On the Huygens principle in the Puu model

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Abstract. We consider the macroeconomic model of T. Puu, describing the fluctuations of gross income in a given region. With a special combination of savings and investment rates, income deviations will take place only for a finite period of time, after which the income will return to a stationary state. This effect is known in mathematical physics as the Huygens principle. We investigated this model using statistical methods of data analysis. The obtained results suggest that the Huygens effect hypothesis in certain historical periods of the Russian economy is plausible. We have also considered the structure of the set of stationary zeros for nontrivial solutions to a stationary equation.

1. Introduction

The Huygens principle is a phenomenon that is well-known among researchers in the hyperbolic partial differential equations and wave processes field. In accordance to J. Hadamard [1] and I.G. Petrovskii (see [2], p. 353–354) an initial-value problem (the Cauchy problem for a hyperbolic equation) satisfies the Huygens principle in the narrow sense if for any point in space, the dimension of the solution dependence domain on the initial data is less then the initial data space dimension. This phenomenon takes place for the 3-d wave equation describing the sound and light propagation. It manifests itself in the fact that vibrations caused by a localized source occur within a finite period of time with no aftereffect, so the wave has a sharp primary wave front and a sharp secondary wave front.

Implementation of the Huygens principle in a narrow sense is rare, it depends on the type of wave process, and on the properties of space (dimension, homogeneity). So, on the plane, the principle ceases to work even for the wave equations, and in three-dimensional space does not hold for sound waves propagating in an inhomogeneous medium. A good example of a secondary wave front absence is the waves spreading on the surface of water: the instantaneous source is fixed at another point on the surface for a long time. You can observe this phenomenon when you throw pebbles in water. A pebble thrown in water creates a lot of concentric circles which are the wave on the water surface. And this wave, reaching another point on the water surface, is observed over there for a long time. Only gradually water calms down again.

Let us state more precisely Hadamard's formulation of the Huygens principle, following [3]. Consider the hyperbolic equation of the second order

$$L[u] \equiv \sum_{i=0}^{n} \sum_{j=0}^{n} g_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0. \qquad (1)$$

Let us set the Cauchy problem for equation (1)

$$L[u] = 0,\ u\mid_M = \varphi,\ \frac{\partial u}{\partial \nu} = \psi. \tag{2}$$

Here M is a space-like manifold of dimension n, called the initial manifold, ν is the unit normal vector for the manifold M. We say that equation (1) (operator L) satisfies the Huygens principle if the solution of the Cauchy problem (2) at each point x depends on initial data values only at the intersection of the initial manifolds M with characteristic conoid with vertex at the point x.

Hadamard [1] formulated the problem consisted in finding all differential equations describing wave processes for which the Huygens principle in a narrow sense fulfilled. From the moment the problem was formulated, a lot of works had appeared to promote its solution. This article is not intended to be an exhaustive review, pursuing only the aim to present the applied results of the authors. Regarding the development of the problem of the Huygens principle, see, for example, results [4–6].

The economic theory recognizes the undulating nature of economic systems development. In this regard it seems appropriate to ask ourselves whether the Huygens principle can appear in the economy.

2. Materials and methods
In this paper we use classical methods of mathematical analysis and theory of partial differential equations.

3. Model description
In accordance to [7], denote by $Y = Y(x, y, t)$ the deviation of the level of income from the stationary state at the point (x, y) at the moment t. To be more precise, $Y = Y(x, y, t)$ is a contribution to gross domestic product, nevertheless, we use terminology of [7]. Suppose that savings S are in a given ratio $s = s(t)$ to income. Let $v = v(t)$ be the ratio between the fixed capital and income. By I we denote investments, which, by definition, are the rate of change in fixed capital. Therefore,

$$I = v\frac{\partial Y}{\partial t}, \tag{3}$$

$$S = s\frac{\partial Y}{\partial t}. \tag{4}$$

Assume that there is an adaptive process in which income increases pro rata to the difference between component, proportional to investments and savings: $\dot{Y} \sim (\alpha I - S)$. Suppose that a similar delay regulates investments:

$$\frac{\partial Y}{\partial t} = \alpha I - S = \alpha I - sY, \tag{5}$$

$$\frac{\partial I}{\partial t} = v\frac{\partial Y}{\partial t} - \beta I. \tag{6}$$

