| \, ||(\lambda_n I - A)^{-1}|| < M_0 < \infty.$$

Then, the Dirichlet problem (21), (22) for the weakly degenerate equation is uniquely solvable and its solution has the form

$$v(t) = -2(\delta t^{1/\delta})^{1-k} \sum_{n=1}^{\infty} \frac{Y_{2-k}(\delta t^{1/\delta}; \lambda_n)}{Y'_{2-k}(1; \lambda_n)} \lambda_n (\lambda_n I - A)^{-1} v_1.$$

As mentioned earlier, the setting of the boundary condition at the degeneracy point $t = 0$ depends on the coefficient b. Let now the coefficient $b > \gamma/2$ in the equation (23). In this case, instead of the Dirichlet conditions (22) the following conditions should be set

$$\lim_{t\to 0+} t^b v'(t) = 0, \quad v(1) = v_1. \tag{25}$$

Similar to Theorem 5, but using Theorem 4 instead of Theorem 2, and in this case we formulate the conditions for unique solvability corresponding boundary value problem.

Theorem 6. *Let* $0 < \gamma < 2$, $b > \gamma/2$, $k = \frac{2(b-1)}{2-\gamma} + 1$, $\delta = \frac{2}{2-\gamma}$ $T = \frac{1}{\delta^\delta}$, *A is a linear closed operator in E,* $v_1 \in D(A^2)$, *and also for all* $n \in \mathbb{N}$ *the zeros of* $\widehat{\lambda}_n$ *defined by the equality (19) of the function* $\Upsilon_k(\lambda)$, *belong to the resolvent set* $\rho(A)$, *and the estimate*

$$\sup_{n\in\mathbb{N}} |\widehat{\lambda}_n| \, ||(\widehat{\lambda}_n I - A)^{-1}|| < M_0 < \infty.$$

Then, the problem (21) and (25) is uniquely solvable and its solution has the form

$$v(t) = -2 \sum_{n=1}^{\infty} \frac{Y_k(\delta t^{1/\delta}; \widehat{\lambda}_n)}{Y'_k(1; \widehat{\lambda}_n)} \widehat{\lambda}_n (\widehat{\lambda}_n I - A)^{-1} v_1.$$

Let us further consider the equation (21) in the case of strong degeneracy, when the parameter $\gamma > 2$. Change of independent variable and unknown function

$$t = \left(-\frac{\tau}{\delta}\right)^{-\delta}, \quad \delta = \frac{2}{2-\gamma} v(t) = v\left(\left(-\frac{\tau}{\delta}\right)^{-\delta}\right) = \hat{w}(\tau)$$

reduces the strongly degenerate equation (21) to the Euler—Poisson—Darboux equation of the form

$$\hat{w}''(\tau) + \frac{p}{\tau}\hat{w}'(\tau) = A\hat{w}(\tau), \quad 0 < \tau < l, \tag{26}$$

where $p = \dfrac{2(b-1)}{\gamma - 2} + 1$, $l = -\delta T^{-1/\delta}$. To simplify notation, in what follows, as before, we will assume that T is chosen so that $l = 1$.

In the case of strong degeneracy, the setting of the boundary conditions at the degeneracy point $t = 0$ also depends on the coefficient b. Sufficient conditions for the unique solvability of boundary value problems for the Euler—Poisson—Darboux equation (26), which reduce considered boundary value problems for strongly degenerate equations are contained in Theorems 2 and 4; therefore, similarly Theorems 5 and 6 establish the following assertions.

Theorem 7. *Let $\gamma > 2$, $b < 1$, $p = \frac{2(b-1)}{2-\gamma} + 1$, $\delta = \frac{2}{2-\gamma}$, $T = \left(\frac{1}{-\delta}\right)^{-\delta}$, A is a linear closed operator in E, $v_1 \in D(A^2)$, and also for all $n \in \mathbb{N}$ the zeros of λ_n defined by the equality (3) of the function $\Upsilon_{2-p}(\lambda)$, belong to the resolvent set $\rho(A)$, and the estimate*

$$\sup_{n\in\mathbb{N}} |\lambda_n|\, ||(\lambda_n I - A)^{-1}|| < M_0 < \infty.$$

