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FRACTIONAL WEIGHTED SPHERICAL MEAN AND MAXIMAL INEQUALITY FOR THE WEIGHTED SPHERICAL MEAN AND ITS APPLICATION TO SINGULAR PDE

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Abstract

In this paper we establish a mean value property for the functions which is satisfied to Laplace–Bessel equation. Our results involve the generalized divergence theorem and the second Green's identities relating the bulk with the boundary of a region on which differential Bessel operators act. Also we design a fractional weighted mean operator, study its boundedness, obtain maximal inequality for the weighted spherical mean and get its boundedness. The connection between the boundedness of the spherical maximal operator and the properties of solutions of the Euler–Poisson–Darboux equation with Bessel operators is given as an application.

Keywords Bessel operator · B-harmonic function · Laplace–Bessel operator · Fractional weighted mean · Maximal inequality · Singular Euler–Poisson–Darboux equation

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Introduction

B-harmonic analysis provides a mathematical theory to deal with problems connected with the singular Bessel differential operator of the form

$$B_{\gamma_j} = \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{\partial}{\partial x_j} = \frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}, \qquad j = 1, \dots, n.$$

We will use notation $\triangle_\gamma = (\triangle_\gamma)_x = \sum_{k=1}^{n} (B_{\gamma_k})_{x_k}$. For $\triangle_\gamma$ the term *Laplace–Bessel operator* is used. A function $u = u(x) = u(x_1, \dots, x_n)$ defined in a domain $\Omega \subset \overline{\mathbb{R}}{}^n_+$ is said to be *B–harmonic* if $u \in C^2(\Omega)$, $\frac{\partial u}{\partial x_j} |_{x_j=0} = 0$ for all $j = 1, \dots, n$ and satisfies the Laplace–Bessel equation $\Delta_\gamma u = 0$ at every point of the domain Ω.

The theory of B–harmonic functions has attracted much interest in the literature during some past decades. Functional spaces adapted to work with Laplace–Bessel operator were studied in [8, 9, 13]. The Bessel potentials generated by the Bessel differential operators were studied and the boundedness in weighted Lebesgue space of such potential was proved in [10, 11]. In [12], the Bessel potentials were characterized of in terms of the B-Lizorkin–Triebel spaces. Weighted inequalities with a general weight for the Littlewood–Paley type functions associated with Laplace-Bessel differential operator were established in [1, 2]. The theory of B-harmonic functions should include generalizations of the classical tools for solving problems with the Laplace-Bessel operator. In this paper we establish a mean value property for the functions which satisfies Laplace-Bessel equation. The paper is organized as follows. In "Definitions" section, we give some definitions in the Bessel setting. The "Generalized divergence theorem and the second Green's formula for the Laplace–Bessel operator" section is to develop a field theory for the case when the Laplace-Bessel operator is used instead of the Laplace operator. Our results involve the generalized divergence theorem and the second Green's identities relating the bulk with the boundary of a region on which differential Bessel operators act. In the "Weighted spherical mean and mean-value theorem for B-harmonic functions" section we obtain a mean-value theorem for B-harmonic functions. This theorem states that the value of a B-harmonic function at a point is equal to its weighted spherical mean over part of a sphere centered at that point. In the "Fractional weighted mean and Hankel transform of its kernel" section, we design a fractional weighted mean operator and study its boundedness. In the "Maximal inequality for the weighted spherical mean" section, we obtain maximal inequality for the weighted spherical mean and get its boundedness. In the "An application" section, the connection between the boundedness of the spherical maximal operator and the properties of solutions of the Euler–Poisson–Darboux equation with Bessel operators is given as an application.

Definitions

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, \ \ x_1>0, \dots, x_n>0\},$$

$$\overline{\mathbb{R}}{}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, \ \ x_1\geq 0, \dots, x_n\geq 0\},$$

$\gamma = (\gamma_1, \dots, \gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i = 1, \dots, n$, and $| \gamma |= \gamma_1 + \dots + \gamma_n$. The part of the sphere of radius r with center at the origin belonging to $\mathbb{R}^n_+$ we will denote $S^+_r(n)$:

$$S^+_r(n) = \{x \in \overline{\mathbb{R}}{}^n_+ : | x |= r\} \cup \{x \in \overline{\mathbb{R}}{}^n_+ : x_i = 0, | x | \leq r, i = 1, \dots, n\}.$$

For the weighed integral by the $S^+_1(n)$ we have formula [21], formula 107, p. 49

$$| S^+_1(n) |_\gamma = \int_{S^+_1(n)} x^\gamma dS = \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1} \Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{1}$$

Let Ω be finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i=0$, $i=1,...,n$, $\Omega_+=\Omega\cap\mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega\cap\overline{\mathbb{R}}^n_+$ where $\overline{\mathbb{R}}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n, x_1\geq 0,\ldots,x_n\geq 0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i=1,\ldots,n$ are continuous up to $x_i=0$. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\Big|_{x_i=0}=0$ for all non-negative integer $k\leq\frac{m-1}{2}$ (see [14], p. 21).
In the following, we will denote $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+)=\bigcap_{m=0}^{\infty}C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+)=C^\infty_{ev}$.
Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ be the space of all functions $f\in C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support. We will use notations $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)=\mathcal{D}_+(\overline{\Omega}_+)$ and $\overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}_+)=\overset{\circ}{C}{}^\infty_{ev}$.
Let $L^\gamma_p(\mathbb{R}^n_+)=L^\gamma_p$, $1\leq p<\infty$, be the space of all measurable in $\mathbb{R}^n_+$ functions even with respect to each variable x_i, $i=1,\ldots,n$ such that

$$\int_{\mathbb{R}^n_+} f(x)|^p x^\gamma dx<\infty,$$

where and further

$$x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $1\leq p<\infty$, the L^γ_p–norm of f is defined by

$$\|f\|_{L^\gamma_p(\mathbb{R}^n_+)}=\|f\|_{p,\gamma}=\left(\int_{\mathbb{R}^n_+} f(x)|^p x^\gamma dx\right)^{1/p}.$$

For $p=\infty$, the L^γ_∞–norm of f is defined by

$$\|f\|_{L^\gamma_\infty(\mathbb{R}^n_+)}=\|f\|_{\infty,\gamma}=\operatorname{ess\,sup}_{x\in\mathbb{R}^n_+}\mathrm{f(x)}.$$

It is known (see [14]) that L^γ_p is a Banach space.
The multi-dimensional Hankel transform of a function $f\in L^\gamma_1(\mathbb{R}^n_+)$ is expressed as

$$\mathbf{F}_\gamma(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^n_+} f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx, \tag{2}$$

where

$$\mathbf{j}_\gamma(x;\xi)=\prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \qquad \gamma_1>0,\ldots,\gamma_n>0,$$

the symbol j_ν is used for the normalized Bessel function of the first kind $j_\nu(x)=\dfrac{2^\nu\Gamma(\nu+1)}{x^\nu}J_\nu(x)$, where J_ν is Bessel function of the first kind [26].
Let $f\in L^\gamma_1(\mathbb{R}_+)$ and of bounded variation in a neighborhood of a point x of continuity of f. Then for $\gamma>0$ the inversion formula

$$\mathbf{F}_\gamma^{-1}(\widehat{f}(\xi))(x) = f(x) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int_{\mathbb{R}_+^n} \mathbf{j}_\gamma(x,\xi)\widehat{f}(\xi)\xi^\gamma\, d\xi$$

holds.

The multi-dimensional Hankel transform can be written using the one-dimensional Hankel transforms:

$$\mathbf{F}_\gamma(f)(\xi) = F_{\gamma_1}...F_{\gamma_n}(f)(\xi_1,\dots,\xi_n),$$

where $x=(x_1,\dots,x_n)$, $\xi=(\xi_1,\dots,\xi_n)$, $i=1,\dots,n$,

$$F_{\gamma_i}(f)(\xi) = \int_0^\infty f(x)\, j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)x_i^{\gamma_i}dx_i.$$

Similar to the Fourier transform, the Hankel transform reduces the Bessel differentiation operation to multiplication by the corresponding arguments (see [14])

$$F_{\gamma_i}((B_{\gamma_i})_{x_i}f)(\xi) = -\mid \xi_i\mid^2 F_{\gamma_i}(f)(\xi), \tag{3}$$

where $(B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ is a Bessel operator and $i=1,\dots,n$.

In [14], p. 20, the next theorem is presented.

Theorem 1 *If* $x^{\frac{\nu}{2}}\varphi \in L_2[0,\infty)$, *then Hankel transform* $x^{\frac{\nu}{2}}F_\nu\varphi \in L_2[0,\infty)$ *and Parseval's formula*

$$\int_0^\infty \mid F_\nu(\varphi)(\xi)\mid^2 \xi^\nu d\xi = 2^{\nu-1}\Gamma^2\left(\frac{\nu+1}{2}\right)\int_0^\infty \mid \varphi(x)\mid^2 x^\nu dx$$

is valid.