Differentiating (5) with respect to t, we get:

$$\ddot{Y} = \dot{\alpha} I + \alpha \dot{I} - \dot{s}Y - s\dot{Y}. \tag{7}$$

Expressing $\dot{I}$ from (6) and substituting in (7), we obtain:

$$\ddot{Y} = (\alpha v - s)\dot{Y} - \dot{s}Y + (\dot{\alpha} - \alpha\beta)I. \tag{8}$$

Now, expressing I from (5), we finally get the equation of income:

$$\alpha\ddot{Y} - (\alpha^2 v - \alpha s + \dot{\alpha} - \alpha\beta)\dot{Y} + (\alpha\beta s + \alpha\dot{s} - \dot{\alpha}s)Y = 0. \tag{9}$$

Equation (9) includes the geographic coordinates as simple parameters, therefore, (9) is an ordinary differential equation. If we add the assumption of inter-regional trade to (5)–(6), then, in the first approximation, the trade balance will be defined by the expression

$$m\Delta Y = m\left(\frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2}\right), \tag{10}$$

where m is the factor of proportionality characterizing propensity to importation. Under these assumptions, equation (9) will be replaced by the partial differential equation:

$$\alpha\ddot{Y} - (\alpha^2 v - \alpha s + \dot{\alpha} - \alpha\beta)\dot{Y} + (\alpha\beta s + \alpha\dot{s} - \dot{\alpha}s)Y = m\Delta Y. \tag{11}$$

In monograph [7] it was assumed that the coefficients s, v were constant, $\alpha = \beta = 1$, thus equation (11) took the form

$$\frac{\partial^2 Y}{\partial t^2} - (v - 1 - s)\frac{\partial Y}{\partial t} + sY = m\Delta Y. \tag{12}$$

4. A manifestation of the Huygens principle

Let us consider a special case of setting the coefficients in equation (11). Assuming that α and β are constant, we put

$$s = s(t) = s_0 e^{-\beta t}, \tag{13}$$

$$v = v(t) = \frac{s_0}{\alpha}e^{-\beta t} + \frac{\beta}{\alpha} - \frac{1}{\alpha t}\ . \tag{14}$$

Then equation (11) takes the form

$$\frac{\partial^2 Y}{\partial t^2} + \frac{1}{t}\frac{\partial Y}{\partial t} = \frac{m}{\alpha}\Delta Y. \tag{15}$$

Equation (15) is a singular equation known as the Euler-Poisson-Darboux equation (see [11]). The left side of this equation contains the Bessel operator $B_\gamma = \partial^2/\partial t^2 + \gamma/x\ \partial/\partial t, \gamma = 1$. Singular partial differential equations with the Bessel operator were studied by I. A. Kipriyanov and his students [8–11].

There is, it should be noted, a qualitative feature that arises when considering equations (15): the Huygens principle appears for the equation with an even number of spatial variables [12–14].

In our case, by direct verification, it is easy to see that the function

$$Y(x,y,t) = \frac{1}{2\pi t\sqrt{m/\alpha}} \oint\limits_{S_{\sqrt{m/\alpha}\ t}(x,y)} \varphi(\xi,\eta)\, dl_{\xi\eta} =$$

$$= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \varphi(x + \sqrt{\frac{m}{\alpha}}\ t\cos\theta, y + \sqrt{\frac{m}{\alpha}}\ t\sin\theta)\, d\theta, \tag{16}$$

where $S_{\sqrt{m/\alpha}\ t}(x,y)$ is a circle on a plane centered at the point (x,y) with the radius $\sqrt{m/\alpha}\ t$, is a regular solution of equation (15) and satisfies the initial conditions

$$\lim_{t\to+0} Y(x,y,t) = \varphi(x,y), \lim_{t\to+0}\frac{\partial Y}{\partial t}(x,y,t) = 0. \tag{17}$$

Table 1. Macroeconomic indicators of Russia for 1992–1995.

