ISSN 0037–4466, Siberian Mathematical Journal, 2023, Vol. 64, No. 3, pp. 707–719. © Pleiades Publishing, Ltd., 2023.
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SOLVING THE EULER–POISSON–DARBOUX EQUATION OF FRACTIONAL ORDER

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UDC 517.95

Abstract: Interest in fractional ordinary and partial differential equations has been steadily increasing in the recent decades. This is due to the necessity of modeling the processes whose current state depends significantly on the previous ones, i.e., the so-called systems with residual memory. We consider the Cauchy problem for the one-dimensional, homogeneous Euler–Poisson–Darboux equation with a differential operator of fractional order in time being the left-sided fractional Bessel operator. At the same time, we use the ordinary differential operator in the space variable of the second order. We reveal the connection between the Meyer and Laplace transform which is obtained by the Poisson transform and presents a special case of the relation with the Obreshkov transformation. We prove the theorem that yields the conditions of the existence of a solution to the problem by using the Meyer transform. In this case, a solution to the problem is represented explicitly in terms of the generalized Green's function that determines the generalized hypergeometric Fox H-function.

DOI: 10.1134/S0037446623030187

Keywords: fractional powers of a Bessel operator, fractional Euler–Poisson–Darboux equation, integral Meyer transform, H-function

1. Introduction

The classical Euler–Poisson–Darboux equation (EPD) is of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t),\ x \in \mathbb{R},\ t > 0,\ \gamma \in \mathbb{R}. \tag{1}$$

The operator, acting on the variable t in (1), agrees with the *Bessel operator* $(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t}$ (see [1]).

Equation (1) serves as a random flight model (see [2–8]). The first contribution to this area was made by Goldstein in 1951 (see [2]). He considered the simplest random walk along a real line in which a particle situated at the origin at time 0 moves with two finite speeds $\pm\lambda$ changing its current speed in accord with the simplest Poisson process with a constant parameter μ. He found out that the distribution of particles at a position x during a time t is a solution to a telegraph equation of the form

$$\frac{\partial^2 u}{\partial t^2} + 2\mu\,\frac{\partial u}{\partial t} = \lambda^2\,\frac{\partial^2 u}{\partial x^2}.$$

Next, this model was studied in details by Kac in [3] and Orsingher in [4, 5]. The articles [6–9] treat the natural generalizations for a Poisson process with the intensity function $\lambda = \lambda(t) \in C^1(\mathbb{R})$ as well as for the multidimensional situation. The random walk models with fractional derivatives are considered in [10, 11].

In [12] there is demonstrated that the Euler–Poisson–Darboux equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t} = \lambda^2\frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t),\ \ a > 0,\ \ t > 0,\ \ x \in \mathbb{R}, \tag{2}$$

The first author was supported by the Ministry of Science and Higher Education of the Russian Federation (Grant no. 075–02–2022–890).

Original article submitted December 10, 2021; accepted November 22, 2022.

determines the probability law of a random walk on $\mathbb{R}$. The explicit distribution $u(x,t)$ of the moving particle positions appears in solving the initial value problems for (2).

In [8], the fractional diffusion wave equation

$$\left(\frac{\partial^2 u}{\partial t^2}+\frac{\gamma}{t}\frac{\partial u}{\partial t}\right)^{\alpha} u=\lambda^2\frac{\partial^2 u}{\partial x^2},\qquad u=u(x,t),\ \ x\in\mathbb{R},\ \ t>0,\ \ 0<\alpha<1, \tag{3}$$

is obtained as a random walk model. If $\alpha\in(0,1/2)$ then a particle moves slowly on the average than in the case of (2) with $\alpha=1$. A particle moves faster on the average if $\alpha\in(1/2,1)$.

In this article we use the operational method for solving (3) with some additional conditions for $0<\alpha\le 1/2$.

2. Special Functions

We start with the definitions of some special functions to be used below.

The *modified Bessel functions* (or hyperbolic Bessel functions) *of the first and second kind* $I_\nu(x)$ and $K_\nu(x)$ are defined as follows (see [13]):

$$I_\nu(x)=i^{-\nu}J_\alpha(ix)=\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu}, \tag{4}$$

$$K_\nu(x)=\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\alpha(x)}{\sin(\nu\pi)}, \tag{5}$$

where ν is a noninteger. To determine these functions for the integer values of α, we need passage to the limit. Obviously, $K_\nu(x)=K_{-\nu}(x)$. For the small values of the argument $0<|x|\ll\sqrt{\nu+1}$, we have the asymptotic formula

$$K_\nu(x)\sim\begin{cases}-\log\left(\frac{x}{2}\right)-\vartheta, & \nu=0,\\ \frac{\Gamma(\nu)}{2^{1-\nu}}\,x^{-\nu}, & \nu>0,\end{cases} \tag{6}$$

where

$$\vartheta=\lim_{n\to\infty}\left(-\log n+\sum_{k=1}^{n}\frac{1}{k}\right)=\int_1^\infty\left(-\frac{1}{x}+\frac{1}{\lfloor x\rfloor}\right)dx$$

is the Euler–Macdonald constant [14].

The asymptotic behavior of the Bessel function $K_\nu(z)$ at infinity is described by the formula

$$K_\nu(z)=\sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}}\left(1+O\left(\frac{1}{z}\right)\right)\qquad\text{as } |z|\to\infty. \tag{7}$$

Given $\nu=\frac{1}{2}$, we infer that

$$K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}e^{-x}. \tag{8}$$

The kernel of the left-sided Bessel derivative on the half-axis is the *hypergeometric Gauss function* inside the disk $|z|<1$ as the sum of the hypergeometric series (see [14, formula 15.3.1])

$${}_2F_1(a,b;c;z)=F(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{9}$$

and as an analytic continuation of this series for $|z|\ge 1$. In (9), the parameters a, b, and c, as well as the variable z can be complex and $c\ne 0,-1,-2,\dots$. The factor $(a)_k$ is the Pochhammer symbol $(z)_n=z(z+1)\dots(z+n-1)$, with $n=1,2,\dots,(z)_0\equiv 1$.

