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# THE CAUCHY PROBLEM FOR THE MULTI-TIME FRACTIONAL DIFFUSION EQUATION

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Consider the equation

$$\sum_{k=1}^{m} \lambda_k \frac{\partial^{\sigma_k}}{\partial y_k^{\sigma_k}} u(x,y) - \Delta_x u(x,y) = f(x,y)\,. \tag{1}$$

Here $x=(x_1,\cdots,x_n)\in\mathbf{R}^n$, $y=(y_1,\cdots,y_m)\in\mathbf{R}^m$ and $\lambda=(\lambda_1,\cdots,\lambda_m)\in\mathbf{R}^m$, $\lambda_k>0$; $\Delta_x$ is the Laplace operator, $\Delta_x=\sum_{i=1}^{n}\partial^2/\partial x_i^2$; $\partial^{\sigma_k}/\partial y_k^{\sigma_k}$ is an operator of fractional partial differentiation of order $\sigma_k$, $\sigma_k\in(0,1)$, with respect to $y_k$ and with origin at $y_k=0$. The fractional differentiation is given by the Dzhrbashyan-Nersesyan operator associated with the sequence $\{\alpha_k,\beta_k\}$, $\alpha_k$, $\beta_k\in(0,1]$, $\sigma_k=\alpha_k+\beta_k-1$, $\partial^{\sigma_k}/\partial y_k^{\sigma_k}=D_{0y_k}^{\{\alpha_k,\beta_k\}}=D_{0y_k}^{\beta_k-1}D_{0y_k}^{\alpha_k}$ (see [1]), where $D_{0y_k}^{\beta_k-1}$ and $D_{0y_k}^{\alpha_k}$ are the Riemann-Liouville fractional integral and derivative.

For a survey on results relating the initial and boundary value problems for a fractional diffusion equation and its generalizations, we refer to papers [2] and [3].

For any element $z\in\mathbf{R}^m$, we denote by $z_k$ the $k$-th coordinate of $z$. Let $z$ and $\zeta$ be elements of $\mathbf{R}^m$. The expressions $z\zeta$, $z^\zeta$, $z_*$ and $z_{*,k}$ denote the vectors $(z_1\zeta_1,\dots,z_m\zeta_m)$ and $(z_1^{\zeta_1},\dots,z_m^{\zeta_m})$, and the quantities $\prod_{i=1}^{m}z_i$ and $\prod_{i=1,\ i\neq k}^{m}z_i$ respectively.

Consider the function

$$f_{m,\delta}\left(z;\sigma;\mu\right)=\int_0^\infty t^{-\delta}e^{-\frac{1}{t}}\prod_{k=1}^{m}\phi\left(-\sigma_k,\mu_k;-z_k t\right)dt\,, \tag{2}$$

where $m\in\mathbf{N}$, $\delta\in\mathbf{R}$, and $z$, $\sigma$, $\mu\in\mathbf{R}^m$, $z_k>0$, $k=\overline{1,m}$. Here, $\phi\left(\xi,\eta;t\right)=\sum_{i=0}^{\infty}\frac{t^i}{i!\Gamma(\xi i+\eta)}$ is the Wright function (see [4]). In terms of function (2), we define the function

$$\Gamma_{m,n}^{\sigma}(x,y)=C_n|x|^{2-n}y_*^{-1}f_{m,n/2}\left(\frac{|x|^2}{4}\lambda y^{-\sigma};\sigma;0\right),\quad \text{where}\quad C_n=\frac{1}{4}\pi^{-n/2}.$$

We put $T=\{y:\,y_k\in(0,T_k),\,k=\overline{1,m}\}$ and $\Omega=\{(x,y):\,x\in\mathbf{R}^n,\,y\in T\}$. By $T_{(k)}$ and $y_{(k)}$ we denote the projections of $T$ and $y\in\mathbf{R}^m$ onto $\mathbf{R}^{m-1}$ along $y_k$. Also we write

$$I_y=(0,y_1)\times\cdots\times(0,y_m),\qquad I_y^{(k)}=(0,y_1)\times\cdots\times(0,y_{k-1})\times(0,y_{k+1})\times\cdots\times(0,y_m).$$

By $\Omega_k$ we denote the interior points of the set $\Omega_k = \partial\Omega \cap \{y_k = 0\}$, $k = \overline{1,m}$.