INDICATORS	1992	1993	1994	1995
GDP nominal, billion rubles (Y)	19000	172000	611000	1429000
Savings, billion rubles (S)	10,024	63,3968	183,1425	396,8088
Fixed investment, billion rubles (I)	2700	27100	108800	267000

Table 2. Macroeconomic indicators of Russia for 1996–1998.

INDICATORS	1996	1997	1998
GDP nominal, billion rubles (Y)	2008000	2343000	2630
Savings, billion rubles (S)	543,487	513,8346	0,50809
Fixed investment, billion rubles (I)	376000	409000	407

Now let's pay attention to the following specificity in the definition of function (16). In order to determine the value of the income deviation Y at the point (x, y) at the time t, it is sufficient to set the values of the function $\varphi(x, y)$ only on the circumference $S_{\sqrt{m/\alpha}\,t}(x, y)$. An open circle with the boundary $S_{\sqrt{m/\alpha}\,t}(x, y)$ will be a lacuna, that is, a set of points, on which a change in the initial data will not entail a change in the solution at the point (x, y) at the time t. The external part of this circle will also be a lacuna. In other words, the domain of dependence of the solution to Cauchy problem (15), (17)at the point (x, y) at the time t is the circle $S_{\sqrt{m/\alpha}\,t}(x, y)$. Therefore, the dimension of the domain of dependence to the solution on the initial data is less than the dimension of the initial data manifold, that is, the Huygens principle takes place [15,16]. From the point of view of the application considered here, it means the following. The initial deviation of income, localized on the plane, will cause the deviation of income localized in time at points in the plane. That is, income deviations will take place only for a finite period of time, after which the income will return to a stationary state. But then a similar situation would mean a kind of futility of the efforts of these subjects. We leave aside the possibility of creating conditions under which the input data could be determined by formulas (13) and (14). It seems to us that with a certain influence of some center on the behavior of subjects involved in commodity exchange, these conditions are quite feasible.

In order to build a specific version of the Puu macroeconomic model based on our assumptions, we take the indicators of the gross domestic product (GDP), population savings and fixed investment in Russia in a discrete measurement with an interval of one year for the period 1992–1998. We made calculations based on the statistical data given in Tables 1–2, which is compiled from the data of the Russian Federal State Statistics Service (Rosstat) [17].

The fact that the savings rate is given by the formula (13), should appear in a discrete measurement with an interval of one year in such a way that the annual values of the savings rate would be geometric progression. Certainly, that's not exactly possible. On the basis of qualitative economic analysis, it can be assumed that such dynamics of the savings rate took place for Russia in 1993–1998. Of course, this period is quite short, but it characterizes a certain stage in the development of the Russian economy. Using Rosstat data [17] on GDP, gross domestic product, savings and investments over the period, we calculated the rate of decline in the savings rate. It turned out that it was falling approximately in geometric progression (we got an adequate model with the coefficient of determination $R^2 = 0,89$). This suggests the *possible* presence of the Huygens principle in the domestic economy in 1993–1998. A similar analysis of data for some other countries of the former USSR (for example, Kazakhstan) over the same years did not suggest the presence of the Huygens principle in their economies.

The linear model considered by us, of course, misses many nuances of dynamics in the subject area. In addition, a partial differential equation satisfying the Huygens principle is a rare phenomenon. In our case, this "rarity" is manifested in the fact that even a small deviation of the values s and v from the form (13)–(14) will lead to the loss of the Huygens principle. In our opinion, the study of the Huygens principle in the economy could help in studying the influence of control actions on the reaction of the economic system in the short and long term, including the analysis of individual macroeconomic markets. In particular, taking into account Oukens law that links fluctuations in unemployment to GDP, we have every reason to believe that the Huygens principle, if it manifests itself in GDP fluctuations, will also appear in labor market fluctuations.