Then, the Dirichlet problem (21), (22) for a strongly degenerate equation is uniquely solvable and its solution has the form

$$v(t) = -2(-\delta t^{-1/\delta})^{1-p} \sum_{n=1}^{\infty} \frac{Y_{2-p}(-\delta t^{-1/\delta}; \lambda_n)}{Y'_{2-p}(1; \lambda_n)} \lambda_n (\lambda_n I - A)^{-1} v_1.$$

Theorem 8. *Let $\gamma > 2$, $b > 2 - \gamma/2$, $p = \frac{2(b-1)}{2-\gamma} + 1$, $\delta = \frac{2}{2-\gamma}$, $T = \left(\frac{1}{-\delta}\right)^{-\delta}$, A is a linear closed operator in E, $v_1 \in D(A^2)$, and also for all $n \in \mathbb{N}$ the zeros of $\widehat{\lambda}_n$ defined by the equality (19) of the function $\Upsilon_p(\lambda)$, belong to the resolvent set $\rho(A)$, and the estimate*

$$\sup_{n\in\mathbb{N}} |\widehat{\lambda}_n|\, ||(\widehat{\lambda}_n I - A)^{-1}|| < M_0 < \infty.$$

Then, the problem

$$t^{\gamma} v''(t) + b t^{\gamma-1} v'(t) = A v(t), \quad \lim_{t\to 0+} t^{2-b} v'(t) = 0, \quad v(1) = v_1$$

is uniquely solvable and its solution has the form

$$v(t) = -2 \sum_{n=1}^{\infty} \frac{Y_p(-\delta t^{-1/\delta}; \widehat{\lambda}_n)}{Y'_p(1; \widehat{\lambda}_n)} \widehat{\lambda}_n (\widehat{\lambda}_n I - A)^{-1} v_1.$$

Finally, we formulate a theorem for the abstract analogue of the degenerate in space variable differential equation with a power character of degeneracy. For $\omega > 0$, consider the equation

$$v''(t) = t^{\omega} A v(t), \quad 0 < t < T \tag{27}$$

and boundary conditions

$$v(0) = 0, \quad v(T) = v_1. \tag{28}$$

If A is the differentiation operator with respect to the spatial variable x, for example, $Av(t,x) = v''_{xx}(t,x)$, then the equation (27) is a degenerate hyperbolic generalization of the Tricomi equation, but has a different character degeneracy compared to the previous degenerate equations. Therefore, the abstract equation (27) is also natural call degenerate.

Change of variable and unknown function

$$t = \left(\frac{\tau}{\sigma}\right)^{\sigma}, \quad \sigma = \frac{2}{\omega+2}, \quad v(t) = \left(\frac{\tau}{\sigma}\right)^{\sigma} \tilde{w}(\tau)$$

for $T = \dfrac{1}{\sigma^{\sigma}}$ reduces the problem (27) and (28) to a boundary value problem for Euler–Poisson–Darboux equations

$$\tilde{w}''(\tau) + \frac{\sigma+1}{\tau} \tilde{w}'(\tau) = A\tilde{w}(\tau) \quad (0 < \tau < 1), \quad \lim_{\tau\to 0} \tau^{\sigma+1} \tilde{w}'(0) = 0, \quad \tilde{w}(1) = \sigma^{\sigma} v_1.$$

Since the parameter of the Euler–Poisson–Darboux equation (1) satisfies the inequality $k = \sigma + 1 > 1$, then by virtue of Theorem 4, the following assertion is true.

Theorem 9. *Let $\omega > 0$, $\sigma = \frac{2}{\omega+2}$, $k = \frac{\omega+4}{\omega+2}$, $T = \frac{1}{\sigma^{\sigma}}$, A is a linear closed operator in E, $v_1 \in D(A^2)$, and also for all $n \in \mathbb{N}$ the zeros of $\widehat{\lambda}_n$ defined by the equality (19) of the function $\Upsilon_k(\lambda)$, belong to the resolvent set $\rho(A)$, and the estimate*

$$\sup_{n\in\mathbb{N}} |\widehat{\lambda}_n|\, ||(\widehat{\lambda}_n I - A)^{-1}|| < M_0 < \infty$$

Then, the problem (27) and (28) is uniquely solvable and its solution has the form

$$v(t) = -2\sigma^\sigma t \sum_{n=1}^{\infty} \frac{Y_k(\sigma t^{1/\sigma}; \widehat{\lambda}_n)}{Y_k'(1; \widehat{\lambda}_n)} \widehat{\lambda}_n (\widehat{\lambda}_n I - A)^{-1} v_1. \tag{29}$$

5. EXAMPLES OF SOLVING BOUNDARY PROBLEMS

Let us give several examples of problems whose solutions can be expressed in integral form.