A function $u(x, y)$ is called a regular solution of equation (1) if $y_*^{1-\nu} u(x, y) \in C\left(\overline{\Omega}\right)$ for some $\nu \in \mathbf{R}^m$ with positive $\nu_k$, $D_{0y_k}^{\alpha_k - 1} u \in C(\Omega \cup \Omega_k)$, $D_{0y_k}^{\{\alpha_k, \beta_k\}} u$ and $u_{x_j x_j}$ belong to $C(\Omega)$, $k = \overline{1,m}$, $j = \overline{1,n}$. This function satisfies equation (1) at all points $(x, y) \in \Omega$.

In this work, we study the following problem: *find a regular solution $u = u(x, y)$ of equation (1) in $\Omega$ such that*

$$\lim_{y_k \to 0} D_{0y_k}^{\alpha_k - 1} u(x, y) = \tau(x, y_{(k)}), \quad x \in \mathbf{R}^m, \quad y_{(k)} \in T_{(k)}, \quad k = \overline{1,m}. \tag{3}$$

Formulate the main results of the work.

**Theorem 1.** *Suppose that $y_{*,k}^{1-\mu} \tau_k(x, y_{(k)}) \in C\left(\mathbf{R}^n \times \overline{T}_{(k)}\right)$ and $y_*^{1-\mu} f(x, y) \in C(\overline{\Omega})$ for some $\mu \in \mathbf{R}^m$ with positive $\mu_k$, and*

$$\lim_{|x| \to \infty} y_{*,k}^{1-\mu} \tau_k(x, y_{(k)}) \exp\left(-\rho_k |x|^{\frac{2}{2-\sigma_k}}\right) = 0, \qquad \lim_{|x| \to \infty} y_*^{1-\mu} f(x, y) \exp\left(-\rho_k |x|^{\frac{2}{2-\sigma_k}}\right) = 0,$$

*where $\rho_k < \left(1 - \frac{\sigma_k}{2}\right)\left(\frac{\sigma_k}{2T_k}\right)^{\frac{\sigma_k}{2-\sigma_k}}$ and $k = \overline{1,m}$. Then a regular solution $u(x, y)$ of problem (1), (3) that satisfies the condition*

$$\lim_{|x| \to \infty} y_*^{1-\nu} u(x, y) \exp\left(-\rho_k |x|^{\frac{2}{2-\sigma_k}}\right) = 0, \quad k = \overline{1,m},$$

*has the form*

$$u(x, y) = \int_{I_y} \int_{\mathbf{R}^n} f(\xi, \eta) \Gamma_{m,n}^{\sigma}(x - \xi, y - \eta)\, d\xi d\eta +$$

$$+ \sum_{k=1}^{m} \lambda_k \int_{I_y^{(k)}} \int_{\mathbf{R}^n} \left[D_{y_k \eta_k}^{\beta_k - 1} \Gamma_{m,n}^{\sigma}(x - \xi, y - \eta)\right]_{\eta_k = 0} \tau_k(\xi, \eta_{(k)})\, d\xi d\eta_{(k)}.$$

**Theorem 2.** *There is at most one regular solution of problem (1), (3) in the class of functions that satisfy the following condition for some positive constant $\rho$:*

$$\lim_{|x| \to \infty} y_*^{1-\nu} u(x, y) \exp\left(-\rho |x|^{\frac{2}{2-\sigma_0}}\right) = 0,$$

*where $\sigma_0 = \min\{\sigma_1, \sigma_2, ..., \sigma_m\}$.*

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# ЗАДАЧА КОШИ ДЛЯ МНОГОВРЕМЕННОГО ДРОБНОГО УРАВНЕНИЯ ДИФФУЗИИ

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**Ключевые слова:** задача Коши, многовременные уравнения, уравнение диффузии.