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Spaces of Random Sets in $\mathbb{R}^d$

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Abstract—The construction of the probabilistic space $\langle \Omega, \mathfrak{B}, \mathsf{P} \rangle$ of random sets X of a general form in the finite-dimensional immersion space $\mathbb{R}^d$ which is based on the concept of its cellular comminution is described. Elementary random events in this space are equivalence classes of X-realizations that have the same right closure. The measurability structure is built on the basis of the introduced concept of Boolean c-systems.

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1. INTRODUCTION

Random sets in $\mathbb{R}^d$, $d \in \mathbb{N}$ are objects traditionally studied by probability theory, both from the point of view of general theory of their probability spaces together with properties of their random realizations (see, for example, [1, 2]) and from the point of view of their specific models analysis for the purpose of their applications.

Until quite recently, in applications, models of random sets located in the *immersion space* $\mathbb{R}^d$ which have the following types were used. Firstly, there are those whose realizations consist of isolated points. It is the so-called *point random fields* [3]. Models of such sets are used, for example, in statistical physics (see, for example, [4, 5]). In the one-dimensional case, models of such sets are used in the queue theory. Random sets with the dimension of their realizations X, which coincides with the dimension of the immersion space, arise, for example, in the well-known Kolmogorov model [6, 7] describing the dynamics of crystallization. In this case realizations X represent some randomly located in $\mathbb{R}^3$ multi-connected regions such that each their connected component has, generally speaking, a random shape. Another application of such probabilistic constructions is represented in the theory of image recognition [2]. Graphs of random processes and random fields with their values in $\mathbb{R}^d$ are some examples of stochastic geometric structures with intermediate topological dimension. In this case, as a rule, it is necessary to abandon the assumption about the continuity of such geometric structures when their "typical" realizations X are under consideration. Methods for constructing of structure of measurability and associated probability distributions for random set models of this type are based on different principles compared to point random fields. This is due to the fact that random realizations $X \subset \mathbb{R}^d$ are uncountable sets. Therefore, in this case, the definition of probability distributions is based on the idea of *separability* [1] of such a stochastic geometric structures, that is, on its unambiguous recoverability by a countable set densely located in $\mathbb{R}^d$. Then, probability distributions are naturally determined by means of point distribution laws $\Pr\{\mathbf{x}_1 \in \Lambda_1, ..., \mathbf{x}_n \in \Lambda_n\}$ for any Borel sets $\Lambda_j \subset \mathbb{R}^d$,

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$j = 1, ..., n,\ n \in \mathbb{N}$ that it is analogous to the case when a probability measure for separable random processes is determined.

In contrast to types of random sets pointed out, in statistical mathematical physics, it is necessary to synthesize probabilistic spaces of random sets with X-realizations which have the continuum cardinality with the probability one, but at the same time they are no longer separable. It is done when it is necessary to describe the so-called *stochastic fractals* [8–10]. For this reason, the structure of measurability in such probability spaces should not be introduced in the same way as it is done in the situations described above. This is due to the fact that the points of a fractal random set cannot be described using any finite or even countable set of parameters. Apparently, the only one way to construct the corresponding probability spaces is the use of the assumption that random X-realizations should be closed with the probability one. Then, each of them can be represented as a limit of a sequence of closed sets with a non-empty interior. In turn, this boils down to the study of random sets represented by *closed covers* of X-realizations. In such a situation, the corresponding measurability structure should be represented by random convergent sequences of closed domains in $\mathbb{R}^d$. In this paper, we introduce a class of probability spaces following this idea.

The constructions described here are a generalization of those that we proposed in the work [11] in the case $d = 1$ and in the works [12–14] devoted to the application to statistical mathematical physics.

2. CELLULAR COMMINUTIONS OF THE SPACE $\mathbb{R}^d$

We will study random subsets X in the space $\mathbb{R}^d$. Further, it is named the *immersion space* of realizations X of a random set.

In our constructions, the concept of the *comminution* of space $\mathbb{R}^d$ will play an important role. Sets $C = \bigotimes_{j=1}^{d}[a_j, a_j + c) \subset \mathbb{R}^d,\ a_k \geq 0,\ k \in I_d \equiv \{1, ..., d\}$ are called the *cells* with size c. For any set $Z \subset \mathbb{R}^d$ of the class of cells $\{C : C \in \mathcal{D}(Z)\}$ not intersecting in pairs, for which it is fulfilled $Z = \bigcup_{C \in \mathcal{D}(Z)} C$ are called its *fragmentation* $\mathcal{D}(Z)$ by cells. The family of all such fragmentations of the set Z is denoted by $\mathfrak{D}(Z)$.

For each cell C, we introduce a partial ordering on the family $\mathfrak{D}(C)$, which we will call the *subordination relation*.

Definition 1. Fragmentation $\mathcal{D}_2(C)$ is called the subordinated one to fragmentation $\mathcal{D}_1(C)$ if for each cell $C_2 \in \mathcal{D}_2(C)$ in the class $\mathcal{D}_1(C)$ there is such a cell $C_1 \in \mathcal{D}_1(C)$ that the inclusion of $C_1 \supset C_2$ takes place.

Lemma 1. *If the fragmentation $\mathcal{D}_2(C)$ is subjected to fragmentation $\mathcal{D}_1(C)$, then for each cell $C' \in \mathcal{D}_1(C)$ there is a subclass in the fragmentation $\mathcal{D}_2(C)$ which is the fragmentation of the cell C'.*

Proof. Define the $\{C'' \in \mathcal{D}_2(C) : C'' \subset C'\}$. Obviously, it is nonempty due to its definition. Let us prove that

$$\bigcup_{C'' \in \mathcal{D}_2(C) : C'' \subseteq C'} C'' = C'. \tag{1}$$

From one side, according to the definition, $\bigcup_{C'' \in \mathcal{D}_2(C) : C'' \subseteq C'} C'' \subseteq C'$, but, from other side, any point

$$\mathbf{x} \in C' \setminus \left(\bigcup_{C'' \in \mathcal{D}_2(C) : C'' \subset C'} C'' \right)$$

must belong to a cell of fragmentation $\mathcal{D}_2(C)$ which is contained certainly in the union in left side of the formula (1). Therefore, it takes place the converse inclusion

$$\bigcup_{C'' \in \mathcal{D}_2(C) : C'' \subseteq C'} C'' \supseteq C'.$$

□

Thus, by virtue of the proved lemma, the *subordination relation* is transitive. It defines a partial ordering on the class $\mathfrak{D}(C)$, which is reflexive and antisymmetric. Therefore, the subordination permits to consider perfectly ordered sequences with components from the class $\mathfrak{D}(C)$.

Definition 2. The sequence $\langle \mathcal{D}^{(m)}(C); m \in \mathbb{N} \rangle \subset \mathfrak{D}(C)$ of fragmentations is called the comminution of the cell C if $\lim_{m\to\infty} c_m = 0$ where c_m is the size of cells in the fragmentation $\mathcal{D}^{(m)}(C)$.

The *cellular comminution* is the special case of comminutions of the space $\mathbb{R}^d$ in which all elements are constructed by a uniform way. Let us fix the point $\mathbf{z} \in \mathbb{R}^d$, the value $c > 0$ which is the size of the cell C of the *primary fragmentation* the space $\mathbb{R}^d$, the number $\nu \in \mathbb{N}, \nu \geq 2$ which is called the *fragmentation parameter*.

For each $m \in \mathbb{N}$ with fixed values $c > 0$, $\nu \geq 2$ and the point $\mathbf{z} \in \mathbb{R}^d$, we introduce the *cellular fragmentation $\mathcal{D}^{(m)}(\mathbb{R}^d)$ of the order m* of the space $\mathbb{R}^d$, which we will denote by the symbol $\mathcal{C}_m \equiv \mathcal{C}_m(c, \nu, \mathbf{z})$. It makes up a class of cells defined by the formula

$$\mathcal{C}_m(c,\nu,\mathbf{z}) = \left\{ C = \mathbf{z} + \bigotimes_{j=1}^{d} \left[\frac{cl_j}{\nu^m}, \frac{c(l_j+1)}{\nu^m} \right) ; l_k \in \mathbb{Z}, k \in I_d \right\}.$$

We denote by $\mathcal{C}(c, \nu, \mathbf{z})$ the union of cells of all fragmentations $\mathcal{C}_m$, $m \in \mathbb{N}$ with the fixed point $\mathbf{z} \in \mathbb{R}^d$ and fixed values of parameters $c \in (0, \infty), \nu \in \mathbb{N}, n \geq 2$.

Definition 3. The sequence $\mathfrak{C}(c, \nu, \mathbf{z}) = \langle \mathcal{C}_m(c, \nu, \mathbf{z});\, m \in \mathbb{N} \rangle$ is called the cellular comminution of the space $\mathbb{R}^d$.

3. BOOLEAN ALGEBRAIC SYSTEMS

The family $\mathfrak{S}$ of classes of subsets X in $\mathbb{R}^d$ with fixed properties with respect to Boolean algebraic operations will be called a Boolean algebraic system or, simply, a system. Elements of the system, which, in the future, are called classes of sets. They will be denote by handwritten letters. If the system $\mathfrak{S}$ contains such a single element Ω that, for any element $\mathcal{S} \in \mathfrak{S}$, the equalities $\Omega \cap \mathcal{S} = \mathcal{S}$, $\Omega \cup \mathcal{S} = \mathcal{S}$ are fulfilled, then this element Ω is the unit of this system. Further, we will denote the units of various systems $\mathfrak{S}$ by the symbol Ω with the addition of an index indicating its belonging to the system, that is $\Omega_{\mathfrak{S}}$.

Definition 4 (see, for example, [17]). Nonempty system $\mathfrak{R}$ is the Boolean semiring, if it contains empty set $\emptyset$, and it satisfies the following two conditions:

1. If elements $\mathcal{A}, \mathcal{B} \in \mathfrak{R}$, then $\mathcal{A} \cap \mathcal{B} \in \mathfrak{R}$.
2. If elements $\mathcal{A}, \mathcal{B} \in \mathfrak{R}$ and $\mathcal{A} \supset \mathcal{B}$, then there is a disjunctive decomposition

$$\mathcal{A} = \bigcup_{k=1}^{n} \mathcal{A}_k, \quad \mathcal{A}_k \in \mathfrak{R}, \quad k \in I_n = \{1, ..., n\},$$

such that the elements $\mathcal{A}_k$, $k \in I_n$ do not intersect in pairs and $\mathcal{A}_1 = \mathcal{B}$.

In future, speaking of a semiring, we will mean that it contains the unit $\Omega_{\mathfrak{R}}$.

Definition 5 (see, for example, [17]). The system $\mathfrak{A}$ with the unit $\Omega_{\mathfrak{A}}$ is called the Boolean algebra, if it contains elements $\mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_1 \cap \mathcal{S}_2$ and $\complement\mathcal{S}_1$, $\complement\mathcal{S}_2$ together with each pair of elements $\mathcal{S}_1$ and $\mathcal{S}_2$.

Definition 6. The system $\mathfrak{B}$ is called the σ-algebra, if it is the algebra and, together with each not contractible sequence $\langle \mathcal{A}_k; k \in \mathbb{N} \rangle$, $\mathcal{A}_k \subseteq \mathcal{A}_{k+1}$, $k \in \mathbb{N}$, it contains also the element which is its limit $\lim\limits_{k\to\infty} \mathcal{A}_k$ in the sense of set theory.

Definition 7. The system $\mathfrak{B}(\mathfrak{S})$ is called the minimal σ-algebra with respect to the system $\mathfrak{S}$, if it is the σ-algebra containing $\mathfrak{S}$ and it is contained in any σ-algebra containing $\mathfrak{S}$.

The unit of the minimal σ-algebra $\mathfrak{B}(\mathfrak{S})$ is the element $\Omega = \bigcup_{\mathcal{S}\in\mathfrak{S}} \mathcal{S}$. The following theorem plays important role in measure theory.

Theorem 1 (see, for example, [17]). *For any nonempty system $\mathfrak{S}$, there is the σ-algebra $\mathfrak{B}(\mathfrak{S})$ being the minimal one with respect to this system.*

Definition 8. Let $\mathfrak{B}$ be the σ-algebra. System $\mathfrak{S}$ of elements is called the generating one for it if $\mathfrak{B} = \mathfrak{B}(\mathfrak{S})$.