The *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ is an entire function of order $1/\alpha$ which is defined for $\operatorname{Re}\alpha > 0$ by the series

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \ \alpha, \beta \in \mathbb{C}, \ \operatorname{Re}\alpha > 0, \ \operatorname{Re}\beta > 0. \tag{10}$$

Let $z, \rho, \beta \in \mathbb{C}$. The function $\varphi(\rho, \beta; z)$ is defined by the formula (see [15, formula E.36′] and [16, formula (7.1.1)])

$$\varphi(\rho, \beta; z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\rho k + \beta)} \frac{z^k}{k!}. \tag{11}$$

If $\rho > -1$ then the series in (11) converges absolutely for all $z \in \mathbb{C}$. For $\rho = -1$, this series converges absolutely for $|z| < 1$. If $\rho = -1$ and $|z| = 1$ then the series in (11) converges absolutely for $\operatorname{Re}\beta > -1$. Moreover, $\varphi(\rho, \beta; z)$ is an entire function of z for $\rho > -1$. For the real values of z, the function of (11) is considered in [17].

For $\rho = 1$ and $\beta = \nu + 1$, the function $\varphi(\alpha, \beta; \pm z^2/4)$ is expressed through the Bessel functions $J_\nu(z)$ and $I_\nu(z)$ as follows:

$$\varphi\left(1, \nu + 1; -\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^\nu J_\nu(z), \qquad \varphi\left(1, \nu + 1; \frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^\nu I_\nu(z).$$

For the integers m, n, p, and q such that $0 \le m \le q$ and $0 \le n \le p$, while $a_i, b_j \in \mathbb{C}$ and for $\alpha_i, \beta_j \in \mathbb{R}_+$ ($i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$) the H-function $H^{m,n}_{p,q}(z)$ is defined through the Mellin–Barnes integral of the form [18]

$$\mathbf{H}^{m,n}_{p,q}(z) = \mathbf{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{Bmatrix} (a_i, \alpha_i)_{1,p}, \\ (b_j, \beta_j)_{1,q} \end{Bmatrix}\right] = \frac{1}{2\pi i} \int\limits_{\mathscr{L}} \mathscr{H}^{m,n}_{p,q}(s)\, z^{-s}\, ds, \tag{12}$$

where

$$\mathscr{H}^{m,n}_{p,q}(s) = \frac{\prod\limits_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod\limits_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod\limits_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod\limits_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}.$$

Put

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j,$$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i,$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.$$

Here the H-function $\mathbf{H}^{m,n}_{p,q}(z)$ makes some sense whenever $\Delta > 0$ and $z \neq 0$, while $\mathscr{L} = \mathscr{L}_{-\infty}$ is a left loop located on a horizontal strip starting at $-\infty + i\varphi_1$ and ending at $-\infty + i\varphi_2$, with $-\infty < \varphi_1 < \varphi_2 < +\infty$. The other cases of existence of $\mathbf{H}^{m,n}_{p,q}(z)$ are given in [18, Theorem 1.1].

Between $\varphi(\rho, \beta; z)$ and $\mathbf{H}^{m,n}_{p,q}(z)$ we have the following connection (see [17, formula 2]):

$$\varphi(\rho, \beta; z) = \mathbf{H}^{1,0}_{0,2}\left[-z \,\middle|\, \begin{matrix} - \\ (0,1), (1-\beta, \rho) \end{matrix}\right]. \tag{13}$$

In [18, p. 33] there is given the formula of differentiation of the H-function

$$\left(\frac{d}{dz}\right)^k \left\{ z^\omega \mathbf{H}^{m,n}_{p,q}\left[cz^\sigma \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right\} = z^{\omega-k} \mathbf{H}^{m,n+1}_{p+1,q+1}\left[cz^\sigma \,\middle|\, \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega, \sigma) \end{matrix}\right]. \tag{14}$$

3. Integral Transforms, the Poisson Operator, and the Fractional Riemann–Liouville Integral

In this section, we describe the Laplace and Meyer transforms and give the formula of their connection by using the Poisson transform. Also we state some theorem that calculates the fractional Riemann–Liouville integral of $\varphi(\rho,\beta;-z)$.

The Laplace transform of $\varphi(\rho,\beta;kt^{\alpha})$ is written as (see [19])

$$\Big(\mathscr{L}_t\, t^{\beta-1}\varphi(\rho,\beta;-kt^{\alpha})\Big)(\tau)=\tau^{-\beta}e^{-kz^{-\rho}}. \tag{15}$$

To apply the operational method for solving differential equations with a fractional power of the Bessel operator, we need a convenient integral transform. In our case we can take the integral transform with a modified Bessel function (5) in the kernel.

Given a function $f:\mathbb{R}_+\to\mathbb{C}$, the integral transform with the Bessel function K_ν, $\nu\geq 0$ in the kernel is the *Meyer transform* defined by the formula (see [20, p. 93])

$$K_\nu[f](\xi)=\int\limits_0^\infty \sqrt{x\xi}\,K_\nu(x\xi)\,f(x)\,dx. \tag{16}$$

The condition $\nu\geq 0$ is not restrictive, since $K_\nu=K_{-\nu}$.