5. The stationary Puu equation

Assuming that in some domain $\Omega \subset \mathbb{R}^2$ the following conditions hold

$$\frac{\partial Y}{\partial t} = 0, \ \frac{\partial^2 Y}{\partial t^2} = 0, \ \alpha = \mathrm{const}, \ \beta = \mathrm{const}, \ s = \mathrm{const}, \ v = \mathrm{const}, \tag{18}$$

we obtain that equation (11) takes the form

$$m\Delta Y - \alpha\beta s Y = 0. \tag{19}$$

Equation (19) describes a stationary income distribution not depending on time. We say that the point (x^0, y^0) is the hopelessness point of the Puu model (19), if

$$Y(x^0, y^0) = 0, \ \nabla Y(x^0, y^0) = 0. \tag{20}$$

Equalities (20) should be regarded as the absence of deviation from the fixed income level and the absence of tendency to deviation. This justifies the name we have chosen. We say that the set of all hopelessness points is the hopelessness set of the Puu model. The following statement was proved in [19].

Theorem. *The set of stationary zeros of a nontrivial solution to the elliptic equation*

$$P(x, D)u \equiv \sum_{|\alpha|\leq 2} p_\alpha(x) D^\alpha u = 0, \quad x \in \Omega \subseteq \mathbb{R}^n, \ n \geq 3, \tag{21}$$

where $p_\alpha \in C^\infty(\Omega)$, $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})$, *in the neighborhood of any stationary zero is contained in a stratified manifold* $G = \bigcup_{k=0}^{r} M^k$, *where* M^k $(k = 0, 1, \ldots, r \leq n-2)$ *is a real analytic manifold of dimension* k. *In particular, when* $n = 3$ *is the intersection of a sufficiently small neighborhood of the stationary zero* x^0 *with the set of all other stationary zeros is a finite set of intervals with a common end at the point* x^0. *For* $n = 2$, *the set of stationary zeros of a nontrivial solution to elliptic equation (21) can contain only isolated points.*

In particular, it follows that the set of hopelessness of the Puu model can consist only of isolated hopelessness points.

6. An approximation of waves satisfying the Huygens principle by diffusion waves

We have already noted that the manifestation of the Huygens principle is a phenomenon quite rare among the varieties of wave processes. Therefore, it is very difficult to select the input data (coefficients) of the model and to implement it practically. Let us pay attention to one more aspect related to this: an approximate setting of the required coefficients will not lead to the Huygens principle. Next, we describe a particular case of such an approximation.

Consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial^2 u}{\partial x_k^2} + c^2 u,\ x \in \mathbb{R}^3,\ t > 0,\ c \geq 0, \tag{22}$$

$$u(x,0) = \varphi(x),\ x \in \mathbb{R}^3, \tag{23}$$

$$\frac{\partial u}{\partial t}(x,0) = \psi(x),\ x \in \mathbb{R}^3. \tag{24}$$

The solution of problem (22)–(24) is denoted by $u_c(x,t)$. It is known [18], that the function $u_c(x,t)$ can be expressed by the formula

$$u_c(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\int\limits_0^t \rho^2 J_0(ic\sqrt{t^2-\rho^2})Q(\varphi,x,\rho)\,d\rho\right) +$$

$$+\frac{1}{t}\frac{\partial}{\partial t}\int\limits_0^t \rho^2 J_0(ic\sqrt{t^2-\rho^2})Q(\psi,x,\rho)\,d\rho, \tag{25}$$

where

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\nu}}{2^{2k+\nu} k! \Gamma(k+\nu+1)}$$

is the Bessel function of the order ν,

$$J_\nu(iz) = \sum_{k=0}^{\infty} \frac{z^{2k+\nu}}{2^{2k+\nu} k! \Gamma(k+\nu+1)}$$

is the Bessel function of the order ν of an imaginary argument,

$$Q(\psi,x,\rho) = \frac{1}{4\pi\rho^2} \oint\limits_{|\xi-x|=\rho} \psi(\xi)\,dS_\xi$$

is the spherical mean of the function $\psi(x)$ over a sphere with the center at the point x and the radius ρ.

Note that the function $u_0(x,t)$ is fundamentally different from the function $u_c(x,t)$ with $c \neq 0$. Let's explain it. For the function $u_0(x,t)$ representation (25) can be converted to the Kirchhoff formula [18]:

$$u_0(x,t) = tQ(\psi,x,t) + \frac{\partial}{\partial t}(tQ(\varphi,x,t)). \tag{26}$$

If $c \neq 0$, the integration in formula (25) is performed on a ball centered at x and with the radius t, while in formula (26) the integration is conducted along the boundary of this ball, that is, the Huygens principle takes place.