A σ-algebra $\mathfrak{B}$ is called the finitely generated if there exists a finite system $\mathfrak{S}$ generating it. A σ-algebra $\mathfrak{B}$ is called the countably generated if there exists a system $\mathfrak{S}$ generating it which has the cardinality $\aleph_0$ but there is not any finite system $\mathfrak{S}'$ generating $\mathfrak{B}$.

The importance of the concept of semiring is determined by the fact that any additive measure, which is given initially on a semiring $\mathfrak{S}$, has an unambiguous extension to the minimal σ-algebra containing $\mathfrak{S}$.

Definition 9. Let $\mathfrak{R}$ be a semiring. The nonnegative function $\mu(\cdot)$ defined on $\mathfrak{R}$ is named the additive measure (simply, in future, the measure), if

1. For any element $\mathcal{A} \in \mathfrak{R}$ it is fulfilled $0 \le \mu(\mathcal{A}) \le \infty$ and $\mu(\emptyset) = 0$;

2. For any collection $\{\mathcal{A}_j \in \mathfrak{R}; j \in I_n\}$ of elements from $\mathfrak{R}$ such that $\mathcal{A}_j \cap \mathcal{A}_k = \emptyset,\ \{j,k\} \subset I_n$, it is fulfilled the equality

$$\mu\left(\bigcup_{j=1}^{n} \mathcal{A}_j\right) = \sum_{j=1}^{n} \mu(\mathcal{A}_j).$$

Theorem 2 (see, for example, [17]). *Let the semiring $\mathfrak{R}$ is closed relative to countable intersections of classes from $\mathfrak{R}$. If the finite additive measure μ on $\mathfrak{R}$ satisfies to the condition σ-semiadditivity,*

$$\mu(\mathcal{A}) \le \sum_{k=1}^{\infty} \mu(\mathcal{A}'_k)$$

for any $\mathcal{A} \in \mathfrak{R}$ and for any countable family $\{\mathcal{A}'_k; k \in \mathbb{N}\}$ of elements of $\mathfrak{R}$ covering the class $\mathcal{A}$, $\mathcal{A} \subseteq \bigcup_{k=1}^{\infty} \mathcal{A}'_k$, then there is the only one σ-additive measure $\bar{\mu}$ on $\mathfrak{B}(\mathfrak{R})$, which satisfies

$$\bar{\mu}\left(\bigcup_{j=1}^{\infty} \mathcal{A}_j\right) = \sum_{j=1}^{\infty} \bar{\mu}(\mathcal{A}_j)$$

for any element $\mathcal{A} \in \mathfrak{B}(\mathfrak{R})$ and for any collection of elements $\mathcal{A}_j \in \mathfrak{B}(\mathfrak{R})$, $j \in \mathbb{N}$ such that

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{A}_k, \quad \mathcal{A}_j \cap \mathcal{A}_k = \emptyset, \quad \{j,k\} \subset \mathbb{N}.$$

For this function it takes place the equality $\mu(\mathcal{A}) = \bar{\mu}(\mathcal{A})$ for all elements $\mathcal{A} \in \mathfrak{R}$.

Here, it is not required that the element $\bigcup_{k=1}^{\infty} \mathcal{A}'_k$ belongs to $\mathfrak{R}$.

Thus, under the conditions of the Theorem 2, the measure is uniquely determined on all elements of the minimal σ-algebra. A system, each element of which can be associated with a certain measure value, can be significantly expanded to a more extensive system containing $\mathfrak{B}$ and contained in 2^{Ω}. This extension is based on the Lebesgue construction. Let $\mathfrak{B}$ be an σ-algebra of elements contained in Ω, on which σ-additive measure μ is defined. Define the functional μ^* on 2^{Ω} by the following formulas which is called the *external measure*. For the given $\mathcal{A}$ we put

$$\mu^*(\mathcal{A}) = \inf\{\mu(\mathcal{B}); \mathcal{B} \in \mathfrak{B}, \mathcal{A} \subset \mathcal{B}\}.$$

The element $\mathcal{A}$ is named the *measurable according to Lebesgue* one (or simply the *measurable* one), if

$$\inf\{\mu^*((\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A})); \mathcal{B} \in \mathfrak{B}\} = 0. \tag{2}$$

We denote the σ-algebra of all such elements by the symbol $\mathfrak{M}(\mathfrak{B}, \mu)$. Continuation of measure on the σ-algebra $\mathfrak{M}(\mathfrak{B}, \mu)$ is defined by the formula $\mu(A) = \mu^*(A)$. It is named the *Lebesgue measure*.

It is evident that $\mathfrak{B} \subset \mathfrak{M}(\mathfrak{B}, \mu)$, since elements of $\mathfrak{B}$ is measurable automatically. But, the σ-algebra $\mathfrak{M}(\mathfrak{B}, \mu)$ is much more wider than $\mathfrak{B}$ if Ω has the cardinality $\aleph_1$. Then, if there is the element $\mathcal{A} \in \mathfrak{B}$ in the σ-algebra $\mathfrak{B}$, which has the cardinality $\aleph_1$ and the null Lebesgue measure, then the σ-algebra $\mathfrak{M}(\mathfrak{B}, \mu)$ has the cardinality $\aleph_2$.

4. BOOLEAN c-SYSTEMS

The construction of the measurability structure and the definition of a σ-additive measure is usually carried out on the basis of the concept of a monotone class (see, for example, [15]). However, some simpler Boolean systems such as the Boolean semiring and the Dynkin systems [16] are used for such a construction. In this paper, in order to construct the structure of measurability in the space $\langle\Omega,\mathfrak{B},\mathsf{P}\rangle$ of random sets in most general case, we introduce the abstract concept of the Boolean c-system on the basis of which the measure P is constructed.

Definition 10. We name the Boolean system $\mathfrak{F}$ of elements of the set 2^Ω, which has the unity Ω, the c-system, if it contains elements Ω, $\emptyset$ and satisfies the following conditions:

(a) if elements $\mathcal{A}$ and $\mathcal{B}$ belong to the system $\mathfrak{F}$, then there is a finite disjunctive collection $\mathfrak{F}(\mathcal{A},\mathcal{B})\subset\mathfrak{F}$ of nonempty elements of $\mathfrak{F}$ such that the element $\mathcal{A}\cap\mathcal{B}$ is represented in the form of decomposition

$$\mathcal{A}\cap\mathcal{B}=\bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{A},\mathcal{B})}\mathcal{F},\quad \mathfrak{F}(\mathcal{A},\mathcal{A})=\{\mathcal{A}\};\tag{3}$$

(b) if $\mathcal{A}\in\mathfrak{F}$, then there is a finite disjunctive collection $\mathfrak{F}(\mathcal{A})\subset\mathfrak{F}$ of nonempty elements of $\mathfrak{F}$ such that $\complement\mathcal{A}=\bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{A})}\mathcal{F}$.

Let us prove the following statement which follows directly from the item (b) of the definition 10.

Lemma 2. *Let $\mathfrak{H}=\langle\mathcal{Q}_1,\dots,\mathcal{Q}_n\rangle$ be an arbitrary finite collection of elements of the c-system $\mathfrak{F}$. Then, the following disjunctive decomposition for their joint intersection takes place*

$$\bigcap_{j=1}^{n}\mathcal{Q}_j=\bigcup_{\mathcal{F}\in\mathfrak{F}_*(\mathfrak{H})}\mathcal{F},\tag{4}$$

where the family $\mathfrak{F}_*(\mathfrak{H})$ is defined by recurrence on $p=Card\,\mathfrak{H}$: $\mathfrak{F}_*(\mathfrak{H})=\mathfrak{H}$ at $p=1$, $\mathfrak{F}_*(\mathfrak{H})=\mathfrak{F}(\mathcal{Q}_1,\mathcal{Q}_2)$ at $p=2$ and

$$\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p,\mathcal{Q}_{p+1})=\bigcup_{\mathcal{F}\in\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p)}\mathfrak{F}(\mathcal{F},\mathcal{Q}_{p+1}),\quad p=2\div n-1.$$

Proof. The proof is carried out by induction on p. Describe the induction step. Supposing that the formula (31) is true at $n=p$, in the case $n=p+1$, the intersection is transformed, using the equality (3),

$$\bigcap_{j=1}^{p+1}\mathcal{Q}_j=\left(\bigcup_{\mathcal{F}\in\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p)}\mathcal{F}\right)\cap\mathcal{Q}_{p+1}=\bigcup_{\mathcal{F}\in\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p)}(\mathcal{F}\cap\mathcal{Q}_{p+1})=\bigcup_{\mathcal{F}\in\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p)}\ \bigcup_{\mathcal{F}'\in\mathfrak{F}(\mathcal{F},\mathcal{Q}_{p+1})}\mathcal{F}'$$
$$=\bigcup_{\mathcal{F}'\in\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p,\mathcal{Q}_{p+1})}\mathcal{F}'.$$

Since the family $\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p)$ consists of elements not intersecting in pairs, then the elements $\mathcal{F}'$ belonging to families $\mathfrak{F}(\mathcal{F}_1,\mathcal{Q}_{p+1})$, $\mathfrak{F}(\mathcal{F}_2,\mathcal{Q}_{p+1})$ generated by different elements $\mathcal{F}_1$ и $\mathcal{F}_2$ of $\mathfrak{F}_*(\mathcal{Q}_1,\dots,\mathcal{Q}_p)$ also not intersecting in pairs. $\square$

It is evident that any Boolean semiring $\mathfrak{F}$ with the unity Ω is the c-system. The basic property of any c-system is expressed by the following statement.

Theorem 3. *Let $\mathfrak{F}$ be a c-system and $\mathfrak{F}_0$ be a family of finite disjunctive collections of elements of $\mathfrak{F}$. Then, the family $\mathfrak{A}(\mathfrak{F})$ defined by the formula*

$$\mathfrak{A}(\mathfrak{F})=\left\{\bigcup_{\mathcal{F}\in\mathfrak{G}}\mathcal{F};\mathfrak{G}\in\mathfrak{F}_0\right\}$$

is the Boolean algebra. This algebra is minimal that contains the system $\mathfrak{F}$.

Proof. It is necessary to verify the closedness of the system $\mathfrak{A}(\mathfrak{F})$ relative to operations $\cup$, $\cap$, $\complement$. Let $\mathcal{A}, \mathcal{B}$ be arbitrary elements of $\mathfrak{A}(\mathfrak{F})$. Then, according to th definition, they are represented in the form

$$\mathcal{A} = \bigcup_{\mathcal{G}\in\mathfrak{G}} \mathcal{G}, \quad \mathcal{B} = \bigcup_{\mathcal{H}\in\mathfrak{H}} \mathcal{H},$$

where $\mathfrak{G}$ и $\mathfrak{H}$ are finite disjunctive collections of $\mathfrak{F}_0$. Therefore, the decomposition

$$\mathcal{A}\cap\mathcal{B} = \bigcup_{\mathcal{G}\in\mathfrak{G}}\bigcup_{\mathcal{H}\in\mathfrak{H}} (\mathcal{G}\cap\mathcal{H}) \tag{5}$$

takes place which is disjunctive since for arbitrary not identical pairs $\langle\mathcal{G},\mathcal{H}\rangle$ and $\langle\mathcal{G}',\mathcal{H}'\rangle$, $\mathcal{G},\mathcal{G}'\in\mathfrak{G}$, $\mathcal{H},\mathcal{H}'\in\mathfrak{H}$ the equality $(\mathcal{G}\cap\mathcal{H})\cap(\mathcal{G}'\cap\mathcal{H}') = (\mathcal{G}\cap\mathcal{G}')\cap(\mathcal{H}\cap\mathcal{H}') = \emptyset$ takes place, due to the fact that either $\mathcal{G}$ does not coincide with $\mathcal{G}'$ and, therefore, $\mathcal{G}\cap\mathcal{G}' = \emptyset$, or $\mathcal{H}$ does not coincide with $\mathcal{H}'$ and, therefore, $\mathcal{H}\cap\mathcal{H}' = \emptyset$. On the basis of the peoperty (a) in Definition 10, for any $\mathcal{G}$ and $\mathcal{H}$, there is a finite collection $\mathfrak{F}(\mathcal{G},\mathcal{H})\in\mathfrak{F}_0$ such that the disjunctive decomposition

$$\mathcal{G}\cap\mathcal{H} = \bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{G},\mathcal{H})} \mathcal{F} \tag{6}$$

takes place.