It is convenient for us to use the modification of the Meyer transform:

$$\mathscr{K}_\gamma[f](\xi)=\int\limits_0^\infty x^{\frac{\gamma+1}{2}}K_{\frac{\gamma-1}{2}}(x\xi)\,f(x)\,dx. \tag{17}$$

Considering (8) and using the fact that $K_\nu=K_{-\nu}$ for $\gamma=0$ and $\gamma=2$, we infer that

$$\mathscr{K}_0[f](\xi)=\sqrt{\frac{\pi}{2\xi}}\int\limits_0^\infty e^{-x\xi}\,f(x)\,dx=\sqrt{\frac{\pi}{2\xi}}\,\mathscr{L}[f(x)](\xi),$$

$$\mathscr{K}_2[f](\xi)=\sqrt{\frac{\pi}{2\xi}}\int\limits_0^\infty xe^{-x\xi}\,f(x)\,dx=\sqrt{\frac{\pi}{2\xi}}\,\mathscr{L}[xf(x)](\xi),$$

where $\mathscr{L}[f(x)](\xi)$ is the Laplace transform.

Assume that

$$f\in L_1^{\mathrm{loc}}(\mathbb{R}_+)\quad\text{and}\quad f(t)=o\big(t^{\beta-\frac{\gamma}{2}}\big)\quad\text{as } t\to+0,$$

where $\beta>\frac{\gamma}{2}-2$ if $\gamma>1$ and $\beta>-1$ if $\gamma=1$. Moreover, let $f(t)=0(e^{at})$ as $t\to+\infty$. In this case the Meyer transform of f exists almost everywhere for $\operatorname{Re}\xi>a$ (see [20, p. 94]). The class of these functions is denoted by $\mathbf{K}_\gamma$.

If $0<\gamma<2$, while $F(\xi)$ is analytic in the half-plane $H_a=\{p\in\mathbb{C}:\operatorname{Re}p\geq a\}$, $a\leq 0$, and $s^{\frac{\gamma}{2}-1}F(\xi)\to 0$, $|\xi|\to+\infty$ uniformly on $\arg s$; then, for every real c such that $c>a$, we have the inverse transform $\mathscr{K}_\gamma^{-1}$ of the form (see [20, p. 94])

$$\mathscr{K}_\gamma^{-1}[\widehat{f}](x)=f(x)=\frac{1}{\pi i}\int\limits_{c-i\infty}^{c+i\infty}\widehat{f}(\xi)i_{\frac{\gamma-1}{2}}(x\xi)\xi^\gamma\,d\xi. \tag{18}$$

Formula (18) is not convenient for calculations due to the constraint $0<\gamma<2$. So we present one more inversion formula by using the Poisson transform.

To simplify the recover of a function from its Meyer transform, we employ the Poisson operator of the form

$$\mathscr{P}_x^\gamma f(x) = (\mathscr{P}_t^\gamma f(t))(x) = \frac{2C(\gamma)}{x^{\gamma-1}} \int\limits_0^x \left(x^2-t^2\right)^{\frac{\gamma}{2}-1} f(t)\,dt, \quad C(\gamma) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}. \tag{19}$$

The left inverse to (19) for $\gamma > 0$ and every function $H(x) \in C^n$ is defined as

$$(\mathscr{P}_x^\gamma)^{-1} H(x) = \frac{2\sqrt{\pi}x}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(n-\frac{\gamma}{2}\right)} \left(\frac{d}{2xdx}\right)^n \int\limits_0^x H(z)(x^2-z^2)^{n-\frac{\gamma}{2}-1} z^\gamma\,dz, \tag{20}$$

where $n = \left[\frac{\gamma}{2}\right]+1$.

The Poisson transform is a particular case of Erdeyi–Kober fractional integration operator and its inversion is analogous to the case of the Erdeyi–Kober fractional derivative [21].

We will use the representation of K_ν in [13, formula (4)] which is of the form

$$K_\nu(x\xi) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)} \left(\frac{\xi}{2x}\right)^\nu \int\limits_x^\infty e^{-\xi z}(z^2-x^2)^{\nu-\frac{1}{2}}\,dz$$

and (19) together with

$$x^{\frac{\gamma+1}{2}} K_{\frac{\gamma-1}{2}}(x\xi) = \frac{\sqrt{\pi}\,x\xi^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_x^\infty e^{-\xi z}(z^2-x^2)^{\frac{\gamma}{2}-1}\,dz.$$

We thus obtain the representation of the Meyer transform

$$\mathscr{K}_\gamma[f](\xi) = \frac{\pi\,\xi^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma+1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \left(\mathscr{L} z^{\gamma-1}\mathscr{P}_z^\gamma z f(z)\right)(\xi).$$

Finally,

$$\mathscr{K}_\gamma[f](\xi) = \frac{\pi\,\xi^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma+1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \left(\mathscr{L} z^{\gamma-1}\mathscr{P}_z^\gamma z f(z)\right)(\xi), \tag{21}$$

where $\mathscr{L}$ is the Laplace transform.

Representation (21) is a particular case of a more general one for the Obreshkov transform with the Poisson–Dimovski operator. The Poisson–Dimovski and Sonin–Dimovski operators generalize the Poisson operator in the sense of the fractional Kiryakova integro-differentiation (see [15, part 3]).

For $\theta > 0$, the fractional Riemann–Liouville integral is defined as follows (see [21]):

$$\left(I_-^\theta f\right)(x) = \frac{1}{\Gamma(\theta)} \int\limits_x^\infty f(t)(t-x)^{\theta-1}\,dt, \quad x>0.$$

Next, we need also the formula of the fractional Riemann–Liouville integral of $z^\omega \varphi(\rho,\beta;-z^\sigma)$ which results from [18, Theorem 2.7].

