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PARTIAL DIFFERENTIAL EQUATIONS

Uniqueness of the Solution of the Cauchy Problem for the General Euler–Poisson–Darboux Equation

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Abstract—For the general Euler–Poisson–Darboux equation, we prove a theorem on the uniqueness of the solution of the Cauchy problem by the energy method. The solution of this problem turns out to be unique only for nonnegative values of the parameter k in the Bessel operator acting with respect to the time variable.

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INTRODUCTION

The main object of study in this article is the general Euler–Poisson–Darboux equation

$$(\Delta_\gamma)_x u = (B_k)_t u, \quad u = u(x,t), \quad t > 0, \quad x = (x_1, \dots, x_n), \tag{1}$$

where B_k is the singular differential Bessel operator (see, e.g., [1, p. 5])

$$(B_k)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t} = \frac{1}{t^k}\frac{\partial}{\partial t} t^k \frac{\partial}{\partial t}, \quad t > 0, \quad k \in \mathbb{R}, \tag{2}$$

and $\triangle_\gamma$ is a B-elliptic operator of the form

$$\triangle_\gamma = (\triangle_\gamma)_x = \sum_{i=1}^{n} (B_{\gamma_i})_{x_i}. \tag{3}$$

The general Euler-Poisson-Darboux equation is studied by methods that generalize the classical ones and has many applications, for example, in electrostatic field theory, hydrodynamics, elasticity theory, etc.

The solution of the singular Cauchy problem for Eq. (1) for an arbitrary real value of the parameter k is the subject of many studies. For $n = 1$ and $\gamma = 0$, Eq. (1) appeared in the work by L. Euler (see [2, p. 227]), later it was studied by S.D. Poisson [3] and G. Darboux [4]. The interest in the multidimensional equation (1) for the case in which the Laplace operator acts on the variable x originally arose in A. Weinstein's papers [5, 6], and its study was continued in the papers [7, 8]. The abstract Euler–Poisson–Darboux equation $Au = (B_k)_t u$, $u = u(x,t;k)$, where A is a linear operator acting only on x, was considered by A.V. Glushak [9, 10]. The books [11–13] studied the solvability of various problems for the classical Euler–Poisson–Darboux equation.

In the present paper, the uniqueness of the solution of the Cauchy problem for Eq. (1) for $k > 0$ is established by the energy method. For $k < 0$, the solution of this problem is not unique, but the set of solutions has a certain structure (see [14]).

1. MAIN DEFINITIONS AND ASSERTIONS

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ x_1 > 0, \ \dots, \ x_n > 0\},$$
$$\overline{\mathbb{R}}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ x_1 \geq 0, \ \dots, \ x_n \geq 0\},$$

$\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multiindex that consists of given positive numbers γ_i, $i = 1, \ldots, n$, and $|\gamma| = \gamma_1 + \ldots + \gamma_n$.

Consider an open set Ω in $\mathbb{R}^n$ symmetric about each hyperplane $x_i = 0$, $i = 1, \ldots, n$. Set $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}^n_+$; then $\Omega_+ \subseteq \mathbb{R}^n_+$ and $\overline{\Omega}_+ \subseteq \overline{\mathbb{R}}^n_+$. Let $C^m(\Omega_+)$ be the set of functions m times differentiable on Ω_+. By $C^m(\overline{\Omega}_+)$ we denote the subset of functions in $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i = 1, \ldots, n$ can be extended continuously to the plane $x_i = 0$. The class $C^m_{ev}(\overline{\Omega}_+)$ consists of functions $f \in C^m(\overline{\Omega}_+)$ such that $\partial^{2k+1} f/\partial x_i^{2k+1}|_{x=0} = 0$ for all nonnegative integers $k \le m$ and for $i = 1, \ldots, n$ (see [1, p. 21 ff.]).

Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be the unit vectors along the axes $x_1, x_2, \ldots, x_n$, respectively, let

$$\nabla'_\gamma = \left(\frac{1}{x_1^{\gamma_1}}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{x_n^{\gamma_n}}\frac{\partial}{\partial x_n}\right) = \sum_{i=1}^n \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i} e_i$$

be the first weighted nabla operator, and let

$$\nabla''_\gamma = \left(x_1^{\gamma_1}\frac{\partial}{\partial x_1}, \ldots, x_n^{\gamma_n}\frac{\partial}{\partial x_n}\right) = \sum_{i=1}^n x_i^{\gamma_i}\frac{\partial}{\partial x_i} e_i$$

be the second weighted nabla operator; then $(\nabla'_\gamma \cdot \nabla''_\gamma) = \Delta_\gamma$. We have

$$\nabla'_\gamma(uv) = u\nabla'_\gamma v + v\nabla'_\gamma u. \tag{4}$$

To prove the uniqueness of the solution of the Cauchy problem for Eq. (1), we need the generalized divergence theorem in [15].