Using Theorem 1, we get Parseval's formula for the multi-dimensional Hankel transform. If $f\in L_2^\gamma(\mathbb{R}_+^n)$, then $\mathbf{F}_\gamma f \in L_2^\gamma(\mathbb{R}_+^n)$ and

$$\int_{\mathbb{R}_+^n} \mid \mathbf{F}_\gamma(f)(\xi)\mid^2 \xi^\gamma d\xi = 2^{|\gamma|-n}\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)\int_{\mathbb{R}_+^n} \mid f(x)\mid^2 x^\gamma dx$$

or

$$\|f\|_{2,\gamma} = C_{n,\gamma}\|\mathbf{F}_\gamma(f)\|_{2,\gamma}, \qquad C_{n,\gamma} = \frac{2^{n-|\gamma|}}{\prod} {}^{n}_{j=1}\Gamma^2\left(\frac{\gamma_j+1}{2}\right). \tag{4}$$

The multi-dimensional generalized translation is defined by the equality

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1}\dots{}^{\gamma_n}T_{x_n}^{y_n}f)(x), \tag{5}$$

where each of one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ acts for i=1, ..., n according to (see [15])

$$({}^{\gamma_i}T_{x_i}^{y_i}f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma_i}{2}\right)}$$

$$\times\int_0^\pi f(x_1,\dots,x_{i-1},\sqrt{x_i^2+\tau_i^2-2x_iy_i\cos\varphi_i},x_{i+1},\dots,x_n)\ \sin^{\gamma_i-1}\varphi_i\, d\varphi_i.$$

Next we will use notation

$$C(\gamma)=\pi^{-\frac{n}{2}}\prod_{i=1}^{n}\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Generalized convolution generated by a multi-dimensional generalized translation ${}^{\gamma}\mathbf{T}_x^y$ is given by

$$(f*g)_\gamma(x)=\int_{\mathbb{R}_+^n} f(y)({}^{\gamma}\mathbf{T}_x^y g)(x)y^\gamma\,dy. \tag{6}$$

Multi-dimensional Poisson operator $\mathbf{P}_x^\gamma$, acts to the integrable function f by the formula

$$\mathbf{P}_x^\gamma f(x)=C(\gamma)\int_0^\pi\ldots\int_0^\pi f(x_1\cos\alpha_1,\ldots,x_n\cos\alpha_n)\prod_{i=1}^{n}\sin^{\gamma_i-1}\alpha_i\,d\alpha_i. \tag{7}$$

Multi-dimensional Hankel transform (2) applied to generalized convolution (6) gives

$$\mathbf{F}_\gamma[(f*g)_\gamma](\xi)=\mathbf{F}_\gamma[f](\xi)\mathbf{F}_\gamma[g](\xi). \tag{8}$$

Integral $\int_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS$ is calculated by the formula (see [21])

$$\int_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}j_{\frac{n+|\gamma|}{2}-1}(r\mid\xi\mid). \tag{9}$$

Generalized divergence theorem and the second Green's formula for the Laplace–Bessel operator

The theory of B-harmonic functions should include generalizations of the classical tools for solving problems with the Laplace-Bessel operator. The aim of this section is to develop a field theory for the case when the Laplace-Bessel operator is used instead of the Laplace operator. To do this we need the following definitions.
Let

$$\nabla'_\gamma=\left(\frac{1}{x_1^{\gamma_1}}\frac{\partial}{\partial x_1},\ldots,\frac{1}{x_n^{\gamma_n}}\frac{\partial}{\partial x_n}\right)$$

be the first weighted operator nabla,

$$\nabla''_\gamma=\left(x_1^{\gamma_1}\frac{\partial}{\partial x_1},\ldots,x_n^{\gamma_n}\frac{\partial}{\partial x_n}\right)$$

be the second weighted operator nabla, then $(\nabla'_\gamma\cdot\nabla''_\gamma)=\Delta_\gamma$, where $\Delta_\gamma=\sum\limits_{j=1}^{n}B_{\gamma_j}$ is Laplace-Bessel operator, $B_{\underset{\rightarrow}{\gamma_j}}=\frac{1}{x_{\underset{\rightarrow}{j}}^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}=\frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$, $j=1,\ldots,n$ is a Bessel operator.
If $F=F(x)=(F_1(x),\ldots,F_n(x))$ is a vector field, then

$$\operatorname{div}'_\gamma\vec{F}=(\nabla'_\gamma\cdot\vec{F})=\frac{1}{x_1^{\gamma_1}}\frac{\partial F_1}{\partial x_1}+\ldots+\frac{1}{x_i^{\gamma_n}}\frac{\partial F_n}{\partial x_n}$$

is the first weighted divergence,

$$\operatorname{div}''_\gamma \vec{F} = (\nabla''_\gamma \cdot \vec{F}) = x_1^{\gamma_1} \frac{\partial F_1}{\partial x_1} + ... + x_n^{\gamma_n} \frac{\partial F_n}{\partial x_n}$$

is the second weighted divergence.
In this case the generalized divergence theorem states that the weighted surface integral of a vector field over a closed surface is equal to the weighted volume integral of the first weighted divergence over the region inside the surface.

Theorem 2 *Let G^+ be a domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i = 0, i = 1, \ldots, n$, either does not intersect G^+ or has one common segment with G^+ (possibly degenerating into a point) of the form*

$$\alpha_i(x') \le x_i \le \beta_i(x'), \qquad x'=(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n), \qquad i = 1, \ldots, n,$$

where α_i, β_i are smooth for $i = 1, \ldots, n$.

If $\vec{g} = (g_1(x), ..., g_n(x))$ is a vector field continuously differentiable in G^+ and $\vec{F} = (F_1(x), ..., F_n(x))$, $F_1(x)=x_1^{\gamma_1} g_1(x), ..., F_n(x)=x_n^{\gamma_n} g_n(x)$, then

$$\int_{G^+} (\nabla'_\gamma \cdot \cdot \vec{F})\, x^\gamma dx = \int_{S^+} \left(\vec{g} \cdot \vec{v}\right) x^\gamma \, dS\,, \tag{10}$$

where $\boldsymbol{\nu} = \mathbf{e}_1 \cos \eta_1 + ... + \mathbf{e}_n \cos \eta_n$ is an outer surface normal vector for S^+, η_i is an angle between vector $\boldsymbol{\nu}$ and an axe x_j, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal basis in $\mathbb{R}^n$.

Proof Let i be the fixed natural number between 1 and n inclusively. The part of surface S^+ defined by equation $x_i = \beta_i(x')$ we denote by S^+_u and the part of the surface S^+ defined by equation $x_i = \alpha_i(x')$ we denote by S^+_d, then

$$(\vec{v}, e_i) = \begin{cases} -\dfrac{1}{\sqrt{1+\left(\frac{\partial \alpha_i}{\partial x_1}\right)^2+\ldots+(\frac{\partial \alpha_i}{\partial x_{i-1}})^2+\left(\frac{\partial \alpha_i}{\partial x_{i+1}}\right)^2+\ldots+\left(\frac{\partial \alpha_i}{\partial x_n}\right)^2}}, & x \in S^+_d \\ \dfrac{1}{\sqrt{1+\left(\frac{\partial \beta_i}{\partial x_1}\right)^2+\ldots+\left(\frac{\partial \beta_i}{\partial x_{i-1}}\right)^2+\left(\frac{\partial \beta_i}{\partial x_{i+1}}\right)^2+\ldots+\left(\frac{\partial \beta_i}{\partial x_n}\right)^2}}, & x \in S^+_u. \end{cases}$$

We have

$$\int_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \sum_{i=1}^{n} \int_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i} x^\gamma dx.$$

Let us consider

$$\int_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i} x^\gamma dx$$

$$= \int_Q x_1^{\gamma_1} \ldots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} \ldots x_n^{\gamma_n} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \int_{\alpha_i(x')}^{\beta_i(x')} \frac{\partial F_i}{\partial x_i} dx_i,$$

where Q is a projection of G^+ to $x_i = 0$. Integrating by x_i we obtain

$$\int_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i} x^\gamma dx$$

$$= \int_Q F_i(x) \,|_{x_i=\alpha_i(x')}^{x_i=\beta_i(x')} \; x_1^{\gamma_1} \dots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} ... x_n^{\gamma_n} dx_1 ... dx_{i-1} dx_{i+1} \dots dx_n.$$