From (5) and (6), we obtain the decomposition

$$\mathcal{A}\cap\mathcal{B} = \bigcup_{\mathcal{G}\in\mathfrak{G}}\bigcup_{\mathcal{H}\in\mathfrak{H}}\bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{G},\mathcal{H})} \mathcal{F},$$

in which all elements are not intersected in pairs either due to the fact that they enter in the same disjunctive collection $\mathfrak{F}(\mathcal{G},\mathcal{H})$ or due to the fact that they enter in different collections $\mathfrak{F}(\mathcal{G},\mathcal{H})$, $\mathfrak{F}(\mathcal{G}',\mathcal{H}')$. Then, introducing the collection

$$\mathfrak{F}(\mathcal{A},\mathcal{B}) \equiv \bigcup_{\mathcal{G}\in\mathfrak{G}}\bigcup_{\mathcal{H}\in\mathfrak{H}} \mathfrak{F}(\mathcal{G},\mathcal{H})$$

of elements for arbitrary $\mathcal{A},\mathcal{B}\in\mathfrak{A}(\mathfrak{F})$, the decomposition (6) is represented in the form $\mathcal{A}\cap\mathcal{B} = \bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{A},\mathcal{B})}\mathcal{F}$ and, consequently, $\mathcal{A}\cap\mathcal{B}\in\mathfrak{A}(\mathfrak{F})$.

Let $\mathcal{A}\in\mathfrak{A}(\mathfrak{F})$. Then, on the basis of the duality law, for the complement of the element

$$\mathcal{A} = \bigcup_{\mathcal{G}\in\mathfrak{G}} \mathcal{G}, \quad \mathfrak{G}\in\mathfrak{F}_0, \quad \complement\mathcal{G} = \bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{G})} \mathcal{F},$$

we obtain

$$\complement\mathcal{A} = \complement\left(\bigcup_{\mathcal{G}\in\mathfrak{G}}\mathcal{G}\right) = \bigcap_{\mathcal{G}\in\mathfrak{G}}\complement\mathcal{G} = \bigcap_{\mathcal{G}\in\mathfrak{G}}\bigcup_{\mathcal{F}\in\mathfrak{F}(\mathcal{G})}\mathcal{F} = \bigcup_{\mathfrak{H}\in\mathfrak{H}_*(\mathcal{A})}\left(\bigcap_{\mathcal{F}\in\mathfrak{H}}\mathcal{F}\right), \tag{7}$$

where the union is carried out on the system $\mathfrak{H}_*(\mathcal{A})$ of finite collections $\mathfrak{H}$ with the length equal to Card $\mathfrak{G}$, which represents cartesian product $\mathfrak{H}_*(\mathcal{A}) = \bigotimes_{\mathcal{G}\in\mathfrak{G}}\mathfrak{F}(\mathcal{G})$. The system of elements represented by these intersections is disjunctive, since for any pair of different collections $\mathfrak{H}$ and $\mathfrak{H}'$, they will have a mismatched component $\mathcal{F}\neq\mathcal{F}'$ and, therefore,

$$\left(\bigcap_{\mathcal{F}\in\mathfrak{H}}\mathcal{F}\right)\cap\left(\bigcap_{\mathcal{F}'\in\mathfrak{H}'}\mathcal{F}'\right) = \bigcap_{\mathcal{F}\in\mathfrak{H}}\bigcap_{\mathcal{F}'\in\mathfrak{H}'}(\mathcal{F}\cap\mathcal{F}') = \emptyset.$$

The last is true due to the fact that all components $\mathcal{F}$ are chosen from the same system $\mathfrak{F}(\mathcal{G})$ being disjunctive according to its definition.

Based on (31), for the intersection of all elements $\mathcal{F}$ of any finite collection $\mathfrak{H}$, a disjunctive decomposition takes place

$$\bigcap_{\mathcal{F}\in\mathfrak{H}}\mathcal{F} = \bigcup_{\mathcal{F}'\in\mathfrak{F}_*(\mathfrak{H})}\mathcal{F}', \quad \mathfrak{H}\in\mathfrak{H}_*(\mathcal{A}),$$

where the union is realized on the finite family $\mathfrak{F}_*(\mathfrak{H})$ defined by induction.

Substituting this decomposition into (7), we obtain the disjunctive decomposition of the $\complement\mathcal{A}$ on elements of the system $\mathfrak{F}$,

$$\complement\mathcal{A} = \bigcup_{\mathfrak{H}\in\mathfrak{H}_*(\mathcal{A})} \bigcup_{\mathcal{F}'\in\mathfrak{F}_*(\mathfrak{H})} \mathcal{F}' = \bigcup_{\mathcal{F}'\in\mathfrak{F}(\mathcal{A})} \mathcal{F}', \quad \mathfrak{F}(\mathcal{A}) \equiv \bigcup_{\mathfrak{H}\in\mathfrak{H}_*(\mathcal{A})} \mathfrak{F}_*(\mathfrak{H}).$$

In each family $\mathfrak{F}_*(\mathfrak{H})$, each pair of elements, according to its construction, is disjunctive. Besides, each pair of elements is disjunctive, if the elements belong to different families $\mathfrak{F}_*(\mathfrak{H}_1)$ and $\mathfrak{F}_*(\mathfrak{H}_2)$, due to disjunctivity of elements $\bigcap_{\mathcal{F}\in\mathfrak{H}_1}\mathcal{F}$ and $\bigcap_{\mathcal{F}\in\mathfrak{H}_2}\mathcal{F}$ generating them. Then, the family $\mathfrak{F}(\mathcal{A})$ which is the union of all families $\mathfrak{F}_*(\mathfrak{H})$, $\mathfrak{H}\in\mathfrak{H}_*(\mathcal{A})$ consists of elements being disjunctive in pairs. Thus, we have $\complement A \in \mathcal{A}(\mathcal{F})$.

At last, we have $\mathcal{A}\cup\mathcal{B}\in\mathfrak{A}(\mathfrak{F})$ for arbitrary elements $\mathcal{A}\in\mathfrak{A}(\mathfrak{F})$, $\mathcal{B}\in\mathfrak{A}(\mathfrak{F})$. It follows on the basis of representation $\mathcal{A}\cup\mathcal{B} = \complement(\complement\mathcal{A}\cap\complement\mathcal{B})$ and on the above-proved closedness of the system $\mathfrak{A}(\mathfrak{F})$ relative to operations $\cap$ and $\complement$.

Since any algebra $\mathfrak{A}$ that contains a c-system $\mathfrak{F}$, by definition, must contain every element $\bigcup_{\mathcal{G}\in\mathfrak{G}}\mathcal{G}$ which has been built on basis of any system $\mathfrak{G}\in\mathfrak{F}_0$, then $\mathfrak{A}(\mathfrak{F})\subseteq\mathfrak{A}$. From here, it follows the minimality of the algebra $\mathfrak{A}(\mathfrak{F})$ among all algebras containing $\mathfrak{F}$. □

Despite the fact that the minimal algebra containing the system $\mathfrak{F}$ consists of all possible finite disjunctive unions of elements from $\mathfrak{F}$, the minimal σ-algebra containing $\mathfrak{F}$ is not reduced to a system of elements that are arbitrary finite or countable disjunctive unions of elements of $\mathfrak{F}$.

5. MEASURE CONTINUATION ON THE MINIMAL σ-ALGEBRA

Explicit building of the minimal algebra $\mathfrak{A}(\mathfrak{F})$ that contains the given c-system $\mathfrak{F}$ permits to prove unambiguity of continuation of additive measure on $\mathfrak{A}(\mathfrak{F})$ which has been defined on $\mathfrak{F}$.

Theorem 4. *If an additive bounded function $\mu(\cdot)$ have been defined on the c-system $\mathfrak{F}$ such that for each finite disjunctive decomposition of the element $\mathcal{A}\in\mathfrak{F}$,*

$$\mathcal{A} = \bigcup_{\mathcal{F}\in\mathfrak{G}} \mathcal{F}, \quad \mathfrak{G}\subset\mathfrak{F}, \tag{8}$$

it is fulfilled

$$\mu(\mathcal{A}) = \sum_{\mathcal{F}\in\mathfrak{G}} \mu(\mathcal{F}), \tag{9}$$

then this function is continued by unambiguous way to the additive function $\bar{\mu}(\cdot)$ defined on the minimal algebra $\mathfrak{A}(\mathfrak{F})$ that contains the c-system $\mathfrak{F}$.

Proof. Let an additive function $\mu(\cdot)$ be defined on the c-system $\mathfrak{F}$. Verification of unambiguity of continuation according to additivity is fulfilled by the same way, as in the case of the continuation of measure from Boolean semiring (see, for example, [17]). Let be an element $\mathcal{A}\in\mathfrak{A}(\mathfrak{F})$, $\mathcal{A}\notin\mathfrak{F}$. This element we represent in the form of the disjunctive decomposition (8) with elements of $\mathfrak{F}$. On the basis of each such an decomposition, we define the values $\bar{\mu}_{\mathfrak{G}}(\mathcal{A}) = \sum_{\mathcal{F}\in\mathfrak{G}}\mu(\mathcal{F})$. We will show now that all such values coincide, that is they do not depend on the choice of $\mathfrak{G}$. Let two elements $\mathfrak{G}_j\subset\mathfrak{F}_0$, $j=1,2$ be such that the following decompositions take place

$$\mathcal{A} = \bigcup_{\mathcal{G}\in\mathfrak{G}_j} \mathcal{G}, \quad j=1,2. \tag{10}$$

They define the values $\bar{\mu}_j(\mathcal{A}) \equiv \bar{\mu}_{\mathfrak{G}_j}(\mathcal{A})$,

$$\bar{\mu}_j(\mathcal{A}) = \sum_{\mathcal{G}\in\mathfrak{G}_j} \mu(\mathcal{G}), \quad j=1,2.$$

On the basis of the decompositions (10), we obtain the following finite disjunctive decompositions for each element $\mathcal{G}_1 \in \mathfrak{G}_1$,

$$\mathcal{G}_1 = \mathcal{G}_1 \cap \mathcal{A} = \mathcal{G}_1 \cap \bigcup_{\mathcal{G}_2 \in \mathfrak{G}_2} \mathcal{G}_2 = \bigcup_{\mathcal{G}_2 \in \mathfrak{G}_2} (\mathcal{G}_1 \cap \mathcal{G}_2) = \bigcup_{\mathcal{G}_2 \in \mathfrak{G}_2} \bigcup_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}_1, \mathcal{G}_2)} \mathcal{F} \tag{11}$$

and for each element $\mathcal{G}_2 \in \mathfrak{G}_2$,

$$\mathcal{G}_2 = \mathcal{G}_2 \cap \mathcal{A} = \mathcal{G}_2 \cap \bigcup_{\mathcal{G}_1 \in \mathfrak{G}_1} \mathcal{G}_1 = \bigcup_{\mathcal{G}_1 \in \mathfrak{G}_1} (\mathcal{G}_2 \cap \mathcal{G}_1) = \bigcup_{\mathcal{G}_1 \in \mathfrak{G}_1} \bigcup_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}_2, \mathcal{G}_1)} \mathcal{F}. \tag{12}$$

The disjunctivity of decompositions (11), (12) follows, firstly, from the fact that the decompositions (10) are disjunctive and, consequently, the decompositions $\bigcup_{\mathcal{G}_1 \in \mathfrak{G}_1} (\mathcal{G}_2 \cap \mathcal{G}_1)$ and $\bigcup_{\mathcal{G}_2 \in \mathfrak{G}_2} (\mathcal{G}_1 \cap \mathcal{G}_2)$ are also disjunctive. Secondly, the disjunctivity of decompositions (11), (12) follows from disjunctivity of decompositions