Theorem 1. *Assume that $\theta > 0$, $\omega \in \mathbb{R}$, $\sigma > 0$, and $\rho < 1$. If $\omega+\theta < 0$, then the fractional integral I_-^θ of $z^\omega\varphi(\rho,\beta;-z^\sigma)$ exists and*

$$\begin{aligned}\left(I_-^\theta p^\omega \varphi(\rho,\beta;-p^\sigma)\right)(w) &= \left(I_-^\theta p^\omega \mathbf{H}_{0,2}^{1,0}\left[p^\sigma \,\middle|\, \begin{matrix} - \\ (0,1),(1-\beta,\rho)\end{matrix}\right]\right)(w) \\ &= w^{\omega+\theta}\mathbf{H}_{1,3}^{2,0}\left[w^\sigma \,\middle|\, \begin{matrix}(-\omega,\sigma) \\ (-\omega-\theta,\sigma),(0,1),(1-\beta,\rho)\end{matrix}\right].\end{aligned} \tag{22}$$

4. Fractional Integrals and Bessel Derivatives

Some explicit definition of the fractional power of the Bessel operator $(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t}$ was given by Sprinkhuizen-Kuyper in [22]. This definition was obtained in terms of Gauss hypergeometric functions with various applications to PDEs. McBride in [23] considers the fractional powers of the hyper-Bessel operator which includes the operators of the present article.

Let $\alpha > 0$ and $\gamma > 0$. The *left-sided fractional Bessel integral on the half-axis* $B^{-\alpha}_{\gamma,0+}$ for $f \in L[0,\infty)$ is defined as

$$(B^{-\alpha}_{\gamma,0+}f)(x) = (IB^{\alpha}_{\gamma,0+}f)(x)$$
$$= \frac{1}{\Gamma(2\alpha)}\int\limits_0^x \left(\frac{y}{x}\right)^\gamma \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{x^2}\right) f(y)\, dy. \tag{23}$$

The properties of (23) are collected in [24].

Let

$$n = [\alpha]+1, \quad f \in L[0,\infty), \quad \text{and} \quad IB^{n-\alpha}_{\gamma,b-}f, IB^{n-\alpha}_{\gamma,b-}f \in C^{2n}(0,\infty).$$

Define the *left-sided fractional Bessel integral on the half-axis* as

$$(\mathscr{B}^{\alpha}_{\gamma,0+}f)(x) = (IB^{n-\alpha}_{\gamma,0+}B^n_\gamma f)(x), \tag{24}$$

where $B^n_\gamma = \left(\frac{\partial^2}{\partial x^2} + \frac{\gamma}{x}\frac{\partial}{\partial x}\right)^n$ is the integrated Bessel operator.

The spaces to use $B^{\alpha}_{\gamma,0+}$, with $\alpha \in \mathbb{R}$, were introduced in [23] as follows:

$$F_p = \left\{\varphi \in C^\infty(0,\infty) :\, x^k \frac{d^k\varphi}{dx^k} \in L^p(0,\infty),\ k = 0,1,2,\dots\right\}, \quad 1 \le p < \infty,$$

$$F_\infty = \left\{\varphi \in C^\infty(0,\infty) :\, x^k \frac{d^k\varphi}{dx^k} \to 0 \text{ as } x \to +,\ x \to \infty,\ k = 0,1,2,\dots\right\},$$

and

$$F_{p,\mu} = \left\{\varphi :\, x^{-\mu}\varphi(x) \in F_p\right\}, \quad 1 \le p \le \infty,\ \mu \in \mathbb{C}.$$

The next theorem is a particular case of that in [23].

Theorem 2. *Let $\alpha \in \mathbb{R}$. For all p, μ and $\gamma > 0$ such that $\mu \ne \frac{1}{p} - 2m, \gamma \ne \frac{1}{p} - \mu - 2m + 1, m = 1,2,\dots$, the operator $B^{\alpha}_{\gamma,0+}$ is a continuous linear mapping from F_p,μ to $F_{p,\mu-2\alpha}$. Moreover, if $2\alpha \ne \mu - \frac{1}{p} + 2m$ and $\gamma - 2\alpha \ne \frac{1}{p} - \mu - 2m + 1$, $m = 1,2\dots$; then $B^{\alpha}_{\gamma,0+}$ is a homeomorphism from F_p,μ onto $F_{p,\mu-2\alpha}$, with the inverse operator $B^{-\alpha}_{\gamma,0+}$.*

Operators (23) and (24) were studied, but any convenient tool was absent for solving differential equations with fractional powers of the Bessel operator. Firstly, some appropriate tool was proposed in [25] as transform (17). We will recall some results of [25] of use in the sequel. Applying the Meyer transform, we suppose that it applies to the functions in $\mathbf{K}_\gamma$.

Theorem 3. *Let $\alpha > 0$. The Meyer transform (17) of $B^{-\alpha}_{\gamma,0+}$ is of the form*

$$\mathscr{K}_\gamma\big[\big(IB^{\alpha}_{\gamma,0+}f\big)(x)\big](\xi) = \xi^{-2\alpha}\mathscr{K}_\gamma f(\xi). \tag{25}$$

Theorem 4. *Assume that $n \in \mathbb{N}$ and $\frac{d}{dx}\left[B_\gamma^{n-k} f(x)\right]$ is bounded. Then the Meyer transform of $B_\gamma^n f$ exists and, for $\gamma \neq 1$, is defined by the formula*

$$\mathscr{K}_\gamma[B_\gamma^n f](\xi) = \xi^{2n}\mathscr{K}_\gamma[f](\xi) - \left(\frac{2}{\xi}\right)^{\frac{\gamma-1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) \sum_{k=1}^{n} \xi^{2k-2} B_\gamma^{n-k} f(0+). \tag{26}$$

If $\frac{d}{dx}\left[B_\gamma^{n-k} f(x)\right] \sim x^\beta$ and $\beta > 0$ as $x \to 0+$ then (26) is true for $\gamma = 1$ as well.