Example 1. Let $k = 0$ and the operator $-A$ be the generator of the operator cosine function $C(t; -A)$, which satisfies the estimate

$$||C(t; -A)|| \leq M e^{\omega t}, \quad M > 1, \quad \omega \geq 0. \tag{30}$$

Then, as is known, the resolvent of the operator $-A$ satisfies the representation

$$\xi(\xi^2 I + A)^{-1} = \int_0^\infty e^{-\xi t} C(t; -A)\, dt, \quad \xi > \omega. \tag{31}$$

Consider the problem

$$u''(t) = Au(t), \quad u(0) = 0, \quad u(1) = u_1 \in D(A^2). \tag{32}$$

If in the inequality (30) $\omega < \pi$, then given the representation (31), according to Theorem 2, we write the solution of the problem (32) in the form

$$u(t) = -2t \sum_{n=1}^{\infty} \frac{Y_2(t; \lambda_n)}{Y_2'(1; \lambda_n)} \lambda_n (\lambda_n I - A)^{-1} u_1 = 2\sqrt{t} \sum_{n=1}^{\infty} \frac{J_{1/2}(t; \mu_n)}{\mu_n J_{3/2}(1; \mu_n)} \lambda_n (\lambda_n I - A)^{-1} u_1$$

$$= 2\sqrt{t} \sum_{n=1}^{\infty} \frac{J_{1/2}(t; \mu_n)}{J_{3/2}(1; \mu_n)} \pi n ((\pi n)^2 I + A)^{-1} u_1 = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sin(\pi n t) \quad \pi n ((\pi n)^2 I + A)^{-1} u_1$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sin(\pi n t) \int_0^\infty e^{-\pi n s} C(s; -A) u_1\, ds, \quad \omega < \pi, \tag{33}$$

where $\lambda_n = -(\pi n)^2$, $\mu_n = \pi n$ are the zeros of the function $J_{1/2}(\mu) = \sqrt{\frac{2}{\pi \mu}} \sin \mu$.

Using series ([16], 5.4.12.1), after summing under the integral sign in the formula (33), we obtain the representation integral solutions

$$u(t) = \sin \pi t \int_0^\infty \frac{C(s; -A) u_1\, ds}{\operatorname{ch} \pi s + \cos \pi t}, \quad \omega < \pi. \tag{34}$$

In particular, if the number $A < 0$, then $C(s; -A) = \operatorname{ch}\left(s\sqrt{-A}\right)$, and calculating the integral (see [16], 2.4.6.7) in the formula (34), we obtain the solution of the Dirichlet problem in the scalar case

$$u(t) = \frac{\sin\left(t\sqrt{-A}\right) u_1}{\sin\left(\sqrt{-A}\right)}.$$

Condition $\omega = \sqrt{-A} < \pi$ required for integral representation (34) in the last equality is no longer required.

Note also that in the case $A < 0$ the sum of the series in the representation (6) of the solution of the Dirichlet problem for $k < 1$ can be found directly, using formula ([13], 5.7.33.4), and we arrive at the expression

$$u(t) = \frac{t^{1/2-k/2} J_{1/2-k/2}(t\sqrt{-A})}{J_{1/2-k/2}(\sqrt{-A})} u_1.$$

Example 2. Let $k=2$ the operator $-A$ be the generator of the operator cosine function $C(t;-A)$, which satisfies the estimate (30). Taking into account Theorem 4, similarly to Example 1 for solving the problem

$$u''(t)+\frac{2}{t}u'(t)=Au(t),\quad \lim_{t\to 0+}t^2u'(t)=0,\quad u(1)=u_1\in D(A^2)$$

the integral representation is set

$$u(t)=\frac{\sin \pi t}{t}\int\limits_0^\infty \frac{C(s;-A)u_1\,ds}{\operatorname{ch}\pi s+\cos\pi t},\quad \omega<\pi.$$