Theorem 1. *Let G^+ be a domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i = 0$ $i = 1, \ldots, n$, either does not meet G^+ or has one common segment with G^+ (possibly degenerating into a point) of the form*

$$\alpha_i(x') \le x_i \le \beta_i(x'), \quad x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad i = 1, \ldots, n.$$

If $\vec{g} = (g_1(x), \ldots, g_n(x))$ is a vector field continuously differentiable in the domain G^+ and $\vec{F} = (F_1(x), \ldots, F_n(x))$, $F_1(x) = x_1^{\gamma_1} g_1(x), \ldots, F_n(x) = x_n^{\gamma_n} g_n(x)$, then one has the formula

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F}) x^\gamma \, dx = \int\limits_{\partial G^+} (\vec{g} \cdot \vec{\nu}) x^\gamma \, dS, \tag{5}$$

where $\vec{\nu} = \vec{e}_1 \cos\eta_1 + \ldots + \vec{e}_n \cos\eta_n$ is the outward normal vector to the surface ∂G^+ and η_i is the angle between the vector $\vec{\nu}$ and the axis Ox_i, $i = 1, \ldots, n$.

In the subspace $\mathbb{R}^n_+$, consider the multidimensional generalized shift corresponding to the multiindex γ; it has the form

$${}^\gamma\mathbf{T}^y_x = {}^{\gamma_1} T^{y_1}_{x_1} \cdots {}^{\gamma_n} T^{y_n}_{x_n},$$

where each of the one-dimensional generalized shifts is defined by the expression

$${}^{\gamma_i}T^{y_i}_{x_i} f(x) = \frac{\Gamma((\gamma_i+1)/2)}{\Gamma(\gamma_i/2)\Gamma(1/2)} \int\limits_0^\pi f\left(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\alpha_i}, x_{i+1}, \ldots, x_n\right) \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i.$$

Based on the multidimensional generalized shift ${}^\gamma\mathbf{T}^y$, we construct a weighted spherical mean of the function f, which, for $n \ge 2$, has the form

$$M^\gamma_t[f(x)] = \frac{1}{\left|S^+_1(n)\right|_\gamma} \int\limits_{S^+_1(n)} {}^\gamma\mathbf{T}^{t\theta}_x f(x) \theta^\gamma \, dS, \tag{6}$$

where $\theta^\gamma=\prod_{i=1}^n\theta_i^{\gamma_i}$, $S_1^+(n)=\{\theta:|\theta|=1,\ \theta\in\mathbb{R}_+^n\}$ is part of a sphere in $\mathbb{R}_+^n$, and

$$\left|S_1^+(n)\right|_\gamma=\frac{\prod_{i=1}^n\Gamma\big((\gamma_i+1)/2\big)}{2^{n-1}\Gamma\big((n+|\gamma|)/2\big)}.$$

For $n=1$, we set

$$M_t^\gamma\big[f(x)\big]={}^\gamma\,\mathbf{T}_x^{t\theta}f(x). \tag{7}$$

Let $L_p^\gamma(\mathbb{R}_+^n)=L_p^\gamma$, $1\le p<\infty$, be the space of all functions measurable on $\mathbb{R}_+^n$, even in each of the variables x_i, $i=1,\dots,n$, and such that

$$\int\limits_{\mathbb{R}_+^n}\big|f(x)\big|^p x^\gamma\,dx<\infty;$$

here and in the following, $x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}$. For real numbers $1\le p<\infty$, the norm of a function f in L_p^γ is defined by the formula

$$\|f\|_{L_p^\gamma(\mathbb{R}_+^n)}=\|f\|_{p,\gamma}=\left(\int\limits_{\mathbb{R}_+^n}\big|f(x)\big|^p x^\gamma\,dx\right)^{1/p}.$$

For $p=\infty$, the norm of the function f in the space L_∞^γ has the form

$$\|f\|_{L_\infty^\gamma(\mathbb{R}_+^n)}=\|f\|_{\infty,\gamma}=\operatorname*{ess\,sup}_{x\in\mathbb{R}_+^n}\big|f(x)\big|.$$

It is well known [1, p. 42] that L_p^γ is a Banach space.