We assume that for any reason we cannot create a prerequisite for fulfilling conditions (22)–(24) with $c = 0$, but we can create prerequisites for fulfillment of these conditions with $c \neq 0$. In this case, we will try to make the parameter c as small as possible. In this sense, c can be considered a "control parameter" by which we will approximate the function $u_0(x,t)$ (that is, the solution of the problem (22)–(24), satisfying the Huygens principle) by functions $u_c(x,t)$ (that is, solutions that do not satisfy the Huygens principle and have the property of wave diffusion).

Below, considering c as a control parameter, we evaluate the proximity of the functions $u_0(x,t)$ and $u_c(x,t)$ for a small $c \neq 0$ at small and large time intervals.

Theorem. *Let*

$$\varphi \in C^3(\mathbb{R}^3),\ \ \psi \in C^2(\mathbb{R}^3),\ \ |\varphi(x)| \leq M,\ \ |\psi(x)| \leq M,\ \ x \in \mathbb{R}^3.$$

Then the following estimates are valid:

$$|u_c(x,t) - u_0(x,t)| \leq M\left(\frac{\operatorname{sh}(ct) - ct}{c} + \operatorname{ch}(ct) - 1 - \frac{c^2t^2}{2}\right),$$

$$|u_c(x,t) - u_0(x,t)| \leq Lc^2t^3,$$

where L doesn't depend on c and t.

Proof. We have:

$$|u_c(x,t) - u_0(x,t)| \leq I_1 + I_2,$$

where

$$I_1 = \left|\frac{1}{t}\frac{\partial}{\partial t}\int\limits_0^t \rho^2 Q(\psi,x,\rho)(J_0(ic\sqrt{t^2-\rho^2}) - 1)\,d\rho\right|,$$

$$I_2 = \left|\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\int\limits_0^t \rho^2 Q(\varphi,x,\rho)(J_0(ic\sqrt{t^2-\rho^2}) - 1)\,d\rho\right)\right|.$$

We evaluate each of the terms I_1, I_2. At first, consider I_1. Calculating the derivative $\frac{\partial}{\partial t}$, taking into account that $J_0(0) = 1$, we obtain

$$I_1 = \left|\frac{1}{t}\int\limits_0^t \rho^2 Q(\psi,x,\rho)\frac{\partial}{\partial t}(J_0(ic\sqrt{t^2-\rho^2}) - 1)\,d\rho\right|.$$

Considering the definition of the Bessel function by means of a generalized power series, we obtain

$$I_1 = \left|\frac{1}{t}\int\limits_0^t \rho^2 Q(\psi,x,\rho)\frac{\partial}{\partial t}\sum_{k=1}^{\infty}\frac{c^{2k}(t^2-\rho^2)^k}{2^{2k}(k!)^2}\,d\rho\right| =$$

$$= \left|\int\limits_0^t \rho^2 Q(\psi,x,\rho)\sum_{k=1}^{\infty}\frac{c^{2k}(t^2-\rho^2)^{k-1}}{2^{2k-1}(k-1)!\,k!}\,d\rho\right|.$$

Taking into account that the function $\psi(x)$ is bounded, we get

$$I_1 \leq M\left|\sum_{k=1}^{\infty}\frac{c^{2k}}{2^{2k-1}(k-1)!\,k!}\int\limits_0^t \rho^2(t^2-\rho^2)^{k-1}\,d\rho\right|.$$

Calculating the integral in the right-hand side of this inequality, we obtain

$$I_1 \leq M\frac{\sqrt{\pi}}{4}\left|\sum_{k=1}^{\infty}\frac{c^{2k}t^{2k+1}}{2^{2k-1}k!\Gamma(k+3/2)}\right|,$$

and summing the series in the resulting inequality, we get the final estimate for I_1:

$$I_1 \leq M\frac{\operatorname{sh}(ct)-ct}{c}. \tag{27}$$

Let us now estimate the term I_2. Firstly, performing internal differentiation, we obtain

$$I_2 = \left|\frac{\partial}{\partial t}\left(\frac{1}{t}\int\limits_0^t \rho^2 Q(\varphi,x,\rho)\frac{\partial}{\partial t}(J_0(ic\sqrt{t^2-\rho^2})-1)\,d\rho\right)\right| =$$

$$= \left|\frac{\partial}{\partial t}\left(\frac{1}{t}\int\limits_0^t \rho^2 Q(\varphi,x,\rho)\sum_{k=1}^{\infty}\frac{c^{2k}(t^2-\rho^2)^{k-1}}{2^{2k-1}(k-1)!\,k!}\,d\rho\right)\right|.$$