The class $C_{ev}^m = C_{ev}^m(\mathbb{R})$ comprises the functions in $C^m(\mathbb{R})$ such that $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x=0} = 0$ for all nonnegative integers $k \leq \frac{m-1}{2}$ (see [1, p. 21]).

Given $\gamma = 0$ and $f \in C_{ev}^{2n}$, we find that

$$\mathscr{K}_0[f(x)](\xi) = \sqrt{\frac{\pi}{2\xi}} \int_0^\infty e^{-x\xi} f(x)\, dx = \sqrt{\frac{\pi}{2\xi}} \mathscr{L}[f(x)](\xi)$$

and

$$\begin{aligned}
\mathscr{L}\left[\frac{d^{2n}}{dx^{2n}} f(x)\right](\xi) &= \xi^{2n}\mathscr{L}[f](\xi) - \sum_{k=0}^{2n-1} \xi^k f^{(2n-k-1)}(0+) \\
&= \xi^{2n}\mathscr{L}[f](\xi) - f^{(2n-1)}(0+) - s f^{(2n-2)}(0+) - s^2 f^{(2n-3)}(0+) \\
-s^3 f^{(2n-4)}(0+) - s^4 f^{(2n-5)}(0+) &- s^5 f^{(2n-6)}(0+) - \cdots - \xi^{2n-2} f'(0+) - \xi^{2n-1} f(0+).
\end{aligned}$$

Since $f \in C_{ev}^{2n}$, we conclude that

$$f'(0+) = f'''(0+) = \cdots = f^{(2n-5)}(0+) = f^{(2n-3)}(0+) = f^{(2n-1)}(0+) = 0$$

and

$$\begin{aligned}
\mathscr{L}\left[\frac{d^{2n}}{dx^{2n}} f(x)\right](\xi) &= \xi^{2n}\mathscr{L}[f](\xi) - s f^{(2n-2)}(0+) \\
&\quad -s^3 f^{(2n-4)}(0+) - s^5 f^{(2n-6)}(0+) - \cdots - \xi^{2n-1} f(0+) \\
&= \xi^{2n}\mathscr{L}[f](\xi) - \sum_{k=1}^{n} s^{2k-1} f^{(2n-2k)}(0+) = \{m = n-k\} \\
&= \xi^{2n}\mathscr{L}[f](\xi) - \sum_{m=0}^{n-1} s^{2(n-m)-1} f^{(2m)}(0+).
\end{aligned}$$

Hence,

$$\mathscr{L}\left[\frac{d^{2n}}{dx^{2n}} f(x)\right](\xi) = \xi^{2n}\mathscr{L}[f](\xi) - \sum_{m=0}^{n-1} s^{2(n-m)-1} f^{(2m)}(0+).$$

On the other hand,

$$\begin{aligned}
\sqrt{\frac{2\xi}{\pi}}\mathscr{K}_0[B_0^n f](\xi) &= \sqrt{\frac{2\xi}{\pi}}\left(\xi^{2n}\mathscr{K}_0[f](\xi) - \sqrt{\frac{\pi\xi}{2}} \sum_{m=0}^{n-1} \xi^{2(n-m)-2} B_0^m f(0+)\right) \\
&= \sqrt{\frac{2\xi}{\pi}}\left(\xi^{2n}\sqrt{\frac{\pi}{2\xi}}\mathscr{L}[f(x)](\xi) - \sqrt{\frac{\pi\xi}{2}} \sum_{m=0}^{n-1} \xi^{2(n-m)-2} B_0^m f(0+)\right) \\
&= \xi^{2n}\,\mathscr{L}[f(x)](\xi) - \sum_{m=0}^{n-1} \xi^{2(n-m)-1} f^{(2m)}(0+).
\end{aligned}$$

This confirms that the Meyer transform is a generalization of the Laplace transform.

Theorem 5. *Assume that $n = [\alpha]+1$ for α fractional, with $n = \alpha$ for $\alpha \in \mathbb{N}$ for all $k \in \mathbb{N}$, and $\frac{d}{dx}[B_\gamma^k f(x)]$ is bounded. Then the Meyer transform $\mathscr{B}^\alpha_{\gamma,0+}f$ exists for $\gamma \neq 1$ and is representable as*

$$\mathscr{K}_\gamma[\mathscr{B}^\alpha_{\gamma,0+}f](\xi) = \xi^{2\alpha}\mathscr{K}_\gamma[f](\xi) - \left(\frac{2}{\xi}\right)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}\xi^{2(\alpha-m)-2}B_\gamma^m f(0+). \tag{27}$$

If $\frac{d}{dx}[B_\gamma^k f(x)] \sim x^\beta$ and $\beta > 0$ as $x \to 0+$ then (27) *is valid for $\gamma = 1$ as well.*

Now, we have some tool for solving differential equations with fractional Bessel operator. Proceed with solving the fractional Euler–Poisson–Darboux equation.