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \bigcup_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}_1, \mathcal{G}_2)} \mathcal{F}, \quad \mathcal{G}_1 \in \mathfrak{G}_1, \mathcal{G}_2 \in \mathfrak{G}_2,$$

according to the definition of c-system. Since $\mathcal{G}_1$ и $\mathcal{G}_2$ belong to $\mathfrak{F}$, then, on the basis (9), we have

$$\mu(\mathcal{G}_j) = \mu\left(\bigcup_{\mathcal{G}_j \in \mathfrak{G}_j} \bigcup_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}_1, \mathcal{G}_2)} \mathcal{F} \right) = \sum_{\mathcal{G}_j \in \mathfrak{G}_j} \sum_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}_1, \mathcal{G}_2)} \mu(\mathcal{F}), \quad j = 1, 2.$$

From here, it follows that

$$\bar{\mu}_j(\mathcal{A}) = \sum_{\mathcal{G}_j \in \mathfrak{G}_j} \mu(\mathcal{G}_j) = \sum_{\mathcal{G}_1 \in \mathfrak{G}_1} \sum_{\mathcal{G}_2 \in \mathfrak{G}_2} \sum_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}_1, \mathcal{G}_2)} \mu(\mathcal{F}),$$

where $\mathfrak{F}(\mathcal{G}_1, \mathcal{G}_2) = \mathfrak{F}(\mathcal{G}_2, \mathcal{G}_1)$. Consequently, $\bar{\mu}_1(\mathcal{A}) = \bar{\mu}_2(\mathcal{A})$. Due to coincidence of all values $\bar{\mu}_{\mathfrak{G}}(\mathcal{A})$, for any element $\mathcal{A} \in \mathfrak{A}(\mathfrak{F})$, the additive function $\bar{\mu}(\cdot)$ on $\mathfrak{A}(\mathfrak{F})$, which coincides with fixed function $\mu(\cdot)$ at its narrowing on $\mathfrak{F}$, is unique. □

Since the algebra $\mathfrak{A}(\mathfrak{F})$ is the Boolean semiring, then, on the basis of the proved theorem and the theorem about the continuation of measure from semiring on its containing the minimal σ-algebra $\mathfrak{B}(\mathfrak{F})$ [17], we may assert that it is true

Theorem 5. *Let a finite additive measure μ on the c-system $\mathfrak{F}$ satisfies the condition of σ-semiadditivity, $\mu(\mathcal{G}) \le \sum_{k=1}^{\infty} \mu(\mathcal{G}_k)$ for any element $\mathcal{G} \in \mathfrak{F}$ and for countable collection $\{\mathcal{G}_k; k \in \mathbb{N}\}$ of elements in $\mathfrak{F}$, which covers the element $\mathcal{G}$, $\mathcal{G} \subseteq \bigcup_{k=1}^{\infty} \mathcal{G}_k$. Then, there is the unique σ-additive measure $\bar{\mu}(\cdot)$ on the minimal σ-algebra $\mathfrak{B}(\mathfrak{F})$ which satisfies the condition*

$$\bar{\mu}\left(\bigcup_{j=1}^{\infty} \mathcal{A}_j \right) = \sum_{j=1}^{\infty} \bar{\mu}(A_j),$$

for any disjunctive collection $\mathcal{A}_j \in \mathfrak{B}(\mathfrak{F})$, $j \in \mathbb{N}$, $\mathcal{A}_k \cap \mathcal{A}_l = \emptyset, l \neq k, \{k, l\} \in \mathbb{N}$ and for which it takes place the equality $\mu(\mathcal{A}) = \bar{\mu}(\mathcal{A})$ for all elements $\mathcal{A} \in \mathfrak{F}$.

Proof. Let the measure $\mu(\cdot)$ on the algebra $\mathfrak{A}(\mathfrak{F})$ be defined according to the formulas (8), (9). It is sufficient to prove the availability of semiadditivity of this measure on the algebra $\mathfrak{A}(\mathfrak{F})$ [17]. If $\mathcal{A} \in \mathfrak{A}(\mathfrak{F})$ and the following inclusion takes place $\mathcal{A} \subseteq \bigcup_{k=1}^{\infty} \mathcal{A}_k$, where $\mathcal{A}_k \in \mathfrak{A}(\mathfrak{F})$, $k \in \mathbb{N}$, then there are some collections $\mathfrak{G} \in \mathfrak{F}_0$ and $\mathfrak{G}_k \in \mathfrak{F}_0$, $k \in \mathbb{N}$ of elements of $\mathfrak{F}$ such that

$$\mathcal{A} = \bigcup_{\mathcal{G} \in \mathfrak{G}} \mathcal{G}, \quad \mathcal{A}_k = \bigcup_{\mathcal{G}' \in \mathfrak{G}_k} \mathcal{G}', \ k \in \mathbb{N}.$$

Consequently, for each element $\mathcal{G} \in \mathfrak{G}$, it is true

$$\mathcal{G} \subseteq \mathcal{G} \cap \left(\bigcup_{k=1}^{\infty} \mathcal{A}_k \right) = \bigcup_{k=1}^{\infty} (\mathcal{G} \cap \mathcal{A}_k) = \bigcup_{k=1}^{\infty} \bigcup_{\mathcal{G}' \in \mathfrak{G}_k} (\mathcal{G} \cap \mathcal{G}') = \bigcup_{k=1}^{\infty} \bigcup_{\mathcal{G}' \in \mathfrak{G}_k} \bigcup_{\mathcal{F} \in \mathfrak{F}(\mathcal{G}, \mathcal{G}')} \mathcal{F}.$$

Since the family $\bigcup_{k=1}^{\infty}\bigcup_{\mathcal{G}'\in\mathfrak{G}_k}\mathfrak{F}(\mathcal{G},\mathcal{G}')$ of elements in $\mathfrak{F}$ forms the coverage of the set $\mathcal{G}$ and the property of σ-semiadditivity is fulfilled for each element of $\mathfrak{F}$, then, on the basis of above pointed inclusions, we obtain the following inequality

$$\mu(\mathcal{G}) \le \sum_{k=1}^{\infty}\sum_{\mathcal{G}'\in\mathfrak{G}_k}\sum_{\mathcal{F}\in\mathfrak{F}(\mathcal{G},\mathcal{G}')}\mu(\mathcal{F}) = \sum_{k=1}^{\infty}\bar{\mu}(\mathcal{G}\cap\mathcal{A}_k)$$

for each element $\mathcal{G}\in\mathfrak{G}$. Summing up these inequalities on $\mathcal{G}\in\mathfrak{G}$ and using the fact that

$$\bar{\mu}(\mathcal{A}) = \sum_{\mathcal{G}\in\mathfrak{G}}\mu(\mathcal{G}), \qquad \sum_{\mathcal{G}\in\mathfrak{G}}\bar{\mu}(\mathcal{G}\cap\mathcal{A}_k) = \bar{\mu}(\mathcal{A}\cap\mathcal{A}_k),$$

we find that it takes place the inequality

$$\bar{\mu}(\mathcal{A}) \le \sum_{\mathcal{G}\in\mathfrak{G}}\sum_{k=1}^{\infty}\bar{\mu}(\mathcal{G}\cap\mathcal{A}_k) = \sum_{k=1}^{\infty}\sum_{\mathcal{G}\in\mathfrak{G}}\bar{\mu}(\mathcal{G}\cap\mathcal{A}_k) = \sum_{k=1}^{\infty}\bar{\mu}(\mathcal{A}\cap\mathcal{A}_k) \le \sum_{k=1}^{\infty}\bar{\mu}(\mathcal{A}_k).$$

□

6. CYLINDRICAL RANDOM EVENTS

Let us described the special Boolean system. Its elements represent some classes of subsets of $\mathbb{R}^d$. It will be the generating system for the σ-algebra $\mathfrak{B}$ in probabilistic space $\langle\Omega,\mathfrak{B},\mathsf{P}\rangle$ of random sets.

Let $\mathbf{C} = \langle C_1,...,C_n\rangle$, $n\in\mathbb{N}$ be a finite ordered collection of intersecting cells of fragmentation $\mathcal{C}_m(c,\nu,\mathbf{z})$ of the space $\mathbb{R}^d$. We will denote by $\mathfrak{C}_n$ each family of collections with fixed length $n\in\mathbb{N}$. Along with ordered collections $\mathbf{C}\in\mathfrak{C}_n$, we enter into consideration the ordered collections $\boldsymbol{\theta} = \langle\theta_1,...,\theta_n\rangle$, $\theta_j\in\{0,1\}$, $j\in I_n$, $n\in\mathbb{N}$ with the same length.

Each pair $\langle\mathbf{C},\boldsymbol{\theta}\rangle$ such that $\mathrm{Card}(\mathbf{C}) = \mathrm{Card}(\boldsymbol{\theta})$ one may consider as unambiguous mapping $\mathsf{V}: \mathfrak{C}_n\mapsto\{0,1\}^n$, $n\in\mathbb{N}$. This mapping we will name the *indicator* one. If the collection $\boldsymbol{\theta}$ is the image of the collection $\mathbf{C}\in\mathfrak{C}_n$, then we will write $\mathsf{V}(\mathbf{C}) = \boldsymbol{\theta}$. At this, it is supposed that each component θ of the collection $\boldsymbol{\theta}$ is obtained by application of the mapping V to the correspondent component C of the collection $\mathbf{C}$, that is $\theta = \mathsf{V}(C)$. We will denote by the symbol $\mathfrak{C}$ the family of all pairs $\{\langle\mathbf{C},\boldsymbol{\theta}\rangle : \mathrm{Card}(\mathbf{C}) = \mathrm{Card}(\boldsymbol{\theta})\}$.

We denote the indicator function of the subset $X\subset\mathbb{R}^d$ by $\chi(X|\cdot)$. It is defined on the class of all cells $C\in\mathcal{C}(c,\nu,\mathbf{z})$. The values of this function are defined by the *fill numbers* (the indicators)

$$\chi(X|C) = \begin{cases} 1, & C\cap X\ne\emptyset, \\ 0, & C\cap X=\emptyset. \end{cases}$$

Definition 11. The elements $\Lambda(\mathbf{C},\boldsymbol{\theta})$ of the family $\mathfrak{F} = \{\Lambda(\mathbf{C},\boldsymbol{\theta}); \langle\mathbf{C},\boldsymbol{\theta}\rangle\in\mathfrak{C}\}$ of classes of random realizations X in Ω defined by the formula

$$\Lambda(\mathbf{C},\boldsymbol{\theta}) = \{X:\, \mathsf{V}(C) = \chi(X|C),\, C\in\mathbf{C}\}, \quad \boldsymbol{\theta} = \mathsf{V}(\mathbf{C}),$$

are named the cylindrical random events.

It is evident that, according to given definition, it is fulfilled $\Lambda(\mathbf{C},\boldsymbol{\theta}) = \emptyset$ in that and only in that case when $\mathbf{C}=\emptyset$. We notice some simplest properties of cylindrical events. At first, for each pair, $\langle\mathbf{C},\boldsymbol{\theta}\rangle\in\mathfrak{C}$ the following formula is fulfilled

$$\Lambda(\mathbf{C},\boldsymbol{\theta}) = \bigcap_{C\in\mathbf{C}}\Lambda(C,\theta), \tag{13}$$

where $\theta = \mathsf{V}(C)$, $C\in\mathbf{C}$. In particular, if the cell $C\in\mathcal{C}_m(c,\nu,\mathbf{z})$ is represented in the form of a disjunctive collection $\mathbf{C} = \mathcal{D}^{(m')}(C)$ of cells of $\mathcal{C}_{m'}(c,\nu,\mathbf{z})$, $m'>m$, then

$$\Lambda(C,0) = \bigcap_{C'\in\mathcal{D}^{(m')}(C)}\Lambda(C',0). \tag{14}$$

Secondly, it follows directly from (2) that the events $\Lambda(C,\theta)$ and $\Lambda(C,\theta')$ are not intersected for any cell C at $\theta \neq \theta'$,

$$\Lambda(C,\theta) \cap \Lambda(C,\theta') = \emptyset, \tag{15}$$

and, thus, the disjunctive decomposition

$$\Lambda(C,0) \cup \Lambda(C,1) = \Omega \tag{16}$$

takes place.