Let $(x')^{\gamma'} = x_1^{\gamma_1} \dots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} \dots x_n^{\gamma_n}$, $dx' = dx_1 ... dx_{i-1} dx_{i+1} \dots dx_n$, then

$$\int_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i} x^\gamma dx = \int_Q F_i(x_1, \dots, x_{i-1}, \beta_i(x'), x_{i+1}, \dots, x_n)(x')^{\gamma'} dx'$$

$$- \int_Q F_i(x_1, \dots, x_{i-1}, \alpha_i(x'), x_{i+1}, \dots, x_n)(x')^{\gamma'} dx'$$

$$= \int_Q F_i(x_1, \dots, x_{i-1}, \beta_i(x'), x_{i+1}, \dots, x_n)(\vec{v}, e_i)$$

$$\times \sqrt{1 + \left(\frac{\partial \beta_i}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial \beta_i}{\partial x_{i-1}}\right)^2 + \left(\frac{\partial \beta_i}{\partial x_{i+1}}\right)^2 + \dots + \left(\frac{\partial \beta_i}{\partial x_n}\right)^2} (x')^{\gamma'} dx'$$

$$+ \int_Q F_i(x_1, \dots, x_{i-1}, \alpha_i(x'), x_{i+1}, \dots, x_n)(\vec{v}, e_i)$$

$$\times \sqrt{1 + \left(\frac{\partial \alpha_i}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial \alpha_i}{\partial x_{i-1}}\right)^2 + \left(\frac{\partial \alpha_i}{\partial x_{i+1}}\right)^2 + \dots + \left(\frac{\partial \alpha_i}{\partial x_n}\right)^2} (x')^{\gamma'} dx'$$

$$= \int_{S_u^+} F_i(x)(\vec{v}, e_i)(x')^{\gamma'} dS_u + \int_{S_d^+} F_i(x)(\vec{v}, e_i)(x')^{\gamma'} dS_d$$

$$= \int_{S_u^+} g_i(x)(\vec{v}, e_i) x^\gamma dS_u + \int_{S_d^+} g_i(x)(\vec{v}, e_i) x^\gamma dS_d$$

$$= \int_{S^+} g_i(x) \cos \eta_i \, x^\gamma dS.$$

Then

$$\int_{G^+} (\nabla'_\gamma \cdot \vec{F}) \, x^\gamma dx = \sum_{i=1}^n \int_{S^+} g_i(x) \cos \eta_i \, x^\gamma dS = \int_{S^+} (\vec{g} \cdot \vec{v}) \; x^\gamma \; dS,$$

which completes the proof.

Remark 1. Suppose that the domain $G^+ \in \overline{\mathbb{R}}_+^n$ is a union of domains $G_1^+, \dots, G_m^+$ without common interior points. Let each domain G_j^+ in $\overline{\mathbb{R}}_+^n$ be such that each line perpendicular to the plane $x_i = 0$, $i = 1, \dots, n$, either does not intersect G_j^+ or has only one common segment with G_j^+ (possibly degenerating into a point) of the form

$$\alpha_i^j(x') \le x_i \le \beta_i^j(x'), \qquad x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \qquad i = 1, \dots, n,$$

where α_i, β_i are smooth for i=1, ..., n and $\vec{F} = (F_1(x), \dots, F_n(x))$, $F_1(x) = x_1^{\gamma_1} g_1(x)$, ..., $F_n(x) = x_n^{\gamma_n} g_n(x)$, $\vec{g} = (g_1(x), \dots, g_n(x))$ is a vector field continuously differentiable in G^+, then the following formula holds:

$$\int_{G^+} (\nabla'_\gamma \cdot \vec{F}) \, x^\gamma dx = \int_{S^+} (\vec{g} \cdot \vec{v}) \; x^\gamma \; dS \,, \tag{11}$$

where $S^+ \in \overline{\mathbb{R}}_+^n$ piecewise smooth surface boundary of G^+, $\vec{v}$ is a normal vector of the surface S^+.

Theorem 3 *Let G^+ satisfy the conditions of Remark 1. If φ and ψ are twice continuously differentiable functions defined on G^+, such that $\frac{\partial \varphi}{\partial x_i} |_{x_i=0} = 0$, $\quad \frac{\partial \psi}{\partial x_i} |_{x_i=0} = 0$, for $i = 1, \dots, n$, then the second Green's formula for the Laplace–Bessel operator of the form*

$$\int_{G^+}(\psi\Delta_\gamma\varphi-\varphi\Delta_\gamma\psi)\,x^\gamma\,dx=\int_{S^+}\left(\psi\frac{\partial\varphi}{\partial\vec{\upsilon}}-\varphi\frac{\partial\psi}{\partial\vec{\upsilon}}\right)x^\gamma\,dS \tag{12}$$

is valid.

Proof Let

$$\begin{aligned}\vec{F}&=\psi\nabla''_\gamma\varphi-\varphi\nabla''_\gamma\psi\\&=\psi\cdot x_1^{\gamma_1}\frac{\partial\varphi}{\partial x_1}-\varphi\cdot x_1^{\gamma_1}\frac{\partial\psi}{\partial x_1},\ldots,\psi\cdot x_n^{\gamma_n}\frac{\partial\varphi}{\partial x_n}-\varphi\cdot x_n^{\gamma_n}\frac{\partial\psi}{\partial x_n}\\&=x_1^{\gamma_1}\left(\psi\frac{\partial\varphi}{\partial x_1}-\varphi\frac{\partial\psi}{\partial x_1}\right),\ldots,x_n^{\gamma_n}\left(\psi\frac{\partial\varphi}{\partial x_n}-\varphi\frac{\partial\psi}{\partial x_n}\right),\end{aligned}$$

then $\vec{F}$ satisfies conditions of Remark 1. Setting

$$\vec{g}=\left(\psi\frac{\partial\varphi}{\partial x_1}-\varphi\frac{\partial\psi}{\partial x_1},\ldots,\psi\frac{\partial\varphi}{\partial x_n}-\varphi\frac{\partial\psi}{\partial x_n}\right),$$

we obtain that $\vec{g}$ is continuously differentiable vector field defined in G^+ and

$$\begin{aligned}(\nabla'_\gamma\cdot\vec{F})&=(\nabla'_\gamma\cdot(\psi\nabla''_\gamma\varphi-\varphi\nabla''_\gamma\psi))\\&=\sum_{i=1}^n\left(\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}\left(\psi\cdot x_i^{\gamma_i}\frac{\partial\varphi}{\partial x_i}\right)-\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}\left(\varphi\cdot x_i^{\gamma_i}\frac{\partial\psi}{\partial x_i}\right)\right)\\&=\sum_{i=1}^n\left(\frac{1}{x_i^{\gamma_i}}\frac{\partial\psi}{\partial x_i}\cdot x_i^{\gamma_i}\frac{\partial\varphi}{\partial x_i}+\psi\cdot\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial\varphi}{\partial x_i}-\right.\\&\left.-\frac{1}{x_i^{\gamma_i}}\frac{\partial\varphi}{\partial x_i}\cdot x_i^{\gamma_i}\frac{\partial\psi}{\partial x_i}-\varphi\cdot\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial\psi}{\partial x_i}\right)\\&=\sum_{i=1}^n\left(\psi B_{\gamma_i}\varphi-\varphi B_{\gamma_i}\psi\right)=\psi\Delta_\gamma\varphi-\varphi\Delta_\gamma\psi,\\(\vec{g}\cdot\vec{\upsilon})&=\left(\psi\frac{\partial\varphi}{\partial x_1}\cos\eta_1+\ldots+\psi\frac{\partial\varphi}{\partial x_n}\cos\eta_n\right)\\&-\left(\varphi\frac{\partial\psi}{\partial x_1}\cos\eta_1+\ldots+\varphi\frac{\partial\psi}{\partial x_n}\cos\eta_n\right)\\&=\psi\frac{\partial\varphi}{\partial\vec{\upsilon}}-\varphi\frac{\partial\psi}{\partial\vec{\upsilon}}.\end{aligned}$$

Now we can easily get (12) by applying (11).