5. The Fractional Euler–Poisson–Darboux Equation

In [26] Gerasimov deduced and solved the partial differential equation for the viscoelasticity problem:

$$\frac{\partial^{2\beta}u}{\partial t^{2\beta}} = D\frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t),\ x \in \mathbb{R},\ t > 0,\ 0 < \beta. \tag{28}$$

Start with considering the simplest one-dimensional case of $u = u(x,t)$, with $x \in \mathbb{R}$ and $t \geq 0$,

$$(\mathscr{B}^\alpha_{\gamma,0+})_t u(x,t) = \lambda^2\frac{\partial^2 u}{\partial x^2}, \quad 0 \leq \alpha < \frac{1}{2},\ \lambda > 0, \tag{29}$$

with the Cauchy data

$$u(x,0) = f(x). \tag{30}$$

Theorem 6. *If $0 < \alpha \leq \frac{1}{2}$ and $\lambda > 0$, then a solution to* (29)–(30) *is of the form*

$$u(x,t) = \int_{-\infty}^{\infty} G^\alpha_\gamma(x-\xi,t)f(\xi)\,d\xi, \tag{31}$$

with

$$G^\alpha_\gamma(x,t) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\lambda\sqrt{\pi}2^{1-\gamma}}t^{-\alpha}\mathbf{H}^{2,0}_{1,3}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\,\begin{matrix}\left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2},\frac{\alpha}{2}\right),(0,1),(\alpha-\gamma,-\alpha)\end{matrix}\right],$$

under the condition that the integral on the right-hand side of (31) *converges.*

Proof. Applying (16) in t and using (30), we derive that

$$\tau^{2\alpha}((\mathscr{K}_\gamma)_t u(x,t))(\tau) - \tau^{2\alpha-1-\gamma}f(x) = \lambda^2((\mathscr{K}_\gamma)_t u_{xx}(x,t))(\tau).$$

Next, applying the Fourier transform in x to both sides of this equation, we can conclude that

$$\tau^{2\alpha}((\mathscr{K}_\gamma)_t F_x u(x,t))(\tau,\xi) - \tau^{2\alpha-1-\gamma}\widehat{f}(\xi) = -\xi^2\lambda^2((\mathscr{K}_\gamma)_t F_x u(x,t))(\tau,\xi)$$

and

$$((\mathscr{K}_\gamma)_t F_x u(x,t))(\tau,\xi) = \frac{\tau^{2\alpha-1-\gamma}}{\tau^{2\alpha}+\lambda^2\xi^2}\widehat{f}(\xi).$$

From formula 6.2.13 of [18] we obtain that

$$\frac{\tau^{2\alpha-1-\gamma}}{\tau^{2\alpha}+\lambda^2\xi^2} = \frac{\tau^{\alpha-\gamma-1}}{2\lambda}\left(F_x e^{-\frac{|x|}{\lambda}\tau^\alpha}\right)(\xi),$$

and the convolution properties yield

$$\big((\mathscr{K}_\gamma)_t F_x u(x,t)\big)(\tau,\xi)=\frac{\tau^{2\alpha-1-\gamma}}{\tau^{2\alpha}+\lambda^2\xi^2}\widehat{f}(\xi)=\frac{\tau^{\alpha-\gamma-1}}{2\lambda}\left(F_x e^{-\frac{|x|}{\lambda}\tau^\alpha}\right)(\xi)\widehat{f}(\xi)$$
$$=\frac{\tau^{\alpha-\gamma-1}}{2\lambda}\left(F_x\left(e^{-\frac{|x|}{\lambda}\tau^\alpha}*_x f(x)\right)\right)(\xi).$$

Applying the inverse Fourier transform, we obtain

$$\big((\mathscr{K}_\gamma)_t u(x,t)\big)(\tau)=\frac{\tau^{\alpha-\gamma-1}}{2\lambda}\left(e^{-\frac{|x|}{\lambda}\tau^\alpha}*_x f(x)\right).$$

Representation (21) yields

$$\frac{\pi}{2^\gamma\Gamma^2\left(\frac{\gamma+1}{2}\right)}\left(\mathscr{L}t^{\gamma-1}\mathscr{P}_t^\gamma tu(x,t)\right)(\tau)=\frac{\tau^{\alpha-\gamma-1}}{2\lambda}\left(e^{-\frac{|x|}{\lambda}\tau^\alpha}*_x f(x)\right)$$

and

$$t^{\gamma-1}\mathscr{P}_t^\gamma tu(x,t)=\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\pi 2^{1-\gamma}\lambda}\left(\left(\mathscr{L}_\tau^{-1}\tau^{\alpha-\gamma-1}e^{-\frac{|x|}{\lambda}\tau^\alpha}\right)(t)*_x f(x)\right).$$

Using the inverse Laplace transform and (15), we infer

$$t^{\gamma-1}\mathscr{P}_t^\gamma tu(x,t)=\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\pi 2^{1-\gamma}\lambda}t^{\gamma-\alpha}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}t^{-\alpha}\right)*_x f(x)$$

and

$$u(x,t)=\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\lambda\pi 2^{1-\gamma}t}\left(\left(\mathscr{P}_t^\gamma\right)^{-1}t^{1-\alpha}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}t^{-\alpha}\right)*_x f(x)\right).$$

Find

$$\left(\mathscr{P}_t^\gamma\right)^{-1}t^{1-\alpha}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}t^{-\alpha}\right)=\frac{2\sqrt{\pi}t}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(n-\frac{\gamma}{2}\right)}\left(\frac{d}{2tdt}\right)^n\mathscr{I}_{\alpha,\gamma}(x,t;\lambda),$$

where $n=\left[\frac{\gamma}{2}\right]+1$,

$$\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)=\int\limits_0^t z^{1+\gamma-\alpha}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}z^{-\alpha}\right)\left(t^2-z^2\right)^{n-\frac{\gamma}{2}-1}dz.$$