Let the collections $\mathbf{C}'$ and $\mathbf{C}$ be such that $\mathbf{C}' \supset \mathbf{C}$ and their cells belong to the same family $\mathcal{C}_m(c,\nu,\mathbf{z})$. Let us introduce the mapping $\mathsf{R}_{\mathbf{C}}(\cdot)$ defined on the class of collections $\boldsymbol{\theta}' = \mathsf{V}(\mathbf{C}')$. It projects such collections on the collection $\boldsymbol{\theta} = \mathsf{V}(\mathbf{C})$ that is $\mathsf{R}_{\mathbf{C}}(\theta') = \theta'$ if $C' \in \mathbf{C}$ and $\mathsf{R}_{\mathbf{C}}(\theta') = \emptyset$ in opposite case. From formulas (13), (15), (16) it follows directly

Theorem 6. *For fixed collection* $\mathbf{C}$*, the events* $\Lambda(\mathbf{C},\boldsymbol{\theta})$*,* $\Lambda(\mathbf{C},\boldsymbol{\theta}')$ *generated by different ordered collections* $\boldsymbol{\theta}, \boldsymbol{\theta}'$ *are not intersected. Besides, the following disjunctive decomposition takes place for any event* $\Lambda(\mathbf{C},\boldsymbol{\theta})$*, which have cells of the collection* $\mathbf{C}$ *belonging to* $\mathcal{C}_m(c,\nu,\mathbf{z})$ *and for any collection* $\mathbf{C}' \supset \mathbf{C}$ *such that its cells also belong to* $\mathcal{C}_m(c,\nu,\mathbf{z})$,

$$\Lambda(\mathbf{C},\boldsymbol{\theta}) = \bigcup_{\boldsymbol{\theta}' \in \{0,1\}^{n'}\,:\,\mathsf{R}_{\mathbf{C}}(\boldsymbol{\theta}')=\boldsymbol{\theta}} \Lambda(\mathbf{C}',\boldsymbol{\theta}'), \tag{17}$$

$n' = \mathrm{Card}(\mathbf{C}')$*, and, in particular, it is true the disjunctive decomposition*

$$\Omega = \bigcup_{\boldsymbol{\theta} \in \{0,1\}^n} \Lambda(\mathbf{C},\boldsymbol{\theta}), \quad n = \mathrm{Card}(\mathbf{C}). \tag{18}$$

Proof. For two different functions $\boldsymbol{\theta} = \mathsf{V}(\mathbf{C})$, $\boldsymbol{\theta}' = \mathsf{V}'(\mathbf{C})$, using the formula (13), we obtain

$$\begin{aligned}\Lambda(\mathbf{C},\boldsymbol{\theta}) \cap \Lambda(\mathbf{C},\boldsymbol{\theta}') &= \left(\bigcap_{C \in \mathbf{C}} \Lambda(C,\theta)\right) \cap \left(\bigcap_{C' \in \mathbf{C}} \Lambda(C',\theta')\right) \\ &= \bigcap_{C \in \mathbf{C}} \bigcap_{C' \in \mathbf{C}} \big(\Lambda(C,\mathsf{V}(C)) \cap \Lambda(C',\mathsf{V}'(C'))\big) = \emptyset,\end{aligned}$$

since, due to the discrepancy of the functions $\mathsf{V}(\cdot)$ and $\mathsf{V}'(\cdot)$, there is at least one cell $C \in \mathbf{C}$ such that in components of the intersection with $C' = C$ it is fulfilled $\mathsf{V}(C) \neq \mathsf{V}'(C)$ and, consequently, the formula (15) may be applied. Similarly, applying the formula (13) to each event in the right-hand side of the formula (17), and after that, using the decomposition (16), we obtain

$$\begin{aligned}\bigcup_{\{\boldsymbol{\theta}' \in \{0,1\}^{n'}\,:\,\mathsf{R}(\boldsymbol{\theta}')=\boldsymbol{\theta}\}} \Lambda(\mathbf{C}',\boldsymbol{\theta}') &= \bigcup_{\{\boldsymbol{\theta}' \in \{0,1\}^{n'}\,:\,\mathsf{R}(\boldsymbol{\theta}')=\boldsymbol{\theta}\}} \bigcap_{C' \in \mathbf{C}'} \Lambda(C',\theta') \\ &= \bigcup_{\{\boldsymbol{\theta}' \in \{0,1\}^{n'}\,:\,\mathsf{R}(\boldsymbol{\theta}')=\boldsymbol{\theta}\}} \left[\left(\bigcap_{C \in \mathbf{C}} \Lambda(C,\theta)\right) \cap \left(\bigcap_{C' \in \mathbf{C}' \setminus \mathbf{C}} \Lambda(C',\theta')\right)\right] \\ = \bigcup_{\boldsymbol{\theta}' \in \{0,1\}^{n'-n}} \left[\Lambda(\mathbf{C},\boldsymbol{\theta}) \cap \left(\bigcap_{C' \in \mathbf{C}' \setminus \mathbf{C}} \Lambda(C',\theta')\right)\right] &= \Lambda(\mathbf{C},\boldsymbol{\theta}) \cap \left[\bigcup_{\boldsymbol{\theta}' \in \{0,1\}^{n'-n}} \bigcap_{C' \in \mathbf{C}' \setminus \mathbf{C}} \Lambda(C',\theta')\right] \\ = \Lambda(\mathbf{C},\boldsymbol{\theta}) \cap \left[\bigcap_{C' \in \mathbf{C}' \setminus \mathbf{C}} \bigcup_{\theta' \in \{0,1\}} \Lambda(C',\theta')\right] &= \Lambda(\mathbf{C},\boldsymbol{\theta}) \cap \left[\bigcap_{C' \in \mathbf{C}' \setminus \mathbf{C}} \Omega\right] = \Lambda(\mathbf{C},\boldsymbol{\theta}).\end{aligned}$$

□

The different pairs $\langle \mathbf{C},\boldsymbol{\theta}\rangle$ and $\langle \mathbf{C}',\boldsymbol{\theta}'\rangle$ connected with fragmentations $\mathcal{C}_m(c,\nu,\mathbf{z})$ and $\mathcal{C}_{m'}(c,\nu,\mathbf{z})$ may be correspond to the same cylindrical event. In connection with this situation, we gives the following

Definition 12. We will name two pairs $\langle \mathbf{C},\boldsymbol{\theta}\rangle$ и $\langle \mathbf{C}',\boldsymbol{\theta}'\rangle$ the equivalent ones if the events $\Lambda(\mathbf{C},\boldsymbol{\theta})$ and $\Lambda(\mathbf{C}',\boldsymbol{\theta}')$ coincide. In this case we will write $\langle \mathbf{C},\boldsymbol{\theta}\rangle \sim \langle \mathbf{C}',\boldsymbol{\theta}'\rangle$.

The equivalence relation decomposes the family $\mathfrak{C}$ into classes of pairs $\langle \mathbf{C}', \boldsymbol{\theta}' \rangle$ being equivalent to each other, which describe the same cylindrical random event $\Lambda(\mathbf{C}, \boldsymbol{\theta})$. Let us find the criterium of equivalence for cylindrical events of the system $\mathfrak{F}$.

Theorem 7. *For each cell $C \in \mathcal{C}_m(c, \nu, \mathbf{z})$ and for any its fragmentation $\mathcal{D}^{(m')}(C)$ by cells from $\mathcal{C}_{m'}(c, \nu, \mathbf{z})$, it takes place the disjunctive decomposition*

$$\Lambda(C, 1) = \bigcup_{\boldsymbol{\theta}: \boldsymbol{\theta} \not\equiv 0, \boldsymbol{\theta} = \mathsf{V}(\mathcal{D}^{(m')}(C))} \Lambda(\mathcal{D}^{(m')}(C), \boldsymbol{\theta}). \tag{19}$$

Proof. In the case if $X \in \Lambda(C, 1)$, $\chi(X|C) = 1$ the proof of fulfilment of the decomposition (19) follows from the fact that there is the cell C' in $\mathcal{D}^{(m')}(C)$ for which $\chi(X|C') = 1$ that is $\boldsymbol{\theta} = \mathsf{V}(\mathcal{D}^{(m')}(C)) \not\equiv 0$. □

For each pair $\langle \mathbf{C}, \boldsymbol{\theta} \rangle \in \mathfrak{C}$, we introduce the collection $\mathbf{N}(\mathbf{C}, \boldsymbol{\theta}) = \{C \in \mathbf{C} : \mathsf{V}(C) = 0\}$ and, on its basis, we define the set

$$N(\mathbf{C}, \boldsymbol{\theta}) = \bigcup_{C \in \mathbf{N}(\mathbf{C}, \boldsymbol{\theta}): \mathsf{V}(C) = 0} C.$$

The desired equivalence criterium of cylindrical random events of he system $\mathfrak{F}$ gives the following

Theorem 8. *The equality $\Lambda(\mathbf{C}, \boldsymbol{\theta}) = \Lambda(\mathbf{C}', \boldsymbol{\theta}')$ takes place if and only if* $\mathbf{N}(\mathbf{C}, \boldsymbol{\theta}) = \mathbf{N}(\mathbf{C}', \boldsymbol{\theta}')$ and $\mathbf{C} \setminus \mathbf{N}(\mathbf{C}, \boldsymbol{\theta}) = \mathbf{C}' \setminus \mathbf{N}(\mathbf{C}', \boldsymbol{\theta}')$.

Proof. For both pairs $\langle \mathbf{C}, \boldsymbol{\theta} \rangle$ and $\langle \mathbf{C}', \boldsymbol{\theta}' \rangle$ we represent the cylindrical random events $\Lambda(\mathbf{C}, \boldsymbol{\theta})$ and $\Lambda(\mathbf{C}', \boldsymbol{\theta}')$ in the following form

$$\begin{aligned} \Lambda(\mathbf{C}, \boldsymbol{\theta}) &= \Bigg(\bigcap_{C \in \mathbf{C}: \mathsf{V}(C) = 1} \Lambda(C, 1) \Bigg) \cap \Bigg(\bigcap_{C \in \mathbf{C}: \mathsf{V}(C) = 0} \Lambda(C, 0) \Bigg) \\ &= \Lambda(\mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 0) \cap \Lambda(\mathbf{C} \setminus \mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 1), \end{aligned} \tag{20}$$

$$\begin{aligned} \Lambda(\mathbf{C}', \boldsymbol{\theta}') &= \Bigg(\bigcap_{C \in \mathbf{C}': \mathsf{V}(C) = 1} \Lambda(C, 1) \Bigg) \cap \Bigg(\bigcap_{C \in \mathbf{C}': \mathsf{V}(C) = 0} \Lambda(C, 0) \Bigg) \\ &= \Lambda(\mathbf{N}(\mathbf{C}', \boldsymbol{\theta}'), 0) \cap \Lambda(\mathbf{C}' \setminus \mathbf{N}(\mathbf{C}', \boldsymbol{\theta}'), 1). \end{aligned} \tag{21}$$

Proof of necessity. Suppose that the equality $\Lambda(\mathbf{C}, \boldsymbol{\theta}) = \Lambda(\mathbf{C}', \boldsymbol{\theta}')$ takes place, but $N(\mathbf{C}', \boldsymbol{\theta}') = N(\mathbf{C}, \boldsymbol{\theta})$ is not fulfilled that is, for certainty, $N(\mathbf{C}, \boldsymbol{\theta}) \setminus N(\mathbf{C}', \boldsymbol{\theta}') \neq \emptyset$. Then, there is such a realization $X \in \Lambda(\mathbf{C}', \boldsymbol{\theta}')$ which has nonempty intersection with $N(\mathbf{C}, \boldsymbol{\theta}) \setminus N(\mathbf{C}', \boldsymbol{\theta}')$. Consequently, there is such a cell $C_0 \subset N(\mathbf{C}, \boldsymbol{\theta})$ in the collection $\langle C \in \mathbf{C}; \chi(C|X) = 0 \rangle$ for which $C_0 \cap X \neq \emptyset$ is fulfilled that is $\chi(X|C_0) = 1 \neq \mathsf{V}(C_0) = 0$, and therefore we obtain the contradiction $X \notin \Lambda(\mathbf{C}; \boldsymbol{\theta})$.