Weighted spherical mean and mean-value theorem for *B*-harmonic functions

In this section we obtain mean-value theorem for *B*-harmonic functions. This theorem states that the value of a *B*-harmonic function at a point is equal to its weighted spherical mean over part of a sphere centered at that point. Weighted spherical mean in this case is constructed with the help of multi-dimensional generalized translation (5).
Weighted spherical mean (see [6, 16, 17, 21]) of function $u(x)$, $x\in\overline{\mathbb{R}}^n_+$ for $n\geq 2$ is

$$(M_t^\gamma u)(x) = (M_t^\gamma)_x[u(x)] = \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} {}^\gamma\mathbf{T}_x^{t\theta} u(x)\theta^\gamma dS, \tag{13}$$

where $\theta^\gamma = \prod_{i=1}^n \theta_i^{\gamma_i}$, $S_1^+(n) = \{\theta : \mid \theta \mid = 1, \theta \in \mathbb{R}_+^n\}$ is a part of a sphere in $\mathbb{R}_+^n$, and $\mid S_1^+(n) \mid_\gamma$ is given by

$$\mid S_1^+(n) \mid_\gamma = \int_{S_1^+(n)} x^\gamma dS = \frac{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{14}$$

For $n = 1$ let $(M_t^\gamma f)(x) = {}^\gamma T_x^t f(x)$.
We define the corresponding weighted maximal function $\mathbf{M}^\gamma$ by

$$\mathbf{M}^\gamma u(x) = \sup_{t>0} \mid M_t^\gamma u(x) \mid . \tag{15}$$

Theorem 4 *If $n > 1$, $n+ \mid \gamma \mid > 2$ and $u = u(x)$ is B-harmonic in a domain Ω and if the part of a sphere $S_{r_0,x}^+(n)$ is contained in Ω, then, for $0 < r \leq r_0$*

$$u(x) = (M_r^\gamma u)(x).$$

Proof Since operator ${}^{\gamma_i}T_{x_i}^{y_i}$ of function $u \in C_{ev}^2$ is a transmutation operator with the following intertwining property

$${}^{\gamma_i}T_{x_i}^{y_i}(B_{\gamma_i})_{x_i} u(x) = (B_{\gamma_i})_{y_i} {}^{\gamma_i}T_{x_i}^{y_i} u(x),$$

then if u is B-harmonic in a domain Ω, then ${}^\gamma\mathbf{T}_x^y u$ is harmonic in some Ω_1. That is, B-harmonicity is preserved under generalized translations. Therefore, we can consider only the case when $x = 0$. Let E be a subdomain of Ω satisfying the conditions of Remark 1 such that ∂E consists of smooth pieces and $\partial E \subset \Omega$. Applying formula (12) from Theorem 3 we obtain

$$\int_{\partial E} \frac{\partial u}{\partial \vec{v}} x^\gamma dS = \int_E \Delta_\gamma u(x) x^\gamma dx = 0, \tag{16}$$

where $\frac{\partial}{\partial \vec{v}}$ is differentiation in the direction of the outward directed normal to ∂E and dS is the element of surface area on ∂E.

Let $x \in \mathbb{R}_n^+$, $n > 1$ and

$$\vec{v}(x) = \begin{cases} \ln |x|, & n + |\gamma| = 2; \\ |x|^{2-n-|\gamma|}, & n + |\gamma| > 2, \end{cases}$$

then for $\mid x \mid > \varepsilon$ $\forall \varepsilon > 0$ we have $\triangle_\gamma v(x) = 0$, so v is B-harmonic in any domain not containing the origin.

Suppose $S_{\varepsilon,0}^+(n)$ and $S_{r,0}^+(n)$ are the surfaces of the parts of spheres centered in origin of radii ε and r correspondingly and Ω^* is the shell domain between $S_{\varepsilon,0}^+(n)$ and $S_{r,0}^+(n)$. Applying formula (12) to the functions u and v we obtain

$$0 = \int_{\Omega^*} (u\Delta_\gamma v - v\Delta_\gamma u) x^\gamma dx = \int_{\partial\Omega^*} \left(u\frac{\partial v}{\partial \vec{v}} - v\frac{\partial u}{\partial \vec{v}} \right) x^\gamma \, dS. \tag{17}$$

On the coordinate planes $x_i = 0$, $i = 1, \dots, n$ the the surface integrals in the right side of (17) are equal to zero. In the parts of the spheres $S_{\varepsilon,0}^+(n)$ and $S_{r,0}^+(n)$ the function $v(x)$ is constant so by (16) we get

$$\int_{\partial\Omega^*} v\frac{\partial u}{\partial \vec{v}} x^\gamma \, dS = 0.$$

Therefore, from (17) for $n+ \mid \gamma \mid > 2$ we obtain

$$\int_{\partial\Omega^*} u \frac{\partial v}{\partial \vec{v}}\, x^\gamma\, dS$$

$$= (2-n-\mid\gamma\mid)\left(\int_{S^+_{r,0}(n)} u(x)\mid x\mid^{1-n-|\gamma|}\; x^\gamma\, dS - \int_{S^+_{\varepsilon,0}(n)} u(x)\mid x\mid^{1-n-|\gamma|}\; x^\gamma\, dS\right) = 0.$$

Consequently,

$$r^{1-n-|\gamma|}\int_{S^+_{r,0}(n)} u(x)\, x^\gamma\, dS = \varepsilon^{1-n-|\gamma|}\int_{S^+_{\varepsilon,0}(n)} u(x)\, x^\gamma\, dS$$

and

$$(M^\gamma_r u)(0) = \frac{1}{\mid S^+_1(n)\mid_\gamma}\int_{S^+_1(n)} u(r\theta)\theta^\gamma\, dS \overset{r\theta=x}{=} \frac{1}{\mid S^+_1(n)\mid_\gamma\, r^{n+|\gamma|-1}}\int_{S^+_{r,0}(n)} u(x)\, x^\gamma\, dS$$

$$= \frac{1}{\mid S^+_1(n)\mid_\gamma\, \varepsilon^{n+|\gamma|-1}}\int_{S^+_{\varepsilon,0}(n)} u(x)\, x^\gamma\, dS \to u(0), \qquad \varepsilon\to 0.$$

This proves the theorem.

Theorem 5 *Let* $u \in L^\gamma_1(\mathbb{R}^n_+)$, *then*

$$\mathbf{F}_\gamma[M^\gamma_t u](x) = j_{\frac{n+|\gamma|}{2}-1}(t\mid x\mid)\, \mathbf{F}_\gamma[u](x). \tag{18}$$

Proof Using the formulas 3.172, p. 156 and 3.190, p. 162 from [21] we get

$$\mathbf{F}_\gamma(M^\gamma_t u)(x) = \int_{\mathbb{R}^n_+} (M^\gamma_t u)(\xi)\, \mathbf{j}_\gamma(x;\xi)\xi^\gamma\, d\xi$$

$$= \frac{1}{\mid S^+_1(n)\mid_\gamma}\int_{\mathbb{R}^n_+}\left(\int_{S^+_1(n)} {}^\gamma\mathbf{T}^{t\theta}_\xi u(\xi)\theta^\gamma\, dS\right)\mathbf{j}_\gamma(x;\xi)\xi^\gamma\, d\xi$$

$$= \frac{1}{\mid S^+_1(n)\mid_\gamma}\int_{S^+_1(n)}\left(\int_{\mathbb{R}^n_+} {}^\gamma\mathbf{T}^{t\theta}_\xi u(\xi)\, \mathbf{j}_\gamma(x;\xi)\xi^\gamma\, d\xi\right)\theta^\gamma\, dS$$

$$= \frac{1}{\mid S^+_1(n)\mid_\gamma}\int_{S^+_1(n)}\left(\int_{\mathbb{R}^n_+} u(\xi)\, {}^\gamma\mathbf{T}^{t\theta}_\xi\, \mathbf{j}_\gamma(x;\xi)\xi^\gamma\, d\xi\right)\theta^\gamma\, dS$$

$$= \int_{\mathbb{R}^n_+} (M^\gamma_t\, \mathbf{j}_\gamma(x;\xi))(\xi) u(\xi)\xi^\gamma\, d\xi$$

$$= j_{\frac{n+|\gamma|}{2}-1}(t\mid x\mid)\int_{\mathbb{R}^n_+} \mathbf{j}_\gamma(x;\xi)(\xi) u(\xi)\xi^\gamma\, d\xi$$

$$= j_{\frac{n+|\gamma|}{2}-1}(t \mid x \mid) \, \mathbf{F}_\gamma[u](x).$$

Fractional weighted mean and Hankel transform of its kernel

Let $\alpha > 0$ and

$$m_\alpha(x) = \begin{cases} \frac{(1-|x|^2)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } \mid x \mid < 1; \\ 0 & \text{if } \mid x \mid \geq 1. \end{cases}$$

For all $\alpha \in \mathbb{C}$ and we will consider weighted generalized function $m_\alpha(x)$ defined by the formula

$$\left(m_\alpha(x), \varphi\right)_\gamma = \frac{1}{\Gamma(\alpha)} \int_{\{|x|<1\}^+} (1- \mid x \mid^2)^{\alpha-1} \varphi(x) x^\gamma dx, \qquad \varphi \in S_{ev}, \tag{19}$$

where $\{\mid x \mid < 1\}^+ = \{x \in \mathbb{R}^n_+ : \mid x \mid < 1\}$.
Let $t > 0$ and

$$\mathbf{M}_t^{n,\gamma,\alpha}(x) = \frac{m_\alpha(x/t)}{t^{n+|\gamma|}}.$$

We consider fractional weighted ball mean defined as a generalized convolution

$$\mathcal{M}_t^{\alpha,\gamma} u(x) = (u * \mathbf{M}_t^{n,\gamma,\alpha})_\gamma(x).$$