In particular, if the number $A<0$, then using formula ([13], 5.7.33.4), we obtain the solution of the problem (20) for $k\geq 0$ in the form

$$u(t)=\frac{t^{1/2-k/2}J_{k/2-1/2}(t\sqrt{-A})}{J_{k/2-1/2}(\sqrt{-A})}u_1, \tag{35}$$

which for $k=2$ has the form

$$u(t)=\frac{\sin\left(t\sqrt{-A}\right)u_1}{t\sin\left(\sqrt{-A}\right)}.$$

Example 3. Let $k=0$ and, as before, the operator $-A$ is the generator of the operator cosine function $C(t;-A)$ which satisfies the estimate (30). Consider the problem

$$u''(t)=Au(t),\quad \lim_{t\to 0+}u'(t)=0,\quad u(1)=u_1\in D(A^2). \tag{36}$$

If in the inequality (30) $\omega<\dfrac{\pi}{2}$, then given the representation (31), according to Theorem 4, we write the solution of the problem (36) in the form

$$u(t)=-2\sum_{n=0}^\infty \frac{C(t;\widehat{\lambda}_n)}{C'(1;\widehat{\lambda}_n)}\widehat{\lambda}_n(\widehat{\lambda}_nI-A)^{-1}u_1=2\sum_{n=0}^\infty\frac{\cos\left(t\widehat{\mu}_n\right)}{\widehat{\mu}_n\sin\widehat{\mu}_n}\widehat{\lambda}_n(\widehat{\lambda}_nI-A)^{-1}u_1$$

$$=2\sum_{n=0}^\infty(-1)^n\cos\left(\frac{\pi t}{2}+\pi nt\right)\int\limits_0^\infty e^{-(\pi/2+\pi n)s}C(s;-A)u_1\,ds,\quad \omega<\frac{\pi}{2}, \tag{37}$$

where $\widehat{\lambda}_n=-\left(\dfrac{\pi}{2}+\pi n\right)^2$, $\widehat{\mu}_n=\dfrac{\pi}{2}+\pi n$ are zeros of the $\cos\mu$ function.

Using series ([16], 5.4.12.4), after summing under the integral sign in the formula (37), we obtain the representation integral solutions

$$u(t)=2\cos\frac{\pi t}{2}\int\limits_0^\infty\frac{\cosh\dfrac{\pi s}{2}\,C(s;-A)u_1\,ds}{\cosh\pi s+\cos\pi t},\quad \omega<\frac{\pi}{2}. \tag{38}$$

In particular, if the number $A<0$, then $C(s;-A)=\cosh\left(s\sqrt{-A}\right)$, and calculating the integral (see [16], 2.4.6.14) in the formula (38), we obtain the solution of the problem (36) in the scalar case

$$u(t)=\frac{2\cos\dfrac{\pi t}{2}}{\sin\pi t}\,\frac{\cos\dfrac{\pi(1-t)}{2}\cos\left(\sqrt{-A}(t+1)\right)-\cos\dfrac{\pi(1+t)}{2}\cos\left(\sqrt{-A}(1-t)\right)}{1+\cos\left(2\sqrt{-A}\right)}$$

$$=\frac{2\cos\dfrac{\pi t}{2}}{\sin\pi t}\,\frac{\sin\dfrac{\pi t}{2}\cos\left(\sqrt{-A}(t+1)\right)+\sin\dfrac{\pi t}{2}\cos\left(\sqrt{-A}(1-t)\right)}{1+\cos\left(2\sqrt{-A}\right)}$$

$$= \frac{\cos\left(\sqrt{-A}(t+1)\right) + \cos\left(\sqrt{-A}(1-t)\right)}{1+\cos\left(2\sqrt{-A}\right)} = \frac{\cos\left(t\sqrt{-A}\right) u_1}{\cos\left(\sqrt{-A}\right)},$$

which agrees with the representation (35) for $k = 0$. Condition $\omega = \sqrt{-A} < \dfrac{\pi}{2}$ required for integral representation (38) in the last equality is no longer required.