Noticing that equality $\Lambda(\mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 0) = \Lambda(\mathbf{N}(\mathbf{C}', \boldsymbol{\theta}'), 0)$ is equivalent to the equality $N(\mathbf{C}', \boldsymbol{\theta}') = N(\mathbf{C}, \boldsymbol{\theta})$, it is follows from (20), (21) and from last equality that it is true

$$\Lambda(\mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 0) \cap \Lambda(\mathbf{C} \setminus \mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 1) = \Lambda(\mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 0) \cap \Lambda(\mathbf{C}' \setminus \mathbf{N}(\mathbf{C}, \boldsymbol{\theta}), 1).$$

Suppose that the equality $\mathbf{C} \setminus \mathbf{N}(\mathbf{C}, \boldsymbol{\theta}) = \mathbf{C}' \setminus \mathbf{N}(\mathbf{C}', \boldsymbol{\theta}')$ does not take place. Then, either there is such a cell $C_* \in \mathbf{C} \setminus \mathbf{N}(\mathbf{C}, \boldsymbol{\theta})$ which is absent in the collection $\mathbf{C}' \setminus \mathbf{N}(\mathbf{C}', \boldsymbol{\theta}')$ or the reverse case is implemented. For certainty, let first situation be realized, that is $C_* \in \mathbf{C} \setminus \mathbf{C}'$. Then, either $C_* \cap \bigcup_{C \in \mathbf{C}'} C = \emptyset$ or there is such a cell C'_* in $\mathbf{C}'$ for which it is fulfilled $C_* \cap C'_* \neq \emptyset$, $C_* \neq C'_*$.

In first case, there is such a realization X in the random event $\Lambda(\mathbf{C}, \boldsymbol{\theta})$ that $X \cap N(\mathbf{C}, \boldsymbol{\theta}) = \emptyset$ and it is fulfilled $X \cap C_* = \emptyset$. But such a realization belongs to the event $\Lambda(\mathbf{C}', \boldsymbol{\theta}')$. At second case, there is such a realization X for which it is fulfilled $X \cap C'_* \neq \emptyset$ and $X \cap C_* = \emptyset$ if $C_* \subset C'_*$ and, converse situation is realized if $C'_* \subset C_*$. In a result, we obtain the contradiction, since, for the event $\Lambda(C'_*, 1)$, it should be fulfilled (19). But in this decomposition, it is absent such a component with the function $\boldsymbol{\theta} = \mathsf{V}(\mathcal{D}^{(m')}(C'_*))$ for which it takes place $\mathsf{V}(C_*) = 0$.

Proof of sufficiency. Now, conversely, let be fulfilled the conditions $N(\mathbf{C},\boldsymbol{\theta}) = N(\mathbf{C}',\boldsymbol{\theta}')$ and $\mathbf{C}\setminus\mathbf{N}(\mathbf{C},\boldsymbol{\theta}) = \mathbf{C}'\setminus\mathbf{N}(\mathbf{C}',\boldsymbol{\theta}')$. From the fulfilment of first condition, it follows that $\Lambda(\mathbf{N}(\mathbf{C}',\boldsymbol{\theta}'),0) = \Lambda(\mathbf{N}(\mathbf{C},\boldsymbol{\theta}),0)$. From the fulfilment of second condition, it follows the equality $\Lambda(\mathbf{C}\setminus\mathbf{N}(\mathbf{C},\boldsymbol{\theta}),1) = \Lambda(\mathbf{C}'\setminus\mathbf{N}(\mathbf{C}',\boldsymbol{\theta}'),1)$. Then, due to (20), (21), it is fulfilled $\Lambda(\mathbf{C}',\boldsymbol{\theta}'),\boldsymbol{\theta}') = \Lambda(\mathbf{C},\boldsymbol{\theta}),\boldsymbol{\theta})$. □

7. ALGEBRAIC PROPERTIES OF THE SYSTEM $\mathfrak{F}$

In this section we will establish some algebraic properties of cylindrical events of the system $\mathfrak{F}$. The presence of these properties shows that $\mathfrak{F}$ is the c-system.

At first, we will set that, for any $m\geq 2$, the family $\mathfrak{F}_m$ of random events $\Lambda(\mathbf{C},\boldsymbol{\theta})$ such that the collection $\mathbf{C}$ consists of the cells from $\mathcal{C}_m(c,\nu,\mathbf{z})$, is the c-systems.

We note that it is true the following formula for the intersection of two cylindrical random events $\Lambda(C,\theta)$ and $\Lambda(C',\theta)$ with the cells C and C' from the same class $\mathcal{C}_m(c,\nu,\mathbf{z})$ and $\theta,\theta'\in\{0,1\}$,

$$\Lambda(C,\theta)\cap\Lambda(C',\theta') = \begin{cases}\emptyset, \quad \text{if } C=C', \theta\neq\theta';\\ \Lambda(\{C,C'\},\{\theta,\theta'\}), \quad \text{if } C\neq C'.\end{cases} \tag{22}$$

Then, it is true

Lemma 3. *If collections $\mathbf{C}_1$ u $\mathbf{C}_2$ consist of cells of the class $\mathcal{C}_m(c,\nu,\mathbf{z})$, then it is fulfilled the following formula*

$$\Lambda(\mathbf{C}_1,\boldsymbol{\theta}_1)\cap\Lambda(\mathbf{C}_2,\boldsymbol{\theta}_2) = \begin{cases}\emptyset, \quad \text{if } \mathsf{V}(C_1)\neq\mathsf{V}(C_2) \text{ at } \{C_1,C_2\}\subset\mathbf{C}_1\cap\mathbf{C}_2;\\ \Lambda(\mathbf{C}_1\cup\mathbf{C}_2,\boldsymbol{\theta}_1\cup\boldsymbol{\theta}_2), \quad \text{if } \mathsf{V}(C_1)=\mathsf{V}(C_2) \text{ at } \{C_1,C_2\}\subset\mathbf{C}_1\cap\mathbf{C}_2,\end{cases} \tag{23}$$

for any collections $\boldsymbol{\theta}_1 = \mathsf{V}(\mathbf{C}_1)$, $\boldsymbol{\theta}_2 = \mathsf{V}(\mathbf{C}_2)$ where $\mathbf{C}_1\cup\mathbf{C}_2\subset\mathcal{C}_m(c,\nu,\mathbf{z})$.

Proof. First equality in (23) follows from first equality in (22). Since $\boldsymbol{\theta}_j = \mathsf{V}(C_j)$, $C_j\in\mathbf{C}_j$, $j\in\{1,2\}$, then, from second equality in (22), it follows that

$$\Lambda(C_1,\theta_1)\cap\Lambda(\mathbf{C}_2,\boldsymbol{\theta}_2) = \Lambda(C_1,\theta_1)\cap\left(\bigcap_{C_2\in\mathbf{C}_2}\Lambda(C_2,\mathsf{V}(C_2))\right) = \bigcap_{C_2\in\mathbf{C}_2}\Lambda(C_1,\mathsf{V}(C_1))\cap\Lambda(C_2,\mathsf{V}(C_2))$$
$$= \bigcap_{C_2\in\mathbf{C}_2}\Lambda(\{C_1\cup C_2\},\{\mathsf{V}(C_1),\mathsf{V}(C_2)\}) = \Lambda(\{C_1\}\cup\mathbf{C}_2,\{\mathsf{V}(C_1)\}\cup\mathsf{V}(\mathbf{C}_2)).$$

Consequently, we may done the following transformations

$$\Lambda(\mathbf{C}_1,\boldsymbol{\theta}_1)\cap\Lambda(\mathbf{C}_2,\boldsymbol{\theta}_2) = \left(\bigcap_{C_1\in\mathbf{C}_1}\Lambda(C_1,\mathsf{V}(C_1))\right)\cap\left(\bigcap_{C_2\in\mathbf{C}_2}\Lambda(C_2,\mathsf{V}(C_2))\right)$$
$$= \bigcap_{C_1\in\mathbf{C}_1}\left(\Lambda(C_1,\mathsf{V}(C_1))\cap\left(\bigcap_{C_2\in\mathbf{C}_2}\Lambda(C_2,\mathsf{V}(C_2))\right)\right) = \bigcap_{C_1\in\mathbf{C}_1}\Lambda(\{C_1\}\cup\mathbf{C}_2,\{\mathsf{V}(C_1)\}\cup\mathsf{V}(\mathbf{C}_2))$$
$$= \Lambda(\mathbf{C}_1\cup\mathbf{C}_2,\mathsf{V}(\mathbf{C}_1\cup\mathbf{C}_2)).$$

□

Lemma 4. *If the collection $\mathbf{C}$ consists of cells of the class $\mathcal{C}_m(c,\nu,\mathbf{z})$, then for any collection $\boldsymbol{\theta} = V(\mathbf{C})$ the following formula takes place*

$$\bigcup_{C\in\mathbf{C}}\Lambda(\mathbf{C},\theta) = \bigcup_{\boldsymbol{\theta}':\boldsymbol{\theta}'\neq\bar{\boldsymbol{\theta}}}\Lambda(\mathbf{C},\boldsymbol{\theta}'), \quad \bar{\boldsymbol{\theta}} = 1-\mathsf{V}(\mathbf{C}), \tag{24}$$

where the collection of random events $\{\Lambda(\mathbf{C},\boldsymbol{\theta}');\boldsymbol{\theta}'\in\{0,1\}^n\}$, $n = Card(\mathbf{C})$ is disjunctive.

Proof. Let be $X\in\Lambda(C,\theta)$ that is $\chi(X|C)=\theta$. Since all events $\Lambda(\mathbf{C},\boldsymbol{\theta}')$ with collections $\boldsymbol{\theta}'\in\{0,1\}^n$, $n=\mathrm{Card}\mathbf{C}$, $\boldsymbol{\theta}'\neq\bar{\boldsymbol{\theta}}$ are included in the union in the right-hand side of the formula (24), then there is such an event among them, in which there is the component $\theta'=\mathsf{V}(C)$. Consequently, $X\in\Lambda(\mathbf{C},\boldsymbol{\theta}')$ with such a collection $\boldsymbol{\theta}'\in\{0,1\}^n$.

Conversely, let be $X \in \Lambda(\mathbf{C}, \boldsymbol{\theta}')$ with something collection $\boldsymbol{\theta}' \neq \bar{\boldsymbol{\theta}}$. Then, it is found such a cell among the cells $C \in \mathbf{C}$ for which it is valid $\mathsf{V}(C) = \theta'$, and therefore, $X \in \Lambda(C, \theta)$. □

The formula (24) permits to prove the closedness of the family of the events $\Lambda(\mathbf{C}, \boldsymbol{\theta})$, $\mathbf{C} \subset \mathcal{C}_m(c, \nu, \mathbf{z})$ relative to the operation of set complement.

Lemma 5. *If the collection* $\mathbf{C}$ *consists of cells of the class* $\mathcal{C}_m(c, \nu, \mathbf{z})$*, then for any collection* $\boldsymbol{\theta} = \mathsf{V}(\mathbf{C})$ *the following decomposition of the random event* $\complement\Lambda(\mathbf{C}, \boldsymbol{\theta})$ *on the disjunctive collection* $\Lambda(\mathbf{C}, \boldsymbol{\theta}')$ *with* $\boldsymbol{\theta}' \neq \boldsymbol{\theta}$ *takes place*

$$\complement\Lambda(\mathbf{C}, \mathsf{V}(\mathbf{C})) = \bigcup_{\boldsymbol{\theta}' : \boldsymbol{\theta}' \neq \boldsymbol{\theta}} \Lambda(\mathbf{C}, \boldsymbol{\theta}'). \tag{25}$$

Proof. Since for any cell $C \in \mathcal{C}_m(c, \nu, \mathbf{z})$ the following formulas are valid $\complement\Lambda(C, 0) = \Lambda(C, 1)$, $\complement\Lambda(C, 1) = \Lambda(C, 0)$, then

$$\complement\Lambda(\mathbf{C}, \mathsf{V}(\mathbf{C})) = \complement \bigcap_{C \in \mathbf{C}} \Lambda(C, \mathsf{V}(C)) = \bigcup_{C \in \mathbf{C}} \complement\Lambda(C, \mathsf{V}(C)) = \bigcup_{C \in \mathbf{C}} \Lambda(C, \bar{\boldsymbol{\theta}}),$$

$\bar{\boldsymbol{\theta}} = 1 - \mathsf{V}(\mathbf{C})$. On the basis of last decomposition, using the formula (24) and also the fact that $\bar{\bar{\boldsymbol{\theta}}} = \boldsymbol{\theta}$, we will build the decomposition (25) of the random event $\complement\Lambda(\mathbf{C}, \mathsf{V}(\mathbf{C}))$ on the disjunctive collection of events $\Lambda(\mathbf{C}, \boldsymbol{\theta}')$, $\boldsymbol{\theta}' \neq \boldsymbol{\theta}$. □

From statements of Lemmas 4 and 5, it follows

Theorem 9. *At any fixed number* $m \in \mathbb{N}$*, the family of random events* $\Lambda(\mathbf{C}, \boldsymbol{\theta})$*,* $\mathbf{C} \subset \mathcal{C}_m(c, \nu, \mathbf{z})$*, Card* $\mathbf{C} < \infty$ *is the Boolean c-system in the sense of the Definition 10.*

We extend the statement of the formulated theorem to the family of events $\Lambda(\mathbf{C}, \boldsymbol{\theta})$, $\mathbf{C} \subset \mathcal{C}(c, \nu, \mathbf{z})$ with finite collections $\mathbf{C}$ in which the degree of fragmentation cells is not fixed.