For $\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)$, we see that

$$\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)=\int\limits_0^t z^{1+\gamma-\alpha}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}z^{-\alpha}\right)(t^2-z^2)^{n-\frac{\gamma}{2}-1}dz$$
$$=\frac{1}{2}\int\limits_0^{t^2}y^{\frac{\gamma-\alpha}{2}}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}y^{-\frac{\alpha}{2}}\right)(t^2-y)^{n-\frac{\gamma}{2}-1}dy.$$

Here the change of variables $z^2=y$ was made. Now, put

$$\left(\frac{|x|}{\lambda}\right)^{\frac{2}{\alpha}}\frac{1}{y}=p.$$

In this case

$$\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)=\frac{1}{2}\left(\frac{|x|}{\lambda}\right)^{\frac{\gamma+2}{\alpha}-1}\int\limits_{\left(\frac{|x|}{\lambda}\right)^{\frac{2}{\alpha}}\frac{1}{t^2}}^{\infty}p^{\frac{\alpha-\gamma}{2}-2}\varphi\left(-\alpha,1+\gamma-\alpha;-p^{\frac{\alpha}{2}}\right)$$
$$\times\left(t^2-\left(\frac{|x|}{\lambda}\right)^{\frac{2}{\alpha}}\frac{1}{p}\right)^{n-\frac{\gamma}{2}-1}dp$$
$$=\frac{t^{2n-\gamma-2}}{2}\left(\frac{|x|}{\lambda}\right)^{\frac{\gamma+2}{\alpha}-1}\int\limits_{\left(\frac{|x|}{\lambda}\right)^{\frac{2}{\alpha}}\frac{1}{t^2}}^{\infty}p^{\frac{\alpha}{2}-n-1}\varphi\left(-\alpha,1+\gamma-\alpha;-p^{\frac{\alpha}{2}}\right)$$
$$\times\left(p-\left(\frac{|x|}{\lambda}\right)^{\frac{2}{\alpha}}\frac{1}{t^2}\right)^{n-\frac{\gamma}{2}-1}dp.$$

Putting $w=\left(\frac{|x|}{\lambda}\right)^{\frac{2}{\alpha}}\frac{1}{t^2}$, we obtain the fractional Riemann–Liouville integral of order $\left(n-\frac{\gamma}{2}\right)$:

$$\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)=\frac{t^{2n-\gamma-2}}{2}\left(\frac{|x|}{\lambda}\right)^{\frac{\gamma+2}{\alpha}-1}\int\limits_{w}^{\infty}p^{\frac{\alpha}{2}-n-1}\varphi\left(-\alpha,1+\gamma-\alpha;-p^{\frac{\alpha}{2}}\right)(p-w)^{n-\frac{\gamma}{2}-1}\,dp$$
$$=\Gamma\left(n-\frac{\gamma}{2}\right)\frac{t^{2n-\gamma-2}}{2}\left(\frac{|x|}{\lambda}\right)^{\frac{\gamma+2}{\alpha}-1}\left(I_{-}^{n-\frac{\gamma}{2}}p^{\frac{\alpha}{2}-n-1}\varphi\left(-\alpha,1+\gamma-\alpha;-p^{\frac{\alpha}{2}}\right)\right)(w).$$

Using (22), we conclude that

$$\theta=n-\frac{\gamma}{2}>0,\quad \sigma=\frac{\alpha}{2}>0,\quad \omega=\frac{\alpha}{2}-n-1,$$
$$\omega+\theta=\frac{\alpha-\gamma}{2}-1<0,\quad \rho=-\alpha<1$$

and

$$\left(I_{-}^{n-\frac{\gamma}{2}}p^{\frac{\alpha}{2}-n-1}\varphi\left(-\alpha,1+\gamma-\alpha;-p^{\frac{\alpha}{2}}\right)\right)(w)$$
$$=w^{\frac{\alpha-\gamma}{2}-1}\mathbf{H}_{1,3}^{2,0}\left[w^{\frac{\alpha}{2}}\,\middle|\,\begin{matrix}\left(n+1-\frac{\alpha}{2},\frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2},\frac{\alpha}{2}\right),(0,1),(\alpha-\gamma,-\alpha)\end{matrix}\right].$$

Therefore,

$$\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)=\Gamma\left(n-\frac{\gamma}{2}\right)\frac{t^{2n-\alpha}}{2}\mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\,\begin{matrix}\left(n+1-\frac{\alpha}{2},\frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2},\frac{\alpha}{2}\right),(0,1),(\alpha-\gamma,-\alpha)\end{matrix}\right].$$

Finally,

$$(\mathscr{P}_t^{\gamma})^{-1}t^{1-\alpha}\varphi\left(-\alpha,1+\gamma-\alpha;-\frac{|x|}{\lambda}t^{-\alpha}\right)=\frac{2\sqrt{\pi}t}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(n-\frac{\gamma}{2}\right)}\left(\frac{d}{2tdt}\right)^{n}\mathscr{I}_{\alpha,\gamma}(x,t;\lambda)$$
$$=\frac{\sqrt{\pi}t}{\Gamma\left(\frac{\gamma+1}{2}\right)}\left(\frac{d}{2tdt}\right)^{n}t^{2n-\alpha}\mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\,\begin{matrix}\left(n+1-\frac{\alpha}{2},\frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2},\frac{\alpha}{2}\right),(0,1),(\alpha-\gamma,-\alpha)\end{matrix}\right].$$