Example 4. Let $\gamma = 1$, $b = \frac{1}{2}$ and operator $-A$ be the generator operator cosine function $C(t; -A)$ that satisfies the estimate (30). Using Theorem 5, we define the parameters used in it are $k = 0$, $\delta = 2$, $T = \frac{1}{4}$, and similarly example 1 to solve the problem

$$t^\gamma v''(t) + bt^{\gamma-1}v'(t) = Av(t), \quad v(0) = 0, \quad v(T) = v_1$$

set the view

$$v(t) = \sin(2\pi\sqrt{t}) \int_0^\infty \frac{C(s; -A)v_1\, ds}{\operatorname{ch}\pi s + \cos(2\pi\sqrt{t})}, \quad \omega < \pi.$$

In particular, if the number $A < 0$, then by calculating the integral (see [13], 5.7.33.4), we obtain the solution of the Dirichlet problem in the scalar case

$$v(t) = \frac{\sin\left(2\sqrt{-At}\right) v_1}{\sin\left(\sqrt{-A}\right)}.$$

Example 5. If the number is $A < 0$ and $\sigma = \dfrac{2}{\omega+2}$, $k = \dfrac{\omega+4}{\omega+2}$, then in this scalar case the sum of the series in the representation (29) of the solution to the problem (27), (28) is found directly, using formula ([13], 5.7.33.4), and we arrive at the expression

$$v(t) = \frac{\sigma^{\sigma/2}\sqrt{t}J_{k/2-1/2}(\sigma\sqrt{-A}t^{1/\sigma})}{J_{k/2-1/2}(\sqrt{-A})}v_1. \tag{39}$$

The form of the solution is consistent with formula ([17], 2.162(10)). In particular, the solution of the boundary value problem for the Airy equation we get from (39) with $\omega = 1$.

6. SOBOLEV TYPE SINGULAR EQUATION

The results of the previous subsections are generalized to the case of a singular equation of Sobolev type

$$B\left(u''(t) + \frac{k}{t}u'(t)\right) = Au(t), \quad 0 < t < 1,$$

where, like A, the operator B is a closed linear operator operator in E whose domain $D(B) \subset E$ is not necessarily dense in E. Information about the uniqueness criterion for the solution of the corresponding boundary value problems is given in [1].

The scheme of proof of the statements is similar to the proof of Theorems 2 and 4. A distinctive feature is the change of equality $Ah = \lambda h$, which determines the eigenvalues of the operator A, onto the operator equation $Ah = \lambda Bh$, as well as replacing the point spectrum $\sigma_p(A)$ with the spectrum $\sigma_p(B, A)$ operator A with respect to B and the resolvent set $\rho(A)$ on resolvent set $\rho(B, A)$ of the operator A with respect to B.

Theorem 10. *Let $k < 1$, A, B be linear closed commuting operators in E, $u_1 \in D(A^2) \bigcap D(B)$, and also for all $n \in \mathbb{N}$ the zeros of λ_n defined by the equality (3) functions $\Upsilon_{2-k}(\lambda)$, belong to the resolvent set $\rho(B, A)$, and the estimate*

$$\sup_{n\in\mathbb{N}} |\lambda_n| ||(\lambda_n B - A)^{-1}|| < M_0 < \infty.$$

Then, the task

$$B\left(u''(t)+\frac{k}{t}u'(t)\right)=Au(t),\quad u(0)=0,\quad u(1)=u_1$$

is uniquely solvable and its solution has the form

$$u(t)=-2t^{1-k}\sum_{n=1}^{\infty}\frac{Y_{2-k}(t;\lambda_n)}{Y'_{2-k}(1;\lambda_n)}\lambda_n B(\lambda_n B-A)^{-1}u_1.$$

Theorem 11. *Let $k\geq 0$, A, B be linear closed commuting operators in E, $u_1\in D(A^2)\bigcap D(B)$, and also for all $n\in\mathbb{N}$ the zeros of $\widehat{\lambda}_n$ defined by the equality (19) functions $\Upsilon_k(\lambda)$, belong to the resolvent set $\rho(B,A)$, and the estimate*

$$\sup_{n\in\mathbb{N}}|\widehat{\lambda}_n|||(\widehat{\lambda}_n B-A)^{-1}||<M_0<\infty.$$

Then, the task

$$B\left(u''(t)+\frac{k}{t}u'(t)\right)=Au(t),\quad \lim_{t\to 0+}t^k u'(t)=0,\quad u(1)=u_1$$

is uniquely solvable and its solution has the form

$$u(t)=-2\sum_{n=1}^{\infty}\frac{Y_k(t;\widehat{\lambda}_n)}{Y'_k(1;\widehat{\lambda}_n)}\widehat{\lambda}_n B(\widehat{\lambda}_n B-A)^{-1}u_1.$$

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