Theorem 10. *The family* $\mathfrak{F} = \{\Lambda(\mathbf{C}, \boldsymbol{\theta}); \mathbf{C} \subset \mathcal{C}(c, \nu, \mathbf{z}),\ \mathrm{Card}\mathbf{C} < \infty,\ \boldsymbol{\theta} = \mathsf{V}(\mathbf{C})\}$ *is the Boolean c-systems.*

Proof. Firstly, we prove that the family of random events $\Lambda(\mathbf{C}, \boldsymbol{\theta})$ defined by the finite collections $\mathbf{C} \subset \mathcal{C}_m(c, \nu, \mathbf{z}) \cup \mathcal{C}_{m+1}(c, \nu, \mathbf{z})$ is the c-system. For the proof of this fact, it is sufficient to set that the event $\Lambda(\mathbf{C}_1, \boldsymbol{\theta}_1) \cap \Lambda(\mathbf{C}_2, \boldsymbol{\theta}_2)$ is represented in the form of the disjunctive decomposition on the events $\Lambda(\mathbf{C}, \mathsf{V}(\mathbf{C}))$ with the cell collections of $\mathcal{C}_m(c, \nu, \mathbf{z}) \cup \mathcal{C}_{m+1}(c, \nu, \mathbf{z})$ for each pair of events $\Lambda(\mathbf{C}_1, \boldsymbol{\theta}_1)$ and $\Lambda(\mathbf{C}_2, \boldsymbol{\theta}_2)$ with $\mathbf{C}_j \subset \mathcal{C}_m(c, \nu, \mathbf{z}) \cup \mathcal{C}_{m+1}(c, \nu, \mathbf{z})$, $j \in \{1, 2\}$.

For definiteness, let be fulfilled $\mathbf{C}_1 \subset \mathcal{C}_m(c, \nu, \mathbf{z})$ and $\mathbf{C}_2 \subset \mathcal{C}_{m+1}(c, \nu, \mathbf{z})$. The random events $\Lambda(\mathbf{C}_1, \boldsymbol{\theta}_1)$ and $\Lambda(\mathbf{C}_2, \boldsymbol{\theta}_2)$ corresponding to these collections are represented in the following form according to (13),

$$\Lambda(\mathbf{C}_1, \boldsymbol{\theta}_1) = \bigcap_{C_1 \in \mathbf{C}_1} \Lambda(C_1, \mathsf{V}(C_1)), \quad \Lambda(\mathbf{C}_2, \boldsymbol{\theta}_2) = \bigcap_{C_2 \in \mathbf{C}_2} \Lambda(C_2, \mathsf{V}(C_2)).$$

On the basis of these formulas, due to the statement of Lemma 3, it is sufficient to set that for each pair of cells C_1 and C_2, the intersection $\Lambda(C_1, \mathsf{V}(C_1)) \cap \Lambda(C_2, \mathsf{V}(C_2))$ is represented in the form of disjunctive decomposition on random events $\Lambda(\mathbf{C}_{1,2}, \boldsymbol{\theta})$, $\boldsymbol{\theta} = \mathsf{V}(\mathbf{C}_{1,2})$ with the suitable collection $\mathbf{C}_{1,2} \subset \mathcal{C}_m(c, \nu, \mathbf{z}) \cup \mathcal{C}_{m+1}(c, \nu, \mathbf{z})$.

Let us compare the collection of cells $\mathbf{C}_{1,2} = \mathcal{D}^{(m+1)}(C_1)$ to the cell C_1. Using it, we apply the formula (19) for the event $\Lambda(C_1, 1)$. In a result, we obtain

$$\Lambda(C_1, \mathsf{V}(C_1)) \cap \Lambda(C_2, \mathsf{V}(C_2)) = \bigcup_{\boldsymbol{\theta} \neq 0 : \boldsymbol{\theta} = \mathsf{V}(\mathbf{C}_{1,2})} \Lambda(\mathbf{C}_{1,2}, \boldsymbol{\theta}).$$

Besides, we note that it is valid $\Lambda(C_1, 0) = \Lambda(\mathbf{C}_{1,2}, 0)$. Then, it is obtained the desired disjunctive decomposition of the event $\Lambda(C_1, \mathsf{V}(C_1)) \cap \Lambda(C_2, \mathsf{V}(C_2))$ in both cases.

Since the family of random events $\Lambda(\mathbf{C}, \boldsymbol{\theta})$, $\mathbf{C} \subset \mathcal{C}_m(c, \nu, \mathbf{z}) \cup \mathcal{C}_{m+1}(c, \nu, \mathbf{z})$, $\mathrm{Card}\mathbf{C} < \infty$ is the c-system, then, reasoning by induction on $n \in \mathbb{N}$, we obtain that families of such random events represent the c-system at $\mathbf{C} \subset \bigcup_{m=1}^{n} \mathcal{C}_m(c, \nu, \mathbf{z})$ at any number $n \in \mathbb{N}$. At last, using the obvious fact that the limit of an expanding sequence of c-systems is also the c-system when the limit is understood in the sense of the set theory, we were coming to the proof the theorem statement. □

8. SPACE OF RANDOM SETS

In this section we complete the construction of the space $\langle \Omega, \mathfrak{B}, \mathsf{P}\rangle$ of random sets in which the measurability structure is determined on the basis of the σ-algebra $\mathfrak{B} = \mathfrak{B}(\mathfrak{F})$ generated by the c-system $\mathfrak{F}$ of cylindrical random events in the immersion space $\mathbb{R}^d$.

Denote by $\mathrm{cl}(\cdot)$ the operator of the topological closure in $\mathbb{R}^d$ which is applied to subsets of $\mathbb{R}^d$.

Definition 13. Probabilistic space $\langle \Omega, \mathfrak{B}, \mathsf{P}\rangle$ is named the space of random sets in the immersion space $\mathbb{R}^d$, if the element Ω is represented by the class of all subsets (random realizations) $X \subset \mathbb{R}^d$, $\mathfrak{B}$ is the minimal σ-algebra which contains the system $\mathfrak{F}_*$ of all classes $\mathcal{F} = \{X \subseteq \mathbb{R}^d : \mathrm{cl}(C) \cap X \neq \emptyset\}$; $C \in \mathcal{C}(c, \nu, \mathbf{z})$ of subsets in $\mathbb{R}^d$. In this case, all random events are represented by Lebesgue-measurable classes $\mathcal{F}$ of such subsets.

Definition 14. The point $\mathbf{z} \in \mathbb{R}$ is named the right limit point of the set $X \subset \mathbb{R}^d$, if there is a nonincreasing sequence $\langle x_l \in X; l \in \mathbb{N}\rangle$ which converges to z at $l \to \infty$.

By other words, if $\mathbf{z} = \langle z_j; j \in I_d\rangle$ is a right limit point of the set X, then for any sufficiently small $\delta > 0$, it takes place $X \cap \bigotimes_{j=1}^{d}[z_j, z_j + \delta)^d \neq \emptyset$.

The operation $\mathrm{cl}_+(\cdot)$ на $2^{\mathbb{R}^d}$ in application to the set X consists of the joining of all its right limit points. The sets $X \subset \mathbb{R}^d$ satisfying the condition $\mathrm{cl}_+(X) = X$ are named the right-closured ones. We note that the following state is true.

Lemma 6. *For each cell $C \in \mathcal{C}_m(c, \nu, \mathbf{z})$ that is for any objects such that the point $\mathbf{z} \in \mathbb{R}^d$, the value $c > 0$ and numbers ν, m not equal to 1, in the case when the realization $X \in \Lambda(\mathbf{C}, \boldsymbol{\theta})$, it takes place* $\mathrm{cl}_+(X) \in \Lambda(\mathbf{C}, \boldsymbol{\theta})$.

Consequently, one may consider that elements of each random event $\Lambda(\mathbf{C}, \boldsymbol{\theta})$ are the equivalence classes $\mathcal{F}$ of realizations $X \subset \mathbb{R}^d$. Each of them is combined all such realizations X where all corresponding sets $\mathrm{cl}_+(X)$ coincide.

Since the $\mathfrak{F}$ is the c-system, then, according to Theorem 3, the family of random events

$$\left\{ \bigcup_{j=1}^{n} \Lambda(\mathbf{C}_j, \boldsymbol{\theta}_j); \langle \mathbf{C}_j, \boldsymbol{\theta}_j\rangle \in \mathfrak{C}, j \in I_n \right\}, \quad n \in \mathbb{N}$$

which are formed by all possible disjunctive finite unions of the random events $\Lambda(\mathbf{C}_j, \boldsymbol{\theta}_j)$, $j \in I_n$, $n \in \mathbb{N}$ from $\mathfrak{F}$, compile the minimal Boolean algebra $\mathfrak{A}(\mathfrak{F})$ that is generated by this system. Then, one of the consequences of the Lemma 6 is the fact that each element of this family consists of classes of realizations being equivalent to each other.

Let $\mathfrak{B}(\mathfrak{F})$ be the minimal σ-algebra generated by the c-system $\mathfrak{F}$ of cylindrical random events. Since it is formed by completion of the algebra $\mathfrak{A}(\mathfrak{F})$ of limits of monotone sequences of algebraic elements in the sense of set theory, by inheritable way, one may consider the elements of the σ-algebra $\mathfrak{B}(\mathfrak{F})$ as such which consist of classes of realizations $X \subset \mathbb{R}^d$ with the same right closure. Therefore, it is natural to consider the component Ω of the triad $\langle \Omega, \mathfrak{B}(\mathfrak{F}), \mathsf{P}\rangle$ defining the probabilistic space, which represents the space of elementary random events, as the aggregate of classes of equivalent realizations.

Now, we suppose that a nonnegative additive function $\mathsf{P}(\mathbf{D}, \boldsymbol{\theta})$ is defined on the c-system $\mathfrak{F}$ of random events. It is done by the equality $\mathsf{P}(\mathbf{D}, \boldsymbol{\theta}) = \Pr\{\Lambda(\mathbf{D}, \boldsymbol{\theta})\}$, $\Lambda(\mathbf{D}, \boldsymbol{\theta}) \in \mathfrak{F}$. Then, it is true

Theorem 11. *In order that the function $P(\mathbf{C}, \boldsymbol{\theta}) = Pr\{\Lambda(\mathbf{C}, \boldsymbol{\theta})\}$ to determine the additive normalized measure on the system $\mathfrak{F}$, it is necessary that it satisfies the equation systems*

$$\mathsf{P}(\mathbf{C}, \boldsymbol{\theta}) = \sum_{\{\boldsymbol{\theta}' \in \{0,1\}^{n'} : \mathsf{R}(\boldsymbol{\theta}') = \boldsymbol{\theta}\}} \mathsf{P}(\mathbf{C}', \boldsymbol{\theta}'), \quad \langle \mathbf{C}, \boldsymbol{\theta}\rangle \subset \mathfrak{C} \tag{26}$$

for any collection $\mathbf{C}'$ such that $\mathsf{R}(\mathbf{C}') = \mathbf{C}$, $n' = \mathrm{Card}(\mathbf{C}')$ and the following equation is fulfilled for any cell $C \in \mathfrak{C}_m(c, \nu, \mathbf{z})$

$$\mathsf{P}(C, 1) = \sum_{\boldsymbol{\theta} : \boldsymbol{\theta} \not\equiv 0, \boldsymbol{\theta} = \mathsf{V}(\mathcal{D}^{(m+1)}(C))} \mathsf{P}(\mathcal{D}^{(m+1)}(C), \boldsymbol{\theta}). \tag{27}$$

In particular, it should be performed

$$1 = \sum_{\boldsymbol{\theta}\in\{0,1\}^n:\mathsf{V}(\mathbf{C})=\boldsymbol{\theta}} \mathsf{P}(\mathbf{C},\boldsymbol{\theta}), \quad n = \mathrm{Card}(\mathbf{C}). \tag{28}$$

Besides, the measure $P(\mathbf{C},\boldsymbol{\theta})$ should be satisfy to the identity $P(\mathbf{C},\boldsymbol{\theta}) = P(\mathbf{C}',\boldsymbol{\theta}')$ for any two equivalent pairs $\langle\mathbf{C},\boldsymbol{\theta}\rangle$, $\langle\mathbf{C}',\boldsymbol{\theta}'\rangle$ of $\mathfrak{C}$.