It is easy to see that

$$\mathcal{M}_t^{\alpha,\gamma} u(x)$$

$$= \int_{\mathbb{R}^n_+} {}^\gamma \mathbf{T}_x^y u(x) \mathbf{M}_t^{n,\gamma,\alpha}(y) y^\gamma dy$$

$$= \frac{1}{t^{n+|\gamma|}} \int_{\mathbb{R}^n_+} {}^\gamma \mathbf{T}_x^y u(x) m_\alpha(y/t) y^\gamma dy$$

$$\overset{y/t=z}{=} \int_{\mathbb{R}^n_+} {}^\gamma \mathbf{T}_x^{tz} u(x) m_\alpha(z) z^\gamma dz = \frac{1}{\Gamma(\alpha)} \int_{B_1^+(n)} {}^\gamma \mathbf{T}_x^{tz} u(x) \, (1 - |z|^2)^{\alpha-1} z^\gamma dz$$

$$= \int_0^1 (1 - \lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} d\lambda \int_{S_1^+(n)} {}^\gamma \mathbf{T}_x^{t\lambda z} u(x) z^\gamma dS.$$

So the formula for the connection between $\mathcal{M}_t^{\alpha,\gamma}$ and M_t^γ is

$$\mathcal{M}_t^{\alpha,\gamma} u(x) = \mid S_1^+(n) \mid_\gamma \int_0^1 (1 - \lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} (M_{t\lambda}^\gamma u)(x) d\lambda. \tag{20}$$

The standard approach to the problems connected with ball and spherical means involves the consideration of appropriate maximal functions. We define the maximal function $\mathbf{M}^{\alpha,\gamma}$ by

$$\mathbf{M}^{\alpha,\gamma} u(x) = \sup_{t>0} \mid \mathcal{M}_t^{\alpha,\gamma} u(x) \mid .$$

Theorem 6 *The following formula holds for* $\alpha \in \mathbb{C}$

$$\mathbf{F}_\gamma\Big(m_\alpha(x)\Big)(\xi)=\mathcal{A}_{n,\gamma,\alpha}\, j_{\frac{n+|\gamma|}{2}+\alpha-1}(|\,\xi\,|),\qquad \mathcal{A}_{n,\gamma,\alpha}=\frac{\prod\limits_{i=1}^{n}\Gamma\Big(\frac{\gamma_i+1}{2}\Big)}{2^n\Gamma\Big(\frac{n+|\gamma|}{2}+\alpha\Big)}. \tag{21}$$

Proof Let first $\operatorname{Re}\alpha>0$. We perform the integration in $\mathbf{F}_\gamma\Big(m_\alpha(x)\Big)(\xi)$ by using spherical coordinates and applying the formula (9):

$$\mathbf{F}_\gamma\Big(m_\alpha(x)\Big)(\xi)=\frac{1}{\Gamma(\alpha)}\int_{B_1^+(n)}\mathbf{j}_\gamma(x,\xi)(1-|\,x\,|^2)^{\alpha-1}x^\gamma\,dx$$

$$\overset{x=r\theta,r=|x|}{=}\frac{1}{\Gamma(\alpha)}\int_0^1(1-r^2)^{\alpha-1}r^{n+|\gamma|-1}dr\int_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS$$

$$=\frac{1}{\Gamma(\alpha)}\frac{\prod\limits_{i=1}^{n}\Gamma\Big(\frac{\gamma_i+1}{2}\Big)}{2^{n-1}\Gamma\Big(\frac{n+|\gamma|}{2}\Big)}\int_0^1(1-r^2)^{\alpha-1}j_{\frac{n+|\gamma|}{2}-1}(r\,|\,\xi\,|)r^{n+|\gamma|-1}dr$$

$$=|\,\xi\,|^{1-\frac{n+|\gamma|}{2}}\frac{1}{\Gamma(\alpha)}2^{\frac{|\gamma|-n}{2}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\int_0^1(1-r^2)^{\alpha-1}J_{\frac{n+|\gamma|}{2}-1}(r\,|\,\xi\,|)r^{\frac{n+|\gamma|}{2}}dr.$$

Using the formula 2.12.4.6 from [18] of the form

$$\int_0^w r^{\nu+1}(w^2-r^2)^{\beta-1}J_\nu(\mu r)dr=\frac{2^{\beta-1}w^{\beta+\nu}\Gamma(\beta)}{\mu^\beta}J_{\beta+\nu}(\mu w), \tag{22}$$

$$w>0,\qquad \operatorname{Re}\beta>0,\qquad \operatorname{Re}\nu>-1,$$

we obtain

$$\int_0^1(1-r^2)^{\alpha-1}J_{\frac{n+|\gamma|}{2}-1}(r\,|\,\xi\,|)r^{\frac{n+|\gamma|}{2}}dr=\frac{2^{\alpha-1}\Gamma(\alpha)}{|\,\xi\,|^\alpha}J_{\frac{n+|\gamma|}{2}+\alpha-1}(|\,\xi\,|)$$

for $\operatorname{Re}\alpha>0$ and

$$\mathbf{F}_\gamma\Big(m_\alpha(x)\Big)(\xi)=\frac{\Gamma(\alpha)\prod\limits_{i=1}^{n}\Gamma\Big(\frac{\gamma_i+1}{2}\Big)}{2^n\Gamma\Big(\frac{n+|\gamma|}{2}+\alpha\Big)}\,j_{\frac{n+|\gamma|}{2}+\alpha-1}(|\,\xi\,|),$$

which coincides with (21). So we get (21) for $\operatorname{Re}\alpha>0$. For other values of α such that $\alpha\neq 0,-1,-2,-3,...$ equality (21) is valid by analytic continuation by α.

Residues of $\frac{(1-|x|^2)_{+,\gamma}^{\alpha-1}}{\Gamma(\alpha)}$ at $\alpha=-m$, $m\in\mathbb{N}\cup\{0\}$ have forms (see [21])

$$\lim_{\alpha\to-m}\frac{(1-|\,x\,|^2)^{\alpha-1}}{\Gamma(\alpha)}=\delta_\gamma^{(m)}(1-|\,x\,|^2),$$

then for $\alpha=-m$ we get

$$\mathbf{F}_\gamma\left(m_\alpha(x)\right)(\xi)$$

$$=\int_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)\delta_\gamma^{(m)}(1-|\,x\,|^2)x^\gamma\,dx=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}-m\right)}\,j_{\frac{n+|\gamma|}{2}-m-1}(|\,\xi\,|).$$

The proof is complete.

Corollary. The following formula holds for $\alpha\in\mathbb{C}$

$$\mathbf{F}_\gamma\left(\mathbf{M}_t^{n,\gamma,\alpha}\right)(\xi)=\mathcal{A}_{n,\gamma,\alpha}\,j_{\frac{n+|\gamma|}{2}+\alpha-1}(|\,t\xi\,|). \tag{23}$$

Proof For $\mathbf{M}_t^{n,\gamma,\alpha}(x)$ using (21) we obtain

$$\mathbf{F}_\gamma\left(\mathbf{M}_t^{n,\gamma,\alpha}\right)(\xi)$$

$$=\int_{\mathbb{R}^n_+}\mathbf{M}_t^{n,\gamma,\alpha}(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma\,dx$$

$$=\frac{1}{t^{n+|\gamma|}}\int_{\mathbb{R}^n_+}m_\alpha(x/t)\,\mathbf{j}_\gamma(x;\xi)x^\gamma\,dx$$

$$\overset{x=ty}{=}\int_{\mathbb{R}^n_+}m_\alpha(y)\,\mathbf{j}_\gamma(ty;\xi)y^\gamma\,dy$$

$$=\mathbf{F}_\gamma\left(m_\alpha(y)\right)(t\xi)=\mathcal{A}_{n,\gamma,\alpha}\,j_{\frac{n+|\gamma|}{2}+\alpha-1}(|\,t\xi\,|).$$

Using asymptotic expansion of Bessel function J_ν for large and for small real arguments we obtain for some $\varepsilon>0$ and $E>0$, respectively

$$|\,\mathbf{F}_\gamma\left(\mathbf{M}_t^{n,\gamma,\alpha}\right)(\xi)\,|=|\,\mathbf{F}_\gamma\left(m_\alpha\right)(t\xi)\,|\leq\mathcal{C}_{n,\gamma,\alpha},\qquad\text{for}\qquad t<\varepsilon \tag{24}$$

and

$$|\,\mathbf{F}_\gamma(\mathbf{M}_t^{n,\gamma,\alpha})(\xi)\,|=|\,\mathbf{F}_\gamma\,(m_\alpha)(t\xi)\,|\leq\frac{\mathcal{C}_{n,\gamma,\alpha}}{|\,t\xi\,|^{\frac{n+|\gamma|-1}{2}+\alpha}},\qquad\text{for}\qquad t>E, \tag{25}$$

where $\mathcal{C}_{n,\gamma,\alpha}$ is some constant depending on n,γ,α and not depending on t and ξ.
Therefore, for $\alpha\in\mathbb{C}$ in general we can also define the operators $\mathcal{M}_t^{\alpha,\gamma}$ by

$$\mathbf{F}_\gamma\left(\mathcal{M}_t^{\alpha,\gamma}u\right)(\xi)=\mathbf{F}_\gamma\left(m_\alpha\right)(t\xi)\mathbf{F}_\gamma[u](\xi),\qquad u\in S_{ev}. \tag{26}$$