The operator M_t^γ is bounded in $L_p^\gamma(\mathbb{R}_+^n)$ for $1\le p\le\infty$. Moreover, one has the inequality

$$\|M_t^\gamma u\|_{p,\gamma}\le\|u\|_{p,\gamma},\quad t>0.$$

In the monograph [1], I.A. Kipriyanov presented a B-polyharmonic function $u=u(x)=u(x_1,\dots,x_n)$ of order p such that $\Delta_\gamma^p u=0$, where Δ_γ is the operator (3). A function that is B-polyharmonic of the first order is said to be B-harmonic.

2. UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR THE GENERAL EULER–POISSON–DARBOUX EQUATION

Consider the Lorentz distance Γ between the points (x,t) and (ξ,τ) of the singular hyperplane,

$$\Gamma(x,t;\xi,\tau)=(t-\tau)^2-\sum_{i=1}^m(x_i-\xi_i)^2.$$

Let (ξ,τ) be a point in $\mathbb{R}_+^{n+1}$. By G^+ we denote the part of a conical domain in $\mathbb{R}_+^{n+1}$ bounded by the lower cavity of the cone $\Gamma(x,t;\xi,\tau)=0$ with vertex at the point (ξ,τ) and by the planes $x_i=0$, $i=1,\dots,n$, and $t=0$.

For $t=0$ we obtain the base of G^+ in $\mathbb{R}_+^n$, which is a ball (part of a ball) $B_n^+(\xi,\tau)$ centered at the point ξ of radius τ, $B_n^+(\xi,\tau)=\{x\in\mathbb{R}_+^n:|x-\xi|\le\tau\}$.

Theorem 2. *Let u be a function in $C_{ev}^2(\overline{G^+})$ satisfying the general Euler–Poisson–Darboux equation*

$$(\Delta_\gamma)_x u=(B_k)_t u,\quad u=u(x,t;k) \tag{8}$$

in G^+, *and let us assume that* $k \geq 0$ *and the functions* u *and* u_t *are zero on the base* G^+; *i.e.*,

$$u(x,0;k) = u_t(x,0;k) = 0, \quad x \in B_n^+(\xi,\tau); \tag{9}$$

then $u(x,t;k)$ *is zero in the domain* $\overline{G^+}$.

Proof. Take an arbitrary point $(\widetilde{x},\widetilde{t})$ inside or on the boundary of the set $\overline{G^+}$ and construct a new cone (part of the cone) $(t-\widetilde{t})^2 = \sum_{i=1}^m (x_i - \widetilde{x}_i)^2$. Denote by D^+ the part of the conical domain in $\mathbb{R}_+^{n+1}$ bounded by the lower cavity of the cone $(t-\widetilde{t})^2 = \sum_{i=1}^m (x_i - \widetilde{x}_i)^2$ with vertex at $(\widetilde{x},\widetilde{t})$ and the planes $x_i = 0$, $i = 1,\dots,n$ and $t = 0$. The domain D^+ is bounded in the plane $t = 0$ by the ball (part of the ball) $B_n^+(\widetilde{x},\widetilde{t})$, which is part of the original ball (part of the ball) $B_n^+(\xi,\tau)$; therefore, relations (9) hold in $B_n^+(\widetilde{x},\widetilde{t})$.

We multiply Eq. (8) by u_t and transform it as follows:

$$\begin{aligned} 0 &= u_t(B_k)_t u - u_t \Delta_\gamma u = u_t \cdot u_{tt} + \frac{k}{t}u_t^2 - (\nabla'_\gamma \cdot u_t \nabla''_\gamma u) + \partial_t\left(\frac{1}{2}|\nabla u|^2\right) \\ &= \partial_t\left(\frac{1}{2}u_t^2\right) + \frac{k}{t}u_t^2 - (\nabla'_\gamma \cdot u_t \nabla''_\gamma u) + \partial_t\left(\frac{1}{2}|\nabla u|^2\right) \\ &= \partial_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right) + \frac{k}{t}u_t^2 - (\nabla'_\gamma \cdot u_t \nabla''_\gamma u). \end{aligned} \tag{10}$$