Proof. The necessity of fulfilment of equalities (26), (27) and also the equality (28) is the consequence of disjunctivity of decomposions in (17), (18), (19) and additivity of the measure on the system $\mathfrak{F}$ defined by the function $\mathsf{P}(\mathbf{C},\boldsymbol{\theta})$. The identity $\mathsf{P}(\mathbf{C},\boldsymbol{\theta}) = \mathsf{P}(\mathbf{C}',\boldsymbol{\theta}')$ is the consequence of requirement of parametrization unambiguity of the random events $\Lambda(\mathbf{C},\boldsymbol{\theta})$ by pairs $\langle\mathbf{C},\boldsymbol{\theta}\rangle$. □

Conditions to the measure P expressed by the equations (26), (27) named the *consistency conditions* and, correspondingly, the formula (28) is named the *normalization condition* of the measure. It is evident that the consistency conditions expressed by (26) are sufficient for the definition of additive measure on each of the c-system $\mathfrak{F}_m$, $m \in \mathbb{N}$. Namely, it is valid

Theorem 12. *In order to define the additive normalized measure P on the c-system $\mathfrak{F}_m$, $m \in \mathbb{N}$, it is sufficient to define the function $P(\mathbf{C},\boldsymbol{\theta})$ on all collections $\mathbf{C} \subset \mathcal{C}_m(c,\nu,\mathbf{z})$ which satisfy the equations (26), (28), and to set* $\mathsf{P}(\mathbf{C},\boldsymbol{\theta}) = \Pr\{\Lambda(\mathbf{C},\boldsymbol{\theta})\}$.

The coordination of the measures P introduced on each of the systems $\mathfrak{F}_m$, $m \in \mathbb{N}$ is based on the following statement.

Theorem 13. *For the definition of the additive nonnegative normalized P on the c-system $\mathfrak{F}$ on the basis of functions $P(\mathbf{C},\boldsymbol{\theta})$ it is sufficient that these functions satisfy the equation (27) for each cell $C \in \mathcal{C}_m(c,\nu,\mathbf{z})$ on each of the system $\mathfrak{F}_m$, $m \in \mathbb{N}$.*

Proof. The sufficiency of the condition (27) for the coordination of measures defined on each of the system $\mathfrak{F}_m$, $m \in \mathbb{N}$ follows from the fact that for any collection $\mathbf{C} \in \mathfrak{C}(c,\nu,\mathbf{z})$ and for any random event $\Lambda(\mathbf{C},\boldsymbol{\theta})$ ia represented in the form of the disjunctive decomposition

$$\Lambda(\mathbf{C},\boldsymbol{\theta}) = \bigcup_{\boldsymbol{\theta}\in\{0,1\}^{\nu^n}:\theta=\mathsf{V}(\mathcal{D}^{(n)}(\mathbf{C}))} \Lambda(\mathcal{D}^{(n)}(\mathbf{C}),\boldsymbol{\theta}), \tag{29}$$

where $\mathcal{D}^{(n)}(\mathbf{C})$ is the fragmentation by cells $C \in \mathcal{C}_n(c,\nu,\mathbf{z})$ of the class $\bigcup_{C\in\mathbf{C}} C$ where n is the maximal degree of the fragmentation $\mathcal{C}_n(c,\nu,\mathbf{z})$ of $\mathbb{R}^d$, whose cells are contained in the collection $\mathbf{C}$. This formula is set step-by-step by application of the decomposition (19) to cells of the collection $\mathbf{C}$. At first step, it is produced the fragmentation of all cells in the collection $\mathbf{C}$ with the smallest degree n_0 and, in a result, the collection $\mathbf{C}_1$ is formed by cells with the minimal fragmentation degree n_0+1. Then, in correspondence with these fragmentation, some decompositions of the random events $\Lambda(C,\theta)$ are built for all fragmented cells C. It is done according to the formulas (14) and (19). At the next step, it is produced the fragmentation of all cells in the collection $\mathbf{C}_1$ which have the fragmentation degree n_0+1. In a result the collection $\mathbf{C}_2$ is obtained which have cells with the minimal fragmentation degree which is equal n_0+2 and, in accordance with this fragmentation, some decompositions of the random events $\Lambda(C,\theta)$ are built for all fragmented cells C.

If on the mth step of such a procedure, when the cellular collection $\mathbf{C}_m$ was produced which has the minimal fragmentation degree n_0+m, then, at the next step, it is produced the fragmentation of all cells with such a fragmentation degree. In a result, the cellular collection $\mathbf{C}_{m+1}$ is obtained which have the minimal fragmentation degree n_0+m+1 and it is produced the correspondent decompositions of the random events $\Lambda(C,\theta)$ according to (19) for all fragmented cells C. At the last step, in a result of the fragmentation, all cells with the fragmentation degree n are obtained and, the formula (29) is obtained due to the application of the formula (19) to all fragmentation cells.

Since on each step of the described procedure, the condition (27) is fulfilled for each fragmentation cells, then, in a result, the following formula is obtained on the basis of (29)

$$\mathsf{P}(\mathbf{C},\boldsymbol{\theta}) = \sum_{\boldsymbol{\theta}\in\{0,1\}^{\nu^n}:\theta=\mathsf{V}(\mathcal{D}^{(n)}(\mathbf{C}))} \mathsf{P}(\mathcal{D}^{(n)}(\mathbf{C}),\boldsymbol{\theta}). \tag{30}$$

It permits to coordinate the measures P defined on each of the systems $\mathfrak{F}_m$, $n_0 \le m \le n$ by unambiguous way. □

The following statement is the consequence of Theorems 11 and 12.

Theorem 14. *If the function $P(\cdot,\cdot)$ of the pairs $\langle \mathbf{C},\boldsymbol{\theta}\rangle \in \mathfrak{C}$ satisfies to equations (26)–(28), then it is continued by unambiguous way on additivity on each of minimal algebras $\mathfrak{A}(\mathfrak{F}_m)$ (and, therefore, on the minimal algebra $\mathfrak{A}(\mathfrak{F})$).*

The countably additive continuation of the measure P from the c-system $\mathfrak{F}$ on the minimal σ-algebra $\mathfrak{B}(\mathfrak{F})$ is possible when the conditions of σ-semiadditivity of this measure on $\mathcal{F}$ are fulfilled [17].

Theorem 15. *Let the function $P(\mathbf{C},\boldsymbol{\theta}) = Pr\{\Lambda(\mathbf{C},\boldsymbol{\theta})\}$ on $\mathfrak{C}$ defines the additive measure P on the c-system $\mathfrak{F}$. It is such that for any cell $C \in \mathcal{C}(c,\nu,\mathbf{z})$ and for any countable disjunctive collection $\{\Lambda(\mathbf{C}_j,\boldsymbol{\theta}_j); j \in \mathbb{N}\}$ of random events such that*

$$\Lambda(C,\theta) = \bigcup_{j=1}^{\infty} \Lambda(\mathbf{C}_j,\boldsymbol{\theta}_j), \quad \theta \in \{0,1\}, \tag{31}$$

the inequality

$$\mathsf{P}(C,\theta) \le \sum_{j=1}^{\infty} \mathsf{P}(\mathbf{C}_j,\boldsymbol{\theta}_j) < \infty, \quad \theta \in \{0,1\} \tag{32}$$

is fulfilled. Then, the measure P is continued by unambiguous way up to the σ-additive measure on the minimal σ-algebra $\mathfrak{B}(\mathfrak{F})$ containing the system $\mathfrak{F}$.

Proof. If the inequality (32) is fulfilled for each cell $C \in \mathfrak{C}$ in the collections $\mathbf{C}_j$ which are defined each random event of the disjunctive collection $\{\Lambda(\mathbf{C}_j,\boldsymbol{\theta}_j); j \in \mathbb{N}\}$ in the decomposition (31), then the similar inequalities occur for any random event $\Lambda(\mathbf{C},\boldsymbol{\theta}) \in \mathfrak{F}$ based on the cells of the collection $\mathbf{C}$ which may be composed of the above-pointed cells C. These inequalities are obtained by use of the decomposition (29) of the event $\Lambda(\mathbf{C},\boldsymbol{\theta})$ with application of the inequality (32) for each summand in the sum in the formula (30).

The presence of inequalities of specified type means that the measure P defined on the system $\mathfrak{F}$ has the property σ-semiadditivity, which ensures the presence of an unambiguous countably additive continuation of the measure P on $\mathfrak{B}(\mathfrak{F})$. □

9. CONCLUSIONS

The constructions studied in the article have both general theoretical significance from the point of view of the general measure theory and from the point of view of applications. In this regard, it is very important to develop the approach proposed in the paper to the study of concrete models of random sets. Our research immediately generates some questions awaiting an answer in future.

1) Under what sufficient conditions the functions $\mathsf{P}(\mathbf{C},\boldsymbol{\theta})$ satisfying the conditions (26)-(28) determine the measure P on the system $\mathfrak{F}$ having the σ-semiadditivity property, that is, under what conditions the inequalities (32) are satisfied?

2) It is necessary to define such a system of variables in terms of which the functions $\mathsf{P}(\mathbf{C},\boldsymbol{\theta})$ were represented by such objects of mathematical analysis that would make it possible to express the conditions (26)–(28) in their terms. In frames of the analytical parametrization of random events $\Lambda(\mathbf{C},\boldsymbol{\theta})$, it would be possible to use the apparatus of mathematical analysis to study the spaces of random sets and synthesize their specific models for the purpose of applications.

We note that in this paper, the probability space $\langle \Omega, \mathfrak{B}(\mathfrak{F}), \mathsf{P}\rangle$ is defined for each fixed set of communition parameters, namely, for the cell size c of the primary fragmentation of the immersion space $\mathbb{R}^d$, for the fragmentation parameter ν, for the point $\mathbf{z} \in \mathbb{R}^d$ which determines the location of the "initial" cell of the primary fragmentation of the space $\mathbb{R}^d$ which contains its "zero" point. The cardinality of the c-system $\mathfrak{F}$ generating the σ-algebra $\mathfrak{B}(\mathfrak{F})$ is equal to $\aleph_0$. Thus, the σ-algebra $\mathfrak{B}(\mathfrak{F})$ is countably generated. However, it is important for applications to construct a probability space that does not depend on the values of fixed parameters. This means that the measurability structure in such a space should be determined on the basis of a c-system which represents as a union of all c-systems $\mathfrak{F}$ with different values of the comminution parameters. The cardinality of such a combined c-system is $\aleph_1$, and the minimal σ-algebra will cease to be countably generated. At the same time, any probability space is subject to the requirement of the countable generality of their σ-algebra which is the basis of its measurability structure [18].

The resulting contradiction is eliminated by imposing certain restrictions on the measure P such that random events in an uncountable generating system are combined into equivalence classes with the same measure and its uncountability is not reflected in any way when calculating the measures of specific random events. For example, in the theory of random processes, such a requirement is the so-called separability [19]. In this regard, the similar question arises within the framework of the developed approach to the study of random sets.

3) What properties should have the functions $\mathsf{P}(\mathbf{C}, \boldsymbol{\theta})$ in order the measure defined by them to be common for all σ-algebras generated by c-systems $\mathfrak{F}_m$ corresponding to its own comminution $\mathfrak{C}_m(c, \nu, \mathbf{z})$?

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