Now we perform the external differentiation:

$$I_2 = \left|\frac{c^2t^2}{2}Q(\varphi,x,\rho) + t\sum_{k=1}^{\infty}\frac{c^{2k}}{2^{2k-2}(k-2)!\,k!}\int\limits_0^t Q(\varphi,x,\rho)\rho^2(t^2-\rho^2)^{k-2}\,d\rho\right|.$$

Further, taking into account boundedness of the function $\varphi(x)$, we get:

$$I_2 \leq M\left|\frac{c^2t^2}{2} + t\sum_{k=1}^{\infty}\frac{c^{2k}}{2^{2k-2}(k-2)!\,k!}\int\limits_0^t \rho^2(t^2-\rho^2)^{k-2}\,d\rho\right| = M\left(\operatorname{ch}(ct)-1-\frac{c^2t^2}{2}\right). \tag{28}$$

From (27) and (28) we finally obtain

$$|u_c(x,t)-u_0(x,t)| \leq M\left(\frac{\operatorname{sh}(ct)-ct}{c}+\operatorname{ch}(ct)-1-\frac{c^2t^2}{2}\right). \tag{29}$$

Taking into consideration the series expansion of the functions $\operatorname{sh}(ct)$ and $\operatorname{ch}(ct)$, we get

$$|u_c(x,t)-u_0(x,t)| \leq Lc^2t^3. \tag{30}$$

The theorem is proved.

Inequality (29) with large values of c and t to some extent characterizes the divergence of the difference $u_c(x,t)-u_0(x,t)$. Moreover, in this case, assessment (29) is exact in the sense that constant functions $\varphi(x)$ and $\psi(x)$ turn inequality (29) into the equality.

For small values c and t, inequality (30), on the contrary, evaluates the proximity of the functions $u_c(x,t)$ and $u_0(x,t)$.

7. On a family of singular hyperbolic equations satisfying the Huygens principle. We will point out one more case when the Huygens principle takes place in the Puu model. Here we assume that the tendency to importing m also depends on time, moreover $m = m(t) \neq 0,\ t > 0$. We write the income distribution equation in the form

$$a(t)\frac{\partial^2 Y}{\partial t^2} + b(t)\frac{\partial Y}{\partial t} + \frac{(s(t)+s'(t))}{m(t)}Y = \Delta Y, \quad t > 0, \tag{31}$$

assuming

$$\mu(t) = s(t) + 1 - v(t),\ a(t) = \frac{1}{m(t)}\ , \qquad b(t) = \frac{\mu(t)}{m(t)}\ . \tag{32}$$

Let

$$s(t) = v(t) = s_0 e^{-t}, \quad m(t) = 1 + \frac{1}{t}. \tag{33}$$

Then in equation (31)

$$a(t) = b(t) = \frac{t}{t+1}\ , \quad s(t) + s'(t) = 0\ . \tag{34}$$

We introduce the function

$$Y(x,y,t) = \frac{1}{2\pi t e^t}\oint\limits_{S_t(x,y)} \varphi(\xi,\eta)dl = \frac{1}{2\pi e^t}\int\limits_{-\pi}^{\pi}\varphi(x + t\cos\theta, y + t\sin\theta)d\theta. \tag{35}$$

By direct substitution, one can verify that the function, defined by formula (35), satisfies equation (31) with coefficients defined by formulas (34) and initial conditions

$$\lim_{t\to 0+} Y(x,y,t) = \varphi(x,y), \quad \lim_{t\to 0+}\left(t\frac{\partial Y(x,y,t)}{\partial t}\right) = 0. \tag{36}$$