Now, using the formula

$$\left(\frac{d}{2tdt}\right)^n t^{2n+\beta} = \frac{\Gamma\left(n+1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)}\, t^{\beta}$$

and the integral of the Mellin–Barnes type (12), we can calculate the derivative

$$(\mathscr{P}_t^{\gamma})^{-1} t^{1-\alpha}\varphi\left(-\alpha, 1+\gamma-\alpha; -\frac{|x|}{\lambda}t^{-\alpha}\right)$$

in the last representation. We conclude that

$$\begin{aligned}
&\left(\frac{d}{2tdt}\right)^n t^{2n-\alpha}\mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\, \begin{matrix}\left(n+1-\frac{\alpha}{2}, \frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2}, \frac{\alpha}{2}\right), (0,1), (\alpha-\gamma, -\alpha)\end{matrix}\right]\\
&\quad= \left(\frac{d}{2tdt}\right)^n t^{2n-\alpha}\frac{1}{2\pi i}\int\limits_{\mathscr{L}} \mathscr{H}_{1,3}^{2,0}(s)\left(\frac{|x|}{\lambda}t^{-\alpha}\right)^{-s} ds\\
&\quad= t^{-\alpha}\frac{1}{2\pi i}\int\limits_{\mathscr{L}} \mathscr{H}_{1,3}^{2,0}(s)\frac{\Gamma\left(n+1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)}{\Gamma\left(1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)}\left(\frac{|x|}{\lambda}t^{-\alpha}\right)^{-s} ds.
\end{aligned}$$

Since

$$\begin{aligned}
\mathscr{H}_{1,3}^{2,0}(s)\frac{\Gamma\left(n+1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)}{\Gamma\left(1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)} &= \frac{\Gamma\left(1-\frac{\alpha-\gamma}{2}+\frac{\alpha}{2}s\right)\Gamma(s)}{\Gamma\left(n+1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)\Gamma(1-\alpha+\gamma+\alpha s)}\\
\times\frac{\Gamma\left(n+1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)}{\Gamma\left(1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)} &= \frac{\Gamma\left(1-\frac{\alpha-\gamma}{2}+\frac{\alpha}{2}s\right)\Gamma(s)}{\Gamma\left(1-\frac{\alpha}{2}+\frac{\alpha}{2}s\right)\Gamma(1-\alpha+\gamma+\alpha s)},
\end{aligned}$$

we infer

$$\begin{aligned}
&\left(\frac{d}{2tdt}\right)^n t^{2n-\alpha}\mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\, \begin{matrix}\left(n+1-\frac{\alpha}{2}, \frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2}, \frac{\alpha}{2}\right), (0,1), (\alpha-\gamma, -\alpha)\end{matrix}\right]\\
&\quad= t^{-\alpha}\mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\, \begin{matrix}\left(1-\frac{\alpha}{2}, \frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2}, \frac{\alpha}{2}\right), (0,1), (\alpha-\gamma, -\alpha)\end{matrix}\right]
\end{aligned}$$

and

$$u(x,t) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\lambda\sqrt{\pi}2^{1-\gamma}}\left(t^{-\alpha}\mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda}t^{-\alpha}\,\middle|\, \begin{matrix}\left(1-\frac{\alpha}{2}, \frac{\alpha}{2}\right)\\ \left(1-\frac{\alpha-\gamma}{2}, \frac{\alpha}{2}\right), (0,1), (\alpha-\gamma, -\alpha)\end{matrix}\right] *_x f(x)\right). \quad \square$$

In [27, Corollary 6.5], some solution to the Cauchy problem

$$\left({}^C D_{0+}^{2\alpha}u\right)(x,t) = \lambda^2\frac{\partial^2 u}{\partial x^2}, \quad x\in\mathbb{R},\ \ t>0,\ \ \lambda>0, \tag{32}$$

$$u(x,0) = f(x), \quad 0<\alpha\le\frac{1}{2} \tag{33}$$

is given in the form

$$u(x,t) = \int\limits_{-\infty}^{\infty} G^{\alpha}(x-\xi,t)f(\xi)\,d\xi,$$

where

$$G^{\alpha}(x,t) = \frac{1}{2\lambda}t^{-\alpha}\varphi\left(-\alpha, 1-\alpha; -\frac{|x|}{\lambda}t^{-\alpha}\right). \tag{34}$$

For $\gamma = 0$, we obtain (32)–(33) rather than (29)–(30) and a solution (31) for $\gamma = 0$ takes the form

$$u(x,t) = \int_{-\infty}^{\infty} G_0^\alpha(x-\xi,t) f(\xi)\, d\xi,$$

where

$$\begin{aligned} G_0^\alpha(x,t) &= \frac{\Gamma\left(\frac{1}{2}\right)}{\lambda\sqrt{\pi}2} t^{-\alpha} \mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda} t^{-\alpha} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right) \\ \left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right),(0,1),(\alpha,-\alpha) \end{matrix}\right] \\ &= \frac{1}{2\lambda} t^{-\alpha} \mathbf{H}_{0,2}^{1,0}\left[\frac{|x|}{\lambda} t^{-\alpha} \,\middle|\, \begin{matrix} - \\ (0,1),(\alpha,-\alpha) \end{matrix}\right], \end{aligned}$$

which agrees with (34). Here we applied formula 2.1.2 in [18, p. 31], formula (13), and $\Gamma(1/2) = \sqrt{\pi}$.

6. Conclusion

The role of higher transcendental functions in pure mathematics and numerous applications grows constantly. The striking example of this kind is the theory of integrals and derivatives of non-integer order (fractional calculus) and its applications. Within this theory, several special cases of higher transcendental functions including the Mittag-Leffler function and its generalizations, the Fox–Wright function, and the H-function become extremely important especially for analytic solutions to fractional ODEs and PDEs. In this article, we represent a solution to fractional differential equations with a fractional Bessel operator in terms of the H-function.

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