For $\alpha=0$ we get

$$\mathbf{F}_\gamma\left(m_0\right)(\xi) = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}\right)} j_{\frac{n+|\gamma|}{2}-1}(\mid \xi \mid).$$

Maximal inequality for the weighted spherical mean

In this section we are interested in the a priori maximal inequalities for $1 \leq p \leq \infty$

$$\|M_t^\gamma u\|_{p,\gamma} \leq \|u\|_{p,\gamma}, \qquad t>0$$

$$\|\mathcal{M}_t^{\alpha,\gamma} u\|_{p,\gamma} \leq 2^\alpha \mid S_1^+(n)\mid_\gamma B(\alpha, n+\mid\gamma\mid)\,\|u\|_{p,\gamma}, \qquad t>0, \qquad \alpha>0,$$

where $B(a,b)=\int_0^1(1-t)^a\, t^b dt$, $a,b>0$ is the beta function.
Let φ be smooth and have a compact support such that $\mathbf{F}_\gamma[\varphi](0)=\int_{\mathbb{R}_+^n}\varphi(x)\mathbf{j}_\gamma(x;0)x^\gamma dx = \int_{\mathbb{R}_+^n}\varphi(x)x^\gamma dx = \mathbf{F}_\gamma\left(m_\alpha(\cdot)\right)(0)$, $\varphi_t(x)=\frac{\varphi(x/t)}{t^{n+|\gamma|}}$. Then it is clear that $\mathbf{F}_\gamma[\varphi_t](\xi)=\mathbf{F}_\gamma[\varphi](t\xi)$. We will be dealing with a given function $g_{\alpha,\gamma}[u](x)$ on $\mathbb{R}_+^n$ of the form

$$g_{\alpha,\gamma}[u](x) = \left(\int_0^\infty \mid \mathcal{M}_t^{\alpha,\gamma}u(x) - (u*\varphi_t)_\gamma \mid^2 \frac{dt}{t}\right)^{1/2}.$$

Theorem 7 *If* $\alpha > \frac{1-n-|\gamma|}{2}$ *and* $u \in L_2^\gamma(\mathbb{R}_+^n)$, *then*

$$\|g_{\alpha,\gamma}[u]\|_{2,\gamma} \leq A_{\alpha,\gamma}\,\|u\|_{2,\gamma}. \tag{27}$$

Proof Let us consider the integral

$$\int_{\mathbb{R}_+^n} \mid g_{\alpha,\gamma}[u](x)\mid^2 x^\gamma dx$$

$$= \int_{\mathbb{R}_+^n} x^\gamma dx \int_0^\infty \mid \mathcal{M}_t^{\alpha,\gamma}u(x) - (u*\varphi_t)_\gamma \mid^2 \frac{dt}{t}$$

$$= \int_{\mathbb{R}_+^n} x^\gamma dx \int_0^\infty \mid (u*(\mathbf{M}_t^{n,\gamma,\alpha}(x)-\varphi_t(x)))_\gamma \mid^2 \frac{dt}{t}$$

$$= \int_0^\infty \frac{dt}{t}\int_{\mathbb{R}_+^n} \mid (u*(\mathbf{M}_t^{n,\gamma,\alpha}(x)-\varphi_t(x)))_\gamma \mid^2 x^\gamma dx$$

$$= \int_0^\infty \left\|(u*(\mathbf{M}_t^{n,\gamma,\alpha}(x)-\varphi_t(x)))_\gamma\right\|_{2,\gamma}^2 \frac{dt}{t}.$$

By Parseval's identity for the Hankel transform (4) and formula (8) we obtain

$$\int_{\mathbb{R}_+^n} \mid g_{\alpha,\gamma}[u](x)\mid^2 x^\gamma dx = C_{n,\gamma}\int_0^\infty \left\|\mathbf{F}_\gamma(u*(\mathbf{M}_t^{n,\gamma,\alpha}(x)-\varphi_t(x)))_\gamma\right\|_{2,\gamma}^2 \frac{dt}{t}$$

$$= C_{n,\gamma} \int_{\mathbb{R}^n_+} |\, \mathbf{F}_\gamma(u)(x)\,|^2\, x^\gamma dx \int_0^\infty |\, (\mathbf{F}_\gamma(\mathbf{M}^{n,\gamma,\alpha}_t)(x) - \mathbf{F}_\gamma(\varphi_t)(x))\,|^2\, \frac{dt}{t}$$

$$= C_{n,\gamma} \int_{\mathbb{R}^n_+} |\, \mathbf{F}_\gamma(u)(x)\,|^2\, x^\gamma dx \int_0^\infty |\, \mathbf{F}_\gamma\left(m_\alpha\right)(tx) - \mathbf{F}_\gamma(\varphi)(tx)\,|^2\, \frac{dt}{t}.$$

Therefore, it suffices to show that $\int_0^\infty |\, \mathbf{F}_\gamma\left(m_\alpha\right)(tx) - \mathbf{F}_\gamma(\varphi)(tx)\,|^2\, \frac{dt}{t}$ is bounded. We have

$$\int_0^\infty |\, \mathbf{F}_\gamma\left(m_\alpha\right)(tx) - \mathbf{F}_\gamma(\varphi)(tx)\,|^2\, \frac{dt}{t} =$$
$$= \int_0^\varepsilon |\, \mathbf{F}_\gamma\left(m_\alpha\right)(tx) - \mathbf{F}_\gamma(\varphi)(tx)\,|^2\, \frac{dt}{t} + \int_\varepsilon^\infty |\, \mathbf{F}_\gamma\left(m_\alpha\right)(tx) - \mathbf{F}_\gamma(\varphi)(tx)\,|^2\, \frac{dt}{t}.$$

Functions $\mathbf{F}_\gamma\left(m_\alpha\right)(tx)$, $\mathbf{F}_\gamma(\varphi)(tx)$ are smooth near the origin and (24) takes place. The first integral converges since $\mathbf{F}_\gamma\left(m_\alpha\right)(0)=\mathbf{F}_\gamma(\varphi)(0)$ and therefore, $|\, \mathbf{F}_\gamma\left(m_\alpha\right)(tx) - \mathbf{F}_\gamma(\varphi)(tx)\,|$ is infinitely small for $t\to 0$. Taking into account (25) and the fact that φ has compact support we can see that the second integral converges for $\alpha > \frac{1-n-|\gamma|}{2}$. The proof is complete.

Theorem 8 *If* $\alpha > \frac{1-n-|\gamma|}{2}$ *and* $u \in L_2^\gamma(\mathbb{R}^n_+)$, *then*

$$\left\| \sup_{s>0} \left(\frac{1}{s} \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x)\,|^2\, dt \right)^{1/2} \right\|_{2,\gamma} \leq A'_{\alpha,\gamma}\, \|u\|_{2,\gamma}. \tag{28}$$

Proof We have

$$\int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x)\,|^2\, dt = \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma + (u * \varphi_t)_\gamma\,|^2\, dt$$
$$\leq \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,|^2\, dt$$
$$+ 2 \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,| \cdot |\,(u * \varphi_t)_\gamma\,|\, dt + \int_0^s |\,(u * \varphi_t)_\gamma\,|^2\, dt.$$

Note that by Hölder's inequality we get

$$\sup_{s>0} \frac{1}{s} \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,|\, dt \leq \sup_{s>0} \frac{1}{\sqrt{s}} \left(\int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,|^2\, dt \right)^{1/2}$$
$$\leq \left(\int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,|^2\, \frac{dt}{t} \right)^{1/2} \leq g_{\alpha,\gamma}[u](x).$$

Therefore,

$$\sup_{s>0} \frac{1}{s} \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x)\,|^2\, dt \leq$$

$$\sup_{s>0} \frac{1}{s} \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,|^2\, dt$$

$$+2 \sup_{s>0} \frac{1}{s} \int_0^s |\, \mathcal{M}^{\alpha,\gamma}_t u(x) - (u * \varphi_t)_\gamma\,| \cdot |\,(u * \varphi_t)_\gamma\,|\, dt + \sup_{s>0} \frac{1}{s} \int_0^s |\,(u * \varphi_t)_\gamma\,|^2\, dt$$

$$\leq (g_{\alpha,\gamma}[u](x))^2 + 2g_{\alpha,\gamma}[u](x)\sup_{t>0} | (u * \varphi_t)_\gamma | + \sup_{t>0} | (u * \varphi_t)_\gamma |^2$$

$$= (g_{\alpha,\gamma}[u](x) + \sup_{t>0} | (u * \varphi_t)_\gamma |)^2$$

and

$$\sup_{s>0}\left(\frac{1}{s}\int_0^s | \mathcal{M}_t^{\alpha,\gamma} u(x) |^2 \, dt\right)^{1/2} \leq | g_{\alpha,\gamma}[u](x) | + \sup_{t>0} | (u * \varphi_t)_\gamma | .$$

Since φ has a compact support we get $\| \sup_{t>0} | (u * \varphi_t)_\gamma \|_{2,\gamma} \leq C\|u\|_{2,\gamma}$. Thus using (27) we obtain (28).