Let us now consider the Cauchy problem of the following form:

$$\frac{\partial^2 u}{\partial t^2} = \Delta_B u + \frac{c}{x_n^2}u = 0,\ x_n > 0,\ t > 0, \tag{37}$$

$$u\ |_{t=0} = \varphi(x), \quad u_t\ |_{t=0} = \psi(x), \tag{38}$$

$$\frac{\partial u}{\partial x_n}\ |_{x_n=0} = 0,\ n = 1(\mathrm{mod}2),\ n \geq 3. \tag{39}$$

Here Δ_B is the B-elliptic operator defined by the formula

$$\Delta_B u = \sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_k^2} + \frac{\gamma}{x_n}\frac{\partial u}{\partial x_n} = \sum_{k=1}^{n-1}\frac{\partial^2 u}{\partial x_k^2} + B_{\gamma,x_n}u, \tag{40}$$

where $B_{\gamma,x_n} = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial u}{\partial x_n} = x_n^{-\gamma}\frac{\partial}{\partial x_n}x_n^{\gamma}\frac{\partial}{\partial x_n}$ is the Bessel operator. B-elliptic operators were studied by I.A. Kipriyanov and his students (see again [8–11]. and a bibliographic review there). It is possible to claim that the theory of B-elliptic operators has taken complete shape. Much less is known about the properties of solutions to hyperbolic equations with degeneration by spatial variables.

In work [4] the authors obtained a reproduction scheme for operators satisfying the Huygens principle. We give it here briefly in the edition necessary for our presentation. Let

$$L_{n+1}[v] \equiv \sum_{k=1}^{n}\frac{\partial^2 v}{\partial x_k^2} + k(x_n)v - \frac{\partial^2 v}{\partial t^2}, \tag{41}$$

$$\widetilde{L_{n+1}}[v] \equiv \sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_k^2} - \left(k(x_n)v + 2\left(\frac{\mu'(x_n)}{\mu(x_n)}\right)^2\right) v - \frac{\partial^2 v}{\partial t^2}, \tag{42}$$

where $\mu(\xi)$ is nontrivial solution of the equation

$$\mu'' + k(\xi)\mu = 0. \tag{43}$$

According to the Lagnes-Stelmacher theorem, if the operator L_{n+1} satisfies the Huygens principle then the operator $\widetilde{L_{n+3}}$ also satisfies the Huygens principle. Based on the results of work [4], it is not difficult to show that equation (37) satisfies the Huygens principle if and only if

$$(\gamma - 1)^2 - 1 - 4c - 4p(p+1) = 0, \quad p = 1, 2 \ldots, \frac{n-3}{2}\ . \tag{44}$$

Further, using the Lagnes-Stelmacher scheme, it is possible to obtain other equations satisfying the Huygens principle. Equation (37) can be written as

$$L_{n+1}[x_n^{\frac{\gamma}{2}} u] = 0, \tag{45}$$

where

$$k(\xi) = \frac{2\gamma - \gamma^2 + 4c}{4\xi^2} = -\frac{p(p+1)}{\xi^2}.$$

Then the function $\mu(\xi)$, determined by equation (43), will have the form

$$\mu(\xi) = A_1\xi^{-p} + A_2\xi^{p+1}, \quad A_1^2 + A_2^2 > 0,$$

therefore, the operator

$$\widetilde{L_{n+3}} = \sum_{k=1}^{n+2} \frac{\partial^2}{\partial x_k^2} - \frac{\partial^2}{\partial t^2} + \frac{p(p+1)}{x_n^2} - \left(\frac{pA_1 - (p+1)A_2 x_n^{2p+1}}{A_1 x_n + A_2 x_n^{2p+2}}\right)^2$$

satisfies the Huygens principle with the equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{n+1} \frac{\partial^2 u}{\partial x_k^2} + B_{\alpha, x_{n+2}} u + \left(\frac{q}{x_{n+2}} - \left(\frac{pA_1 - (p+1)A_2 x_n^{2p+1}}{A_1 x_n + A_2 x_n^{2p+2}}\right)^2\right) u,$$

where α and q are related by ratio

$$4q - (\alpha - 1)^2 + 1 = 4p(p+1).$$

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