Here we have used relations obtained from (4); namely

$$u_t \Delta_\gamma u = (\nabla'_\gamma \cdot u_t \nabla''_\gamma u) - (\nabla'_\gamma u_t \cdot \nabla''_\gamma u)$$

and

$$\begin{aligned} (\nabla'_\gamma u_t \cdot \nabla''_\gamma u) &= \left(\frac{1}{x_1^{\gamma_1}}\frac{\partial u_t}{\partial x_1},\dots,\frac{1}{x_n^{\gamma_n}}\frac{\partial u_t}{\partial x_n}\right) \cdot \left(x_1^{\gamma_1}\frac{\partial u}{\partial x_1},\dots,x_n^{\gamma_n}\frac{\partial u}{\partial x_n}\right) = \sum_{i=1}^n \left(\frac{1}{x_i^{\gamma_i}}\frac{\partial u_t}{\partial x_i}\right) \cdot \left(x_i^{\gamma_i}\frac{\partial u}{\partial x_i}\right) \\ &= \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} = \sum_{i=1}^n \partial_t\left(\frac{1}{2}\frac{\partial u}{\partial x_i}\right)^2 = \partial_t \sum_{i=1}^n \left(\frac{1}{2}\frac{\partial u}{\partial x_i}\right)^2 = \partial_t\left(\frac{1}{2}|\nabla u|^2\right). \end{aligned}$$

Let us integrate Eq. (10) over the domain D^+ and apply formula (5) by setting

$$\begin{aligned} \vec{F} &= \left(\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right),\ -u_t x_1^{\gamma_1}\frac{\partial u}{\partial x_1},\dots,\ -u_t x_n^{\gamma_n}\frac{\partial u}{\partial x_1}\right), \\ \vec{g} &= \left(\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right),\ -u_t\frac{\partial u}{\partial x_1},\dots,\ -u_t\frac{\partial u}{\partial x_n}\right); \end{aligned}$$

as a result, we obtain

$$\begin{aligned} 0 &= \int\limits_{D^+} \left(\partial_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right) + \frac{k}{t}u_t^2 - (\nabla'_\gamma \cdot u_t \nabla''_\gamma u)\right) x^\gamma \, dt\, dx \\ &= \int\limits_{\partial D^+} \left(\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)\cos\eta_0 - \sum_{i=1}^n u_t \frac{\partial u}{\partial x_i}\cos\eta_i\right) x^\gamma \, dS + \int\limits_{D^+} \frac{k}{t}u_t^2 x^\gamma \, dt\, dx \\ &= \frac{1}{2}\int\limits_{\partial D^+} \left(u_t^2\cos\eta_0 - 2u_t\sum_{i=1}^n \frac{\partial u}{\partial x_i}\cos\eta_i + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 \cos\eta_0\right) x^\gamma \, dS + \int\limits_{D^+} \frac{k}{t}u_t^2 x^\gamma \, dt\, dx, \end{aligned}$$

where $\vec{n} = (\cos\eta_0, \cos\eta_1, \dots, \cos\eta_n)$ is the outward normal vector to the surface ∂D^+, η_0 is the angle between the vector $\vec{n}$ and the axis Ot, and η_i is the angle between the vector $\vec{n}$ and the axis Ox_i, $i = 1, \dots, n$; moreover, $\cos\eta_0 = 1/\sqrt{2}$. Let us multiply the last equality by $\cos\eta_0$. Taking into account the fact that $\sum_{i=0}^n \cos\eta_i^2 = 1$ and $1/2 = \cos\eta_0^2 = 1 - \cos\eta_0^2 = \sum_{i=1}^n \cos\eta_i^2$, we have

$$0 = \frac{1}{2}\sum_{i=1}^n \int\limits_{\partial D^+} \left(u_t \cos\eta_i - \frac{\partial u}{\partial x_i}\cos\eta_0\right)^2 x^\gamma\, dS + \int\limits_{D^+} \frac{k}{t} u_t^2 x^\gamma\, dt\, dx.$$

On the plane $t = 0$, we have $u_t(x, 0) = 0$. Since $k \ge 0$, $t > 0$, from the last equality we conclude that on the lateral surface of the cone (part of the cone) ∂D^+ we have the identities

$$u_t \cos\eta_i - \frac{\partial u}{\partial x_i}\cos\eta_0 \equiv 0$$

and $u_t \equiv 0$ in D^+. It follows that $\partial u/\partial x_i \equiv 0$, $i = 1, \dots, n$. This means that on the lateral surface of the cone (part of the cone) ∂D^+ the vector $\operatorname{grad} u$ is parallel to the normal. Take an arbitrary point (x, t) on ∂D^+ and draw the generator ℓ through it. The vector $\operatorname{grad} u$ is orthogonal to ℓ, and so $\partial u/\partial\ell = 0$. This means that u is constant along any generator of the lateral surface of the cone (part of the cone) ∂D^+ and the value of u at the vertex $(\widetilde{x}, \widetilde{t})$ is equal to the value u at a point of the generatrix ℓ that lies in the plane $t = 0$. However, by conditions (9) we have $u(x, 0; k) = 0$, and hence $u(\widetilde{x}, \widetilde{t}; k) = 0$. Since the point $(\widetilde{x}, \widetilde{t})$ was taken arbitrarily in $\overline{G^+}$, we conclude that $u(x, t; k) \equiv 0$ in $\overline{G^+}$. The proof of the theorem is complete.