Theorem 9 *Let us take* α' *such that* $\alpha' < \alpha$. *We can move from the smaller values of* α *to the larger values using the rising operator. Namely*

$$\mathcal{M}_t^{\alpha,\gamma} u(x) = \frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 \mathcal{M}_{ts}^{\alpha',\gamma} u(x)(1-s^2)^{\alpha-\alpha'-1} s^{n+|\gamma|+2\alpha'-1} ds. \tag{29}$$

Proof We have

$$\begin{aligned}
&\frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 \mathcal{M}_{ts}^{\alpha',\gamma} u(x)(1-s^2)^{\alpha-\alpha'-1} s^{n+|\gamma|+2\alpha'-1} ds \\
&= \frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 (1-s^2)^{\alpha-\alpha'-1} s^{n+|\gamma|+2\alpha'-1}\left(\frac{1}{(ts)^{n+|\gamma|}}\int_{\mathbb{R}_+^n} {}^\gamma\mathbf{T}_x^y u(x) m_{\alpha'}\left(\frac{y}{st}\right) y^\gamma dy\right) ds \\
&= \frac{2}{\Gamma(\alpha-\alpha')}\frac{1}{t^{n+|\gamma|}}\int_{\mathbb{R}_+^n} {}^\gamma\mathbf{T}_x^y u(x)\left(\int_0^1 (1-s^2)^{\alpha-\alpha'-1} s^{2\alpha'-1} m_{\alpha'}\left(\frac{y}{st}\right) ds\right) y^\gamma dy \\
&= \frac{2}{\Gamma(\alpha')\Gamma(\alpha-\alpha')}\frac{1}{t^{n+|\gamma|}}\int_{\mathbb{R}_+^n} {}^\gamma\mathbf{T}_x^y u(x)\left(\int_{|\frac{y}{t}|}^1 (1-s^2)^{\alpha-\alpha'-1} s^{2\alpha'-1}\left(1 - |\frac{y}{st}|^2\right)^{\alpha'-1} ds\right) y^\gamma dy.
\end{aligned}$$

Let us calculate the inner integral. For $|t| < |y|$ we get

$$\begin{aligned}
&\int_{|\frac{y}{t}|}^1 (1-s^2)^{\alpha-\alpha'-1} s^{2\alpha'-1}\left(1 - |\frac{y}{st}|^2\right)^{\alpha'-1} ds \\
&= \frac{1}{|t|^{2\alpha'-2}}\int_{|\frac{y}{t}|}^1 (1-s^2)^{\alpha-\alpha'-1}\left(t^2s^2 - |y|^2\right)^{\alpha'-1} s ds \\
&= \frac{1}{|t|^{2\alpha'-2}}\frac{(t^2-|y|^2)^{\alpha-1}|t|^{2(\alpha'-\alpha)}\Gamma(\alpha')\Gamma(\alpha-\alpha')}{2\Gamma(\alpha)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 \mathcal{M}_{ts}^{\alpha',\gamma} u(x)(1-s^2)^{\alpha-\alpha'-1} s^{n+|\gamma|+2\alpha'-1} ds \\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{t^{n+|\gamma|+2\alpha-2}}\int_{\mathbb{R}_+^n, |t|<|y|} {}^\gamma\mathbf{T}_x^y u(x)(t^2-|y|^2)^{\alpha-1} y^\gamma dy \\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{t^{n+|\gamma|}}\int_{\mathbb{R}_+^n, |t|<|y|} {}^\gamma\mathbf{T}_x^y u(x)\left(1 - \frac{|y|^2}{t^2}\right)^{\alpha-1} y^\gamma dy \\
&= \frac{1}{t^{n+|\gamma|}}\int_{\mathbb{R}_+^n} {}^\gamma\mathbf{T}_x^y u(x) m_\alpha(y/t) y^\gamma dy = \mathcal{M}_t^{\alpha,\gamma} u(x).
\end{aligned}$$

Theorem 10 *If* $\alpha > 1 - \frac{n+|\gamma|}{2}$ *and* $u \in L_2^\gamma(\mathbb{R}_+^n)$, *then*

$$\left\| \sup_{t>0} \mid \mathcal{M}_t^{\alpha,\gamma} u \mid \right\|_{2,\gamma} \leq A''_{\alpha,\gamma} \|u\|_{2,\gamma}. \tag{30}$$

Proof Let $\alpha > \alpha' + \frac{1}{2}$, then by (29) applying Hölder's inequality we get

$$\mid \mathcal{M}_t^{\alpha,\gamma} u(x) \mid = \frac{2}{\Gamma(\alpha-\alpha')} \mid \int_0^1 \mathcal{M}_{ts}^{\alpha',\gamma} u(x)(1-s^2)^{\alpha-\alpha'-1} s^{n+|\gamma|+2\alpha'-1} ds \mid$$

$$\leq \frac{2}{\Gamma(\alpha-\alpha')} \left(\int_0^1 \mid \mathcal{M}_{ts}^{\alpha',\gamma} u(x) \mid^2 ds \right)^{1/2} \left(\int_0^1 \mid (1-s^2)^{\alpha-\alpha'-1} s^{n+|\gamma|+2\alpha'-1} \mid^2 ds \right)^{1/2}$$

$$= \frac{2}{\Gamma(\alpha-\alpha')} \left(\frac{\Gamma(2\alpha-2\alpha'-1)\Gamma\left(2\alpha'+n+\mid\gamma\mid-\frac{1}{2}\right)}{2\Gamma\left(n+\mid\gamma\mid+2\alpha-\frac{3}{2}\right)} \right)^{1/2} \left(\frac{1}{t} \int_0^t \mid \mathcal{M}_s^{\alpha',\gamma} u(x) \mid^2 ds \right)^{1/2}.$$

For $\alpha' > \frac{1-n-|\gamma|}{2}$ by (28) we obtain

$$\left\| \sup_{t>0} \mid \mathcal{M}_t^{\alpha,\gamma} u(x) \mid \right\|_{2,\gamma} \leq C(n,\gamma,\alpha',\alpha) \left\| \sup_{s>0} \left(\frac{1}{s} \int_0^s \mid \mathcal{M}_t^{\alpha,\gamma} u(x) \mid^2 dt \right)^{1/2} \right\|_{2,\gamma} \leq A''_{\alpha,\gamma} \|u\|_{2,\gamma}.$$

Therefore, for $\alpha > \alpha' + \frac{1}{2} > \frac{1-n-|\gamma|}{2} + \frac{1}{2} = 1 - \frac{n+|\gamma|}{2}$ we get the statement of Theorem.

The next result follows from Lemma 10.

Theorem 11 *If* $\alpha > 1 - \frac{n+|\gamma|}{2}$ *and* $u \in L_2^\gamma(\mathbb{R}_+^n)$, *then the following inequalities hold*

$$\|\mathbf{M}^{\alpha,\gamma} u\|_{2,\gamma} \leq C_{\alpha,\gamma} \|u\|_{2,\gamma}, \tag{31}$$

$$\|\mathbf{M}^\gamma u\|_{2,\gamma} \leq C_\gamma \|u\|_{2,\gamma}, \tag{32}$$

where $C_{\alpha,\gamma}$ *and* C_γ *are some constants,* $\mathbf{M}^\gamma u$ *is weighted maximal function* (15).

Let us establish important properties of $M_t^\gamma u$.

Theorem 12 *The operator* M_t^γ *is bounded on* $L_p^\gamma(\mathbb{R}_+^n)$ *for* $1 \leq p \leq \infty$. *Moreover,*

$$\|M_t^\gamma u\|_{p,\gamma} \leq \|u\|_{p,\gamma}, \quad t > 0.$$

Proof For $1 \leq p < \infty$ applying the Minkowski inequality we get

$$\|M_t^\gamma u\|_{p,\gamma} = \left(\int_{\mathbb{R}_+^n} \mid \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} {}^\gamma\mathbf{T}_x^{t\theta} u(x) \theta^\gamma dS \mid^p x^\gamma dx \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} \left(\int_{\mathbb{R}_+^n} \mid^\gamma \mathbf{T}_x^{t\theta} u(x) \mid^p x^\gamma dx \right)^{\frac{1}{p}} \theta^\gamma dS$$

$$\leq \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} \Big(\int_{\mathbb{R}^n_+} {}^\gamma\mathbf{T}_x^{t\theta} \big(|u(x)|^p\big) x^\gamma dx \Big)^{\frac{1}{p}} \theta^\gamma dS$$

$$\leq \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} \Big(\int_{\mathbb{R}^n_+} |u(x)|^p x^\gamma dx \Big)^{\frac{1}{p}} \theta^\gamma dS$$

$$= \|u\|_{p,\gamma} \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} \theta^\gamma dS = \|u\|_{p,\gamma}.$$

Analogously, for $p = \infty$

$$\|M_t^\gamma u\|_{\infty,\gamma} = \frac{1}{\mid S_1^+(n) \mid_\gamma} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+} \mid \int_{S_1^+(n)} {}^\gamma\mathbf{T}_x^{t\theta} u(x)\theta^\gamma dS \mid$$

$$\leq \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} \|u\|_{\infty,\gamma} \theta^\gamma dS$$

$$= \|u\|_{\infty,\gamma} \frac{1}{\mid S_1^+(n) \mid_\gamma} \int_{S_1^+(n)} \theta^\gamma dS = \|u\|_{\infty,\gamma}.$$

Let us establish important properties of $\mathcal{M}_t^{\alpha,\gamma}$.