Corollary. *Let $(\widetilde{x}, \widetilde{t})$ be a point, and let G^+ be the domain described in Theorem 2. Assume that two functions u_l and u_2 in the class $C^2_{ev}(\overline{G^+})$ satisfy Eq. (8) in G^+; moreover, $u_1(x, 0) = u_2(x, 0)$ and $\partial u_1/\partial t|_{t=0} = \partial u_2/\partial t|_{t=0} = 0$. Then $u_1 \equiv u_2$ in $\overline{G^+}$.*

Combining the result in Theorem 2 and the results in [16], we obtain the following assertions.

Theorem 3. *Assume that the domain G^+ has the form described in Theorem 2, the point (x, t) is inside or on the boundary of the set $\overline{G^+}$, and $u \in C^2_{ev}(\overline{G^+})$. Then for $k \ge n + |\gamma| - 1$ the unique solution of the problem*

$$\triangle_\gamma u(x, t) = (B_k)_t u, \quad u = u(x, t; k),$$
$$u(x, 0; k) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

has the form

$$u(x, t; k) = \frac{2t^{1-k}\Gamma\big((k+1)/2\big)}{\Gamma\big((k - n - |\gamma| + 1)/2\big)\Gamma\big((n + |\gamma|)/2\big)} \int\limits_0^t (t^2 - r^2)^{(k-n-|\gamma|-1)/2} r^{n+|\gamma|-1} M_r^\gamma f(x)\, dr,$$

where $M_t^\gamma f(x)$ is the weighted spherical mean defined by Eq. (6) or (7).

Let $k \ge n + |\gamma| - 1$ and $1 \le p \le \infty$; then the solution of the Cauchy problem $u = u(x, t; k)$ in Theorem 3 with the initial function $f \in L_p^\gamma(\mathbb{R}^n_+)$ admits the estimate

$$\big\|u(\,\cdot\,, t; k)\big\|_{p,\gamma} \le C_{n,\gamma,k}\|f\|_{p,\gamma}, \quad t > 0.$$

Moreover, $\lim\limits_{t\to 0} u(x, t; k) = f(x)$ for almost all $x \in \mathbb{R}^n_+$.

Theorem 4. *Let the domain G^+ have the form described in Theorem 2; let the point (x, t) be located inside or on the boundary of the set $\overline{G^+}$, and let $u \in C_{ev}^{2+[(n+|\gamma|-k)/2]}(\overline{G^+})$. The solution of the Cauchy problem*

$$\triangle_\gamma u(x, t) = (B_k)_t u, \quad u = u(x, t; k), \tag{11}$$

$$u(x,0;k)=f(x),\qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0 \tag{12}$$

for $k<n+|\gamma|-1,\ k\neq -1,-3,-5,\ldots,$ *has the form*

$$u(x,t;k)=t^{1-k}\left(\frac{\partial}{t\,\partial t}\right)^{m}(t^{k+2m-1}u(x,t;k+2m)), \tag{13}$$

where m *is the minimum integer such that* $m\geq \dfrac{n+|\gamma|-k-1}{2}$ *and* $u(x,t;k+2m)$ *is the solution of the Cauchy problem*

$$(B_{k+2m})_t u=(\Delta_\gamma)_x u,$$
$$u(x,0;k+2m)=\frac{f(x)}{(k+1)(k+3)\cdots(k+2m-1)},\qquad u_t(x,0;k+2m)=0.$$

The solution (13) is unique for $k\geq 0$ *and nonunique for* $k<0$. *If* f *is a* B*-polyharmonic function of order* $(1-k)/2$ *and* $f\in C^{1-k}_{ev}$, *then one of the solutions of the Cauchy problem* (11), (12) *for* $k=-1,-3,-5,\ldots$ *has the form*

$$u(x,t;k)=f(x),\qquad k=-1,$$
$$u(x,t;k)=f(x)+\sum_{h=1}^{-(k+1)/2}\frac{\Delta_\gamma^h f}{(k+1)\cdots(k+2h-1)}\,\frac{t^{2h}}{2\cdot 4\cdots 2h},\qquad k=-3,-5,\ldots$$

CONCLUSIONS

The above theorem on the uniqueness of the solution of the Cauchy problem for the general Euler–Poisson–Darboux equation, proved by the energy method, supplements the results of studies of problems for singular hyperbolic equations.

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