Theorem 13 *The operator $\mathcal{M}_t^{\alpha,\gamma}$ is bounded on $L_p^\gamma(\mathbb{R}^n_+)$ for $1 \leq p \leq \infty$. Moreover,*

$$\|\mathcal{M}_t^{\alpha,\gamma} u\|_{p,\gamma} \leq 2^\alpha \mid S_1^+(n) \mid_\gamma B(\alpha, n+ \mid \gamma \mid) \|u\|_{p,\gamma}, \quad t > 0, \ \alpha > 0.$$

Proof For $1 \leq p < \infty$ applying the Minkowski inequality we get

$$\|\mathcal{M}_t^{\alpha,\gamma} u\|_{p,\gamma} = \left(\int_{\mathbb{R}^n_+} \mid \mid S_1^+(n) \mid_\gamma \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} (M_{t\lambda}^\gamma u)(x) d\lambda \mid^p \ x^\gamma dx \right)^{\frac{1}{p}}$$

$$\leq \mid S_1^+(n) \mid_\gamma \int_0^1 \Big(\int_{\mathbb{R}^n_+} \mid (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} (M_{t\lambda}^\gamma u)(x) \mid^p \ x^\gamma dx \Big)^{\frac{1}{p}} d\lambda$$

$$= \mid S_1^+(n) \mid_\gamma \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} \left(\int_{\mathbb{R}^n_+} \mid (M_{t\lambda}^\gamma u)(x) \mid^p \ x^\gamma dx \right)^{\frac{1}{p}} d\lambda$$

$$\leq \mid S_1^+(n) \mid_\gamma \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} d\lambda \ \|u\|_{p,\gamma}$$

$$\leq 2^{\alpha} \ | S_1^+(n) |_{\gamma} \ \|u\|_{p,\gamma} \int_0^1 (1-\lambda)^{\alpha-1} \lambda^{n+|\gamma|-1} d\lambda$$

$$= 2^{\alpha} \ | S_1^+(n) |_{\gamma} \ B(\alpha, n+ | \gamma |) \|u\|_{p,\gamma}.$$

Analogously, for $p = \infty$

$$\|\mathcal{M}_t^{\alpha,\gamma} u\|_{\infty,\gamma} =| S_1^+(n) |_{\gamma} \ \operatorname*{ess\,sup}_{x \in \mathbb{R}_+^n} \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} (M_{t\lambda}^{\gamma} u)(x) d\lambda$$

$$\leq | S_1^+(n) |_{\gamma} \ \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} \ \|M_{t\lambda}^{\gamma} u\|_{\infty,\gamma} \, d\lambda$$

$$\leq | S_1^+(n) |_{\gamma} \ \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} d\lambda \, \|u\|_{\infty,\gamma}$$

$$\leq 2^{\alpha} \ | S_1^+(n) |_{\gamma} \ \int_0^1 (1-\lambda)^{\alpha-1} \lambda^{n+|\gamma|-1} d\lambda \, \|u\|_{\infty,\gamma}$$

$$= 2^{\alpha} \ | S_1^+(n) |_{\gamma} \ B(\alpha, n+ | \gamma |) \|u\|_{\infty,\gamma}.$$

An application

Spherical averages often make their appearance as solutions of certain partial differential equations. In this section we will use in [21], "An application'', the solution representations to the Cauchy problem for a general form of the Euler–Poisson–Darboux equation with Bessel operators via generalized translation and spherical mean operators. [21], "An application'', also contains a short historical introduction on differential equations with Bessel operators and a rather detailed reference list of monographs and papers on mathematical theory and applications of this class of differential equations.
The classical Euler–Poisson–Darboux equation is defined by

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} = \triangle_x u, \quad u = u(x,t;k), \quad x \in \mathbb{R}^n, \quad t > 0, \quad k \in \mathbb{R}. \tag{33}$$

The operator acting by variable t in (33) is the Bessel operator

$$\left(B_k\right)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t} = \frac{1}{t^k}\frac{\partial}{\partial t} t^k \frac{\partial}{\partial t}.$$

When $n = 1$, Eq. (33) appears in Leonard Euler's work (see [7, p. 227]) and later it was studied by Simeon Denis Poisson in [19], by Gaston Darboux in [5], and by Bernhard Riemann in [20].
For the Cauchy problem initial conditions to the solution of Eq. (33) are added:

$$u(x, 0; k) = f(x), \qquad \frac{\partial u(x,t;k)}{\partial t} \ |_{t=0} = 0. \tag{34}$$

The interest in the multi-dimensional equation (33) has increased significantly after Alexander Weinstein's papers [27–30]. In [27, 28] the Cauchy problem for (33) is considered with $k \in \mathbb{R}$, the first initial condition being nonzero and the second initial condition equaling zero. A solution of the Cauchy problem (33)–(34) in the classical sense was obtained in [28–30] and in the distributional sense in [3, 4]. S. A. Tersenov in [25] solved the Cauchy problem for (33) in the general form where the first and the second conditions are nonzeros. Singular and degenerate hyperbolic equations of one-dimensional EPD-type were considered in [22–24]. Different problems for Eq. (33) with many applications to gas dynamics, hydrodynamics, mechanics, elasticity, plasticity, and so on were also studied, see [21, section "An application"] for references. In this section we consider the singular with respect to all variables hyperbolic differential equation, which is a generalization of the multi-dimensional Euler-Poisson-Darboux equation (33):

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} = \left(\triangle_\gamma \right)_x u, \quad u = u(x,t;k), \quad x \in \mathbb{R}^n, \quad t > 0, \quad k \in \mathbb{R}, \tag{35}$$

with the singular elliptic operator defined by $\left(\triangle_\gamma \right)_x = \sum\limits_{j=1}^{n} \left(B_{\gamma_j}\right)_{x_j}$ is Laplace-Bessel operator together with initial conditions

$$u(x,0;k) = f(x), \quad \lim_{t\to 0} t^k\, u_t(x,t;k) = g(x).$$

Note that, Eq. (35) in the general form is called the Euler-Poisson-Darboux equation.
We start using a solution to the first Cauchy problem above,

$$\left(B_k\right)_t = \left(\triangle_\gamma \right)_x u, \quad u = u(x,t;k), \quad x \in \mathbb{R}^n_+, \quad t > 0, \quad k \in \mathbb{R}, \tag{36}$$

$$u(x,0;k) = f(x), \quad u_t(x,0;k) = 0 \tag{37}$$

in the compact integral form via generalized translation and spherical mean operators for all values of the parameter k, including also exceptional odd negative values, which have been studied in [21], Theorem 75].

Theorem 14 *Let $f = f(x) \in C^2_{ev}$, $x \in \mathbb{R}^n_+$. Then for all the cases $k > n + |\gamma| - 1$ the unique solution to (36)–(37) is*

$$u(x,t;k) = C_{n,\gamma,k}\, \mathcal{M}^{\alpha,\gamma}_t f(x), \tag{38}$$

where $\mathcal{M}^{\alpha,\gamma}_t$ is given by (20), $\alpha = \frac{k-n-|\gamma|+1}{2}$, $C_{n,\gamma,k} = \frac{2^n \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right) \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}$. The unique solution of the problem (36)–(37) for $k = n + |\gamma| - 1$ is the weighted spherical mean $M^\gamma_t f(x)$.

From Theorems 12, 13 and 14 we get the following corollary:
Corollary. Let $k \geq n + |\gamma| - 1$ and $1 \leq p \leq \infty$, then for the weak solution $u = u(x,t;k)$ of the problem (36)–(37) with the initial data in $f \in L^\gamma_p(\mathbb{R}^n_+)$, we have the following a priori estimate:

$$\|u(\cdot,t;k)\|_{p,\gamma} \leq C_{n,\gamma,k} \|f\|_{p,\gamma}, \qquad t > 0.$$

Also, $\lim\limits_{t\to 0} u(x,t;k) = f(x)$ a.e. $x \in \mathbb{R}^n_+$.

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