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Discrete Boundary Value Problems as Approximate Constructions

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Abstract—We study some discrete boundary value problems for discrete elliptic pseudo-differential equations in a half-space. These statements are related with a special periodic factorization of an elliptic symbol and a number of boundary conditions depends on an index of periodic factorization. This approach was earlier used by authors for studying special types of discrete convolution equations. Here we consider more general equations and functional spaces.

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1. INTRODUCTION

We will consider a certain class of discrete operators and equations in some so-called canonical domains from Euclidean space $\mathbb{R}^m$. These operators are defined by a given function on the m-dimensional cube $\mathbb{T}^m = [-\pi, \pi]^m$, such a function is called a symbol of the discrete operator. Simple examples of such operators have the form

$$u_d(\tilde{x}) \longmapsto \sum_{\tilde{x} \in D_d} A_d(\tilde{x} - \tilde{y}) u_d(\tilde{y}), \quad \tilde{x} \in D_d,$$

where $D_d = h\mathbb{Z}^m \cap D$, h>0, D is a domain $D \subset \mathbb{R}^m$, A_d, u_d are functions of a discrete variable $\tilde{x} \in h\mathbb{Z}^m$, and the given function $A_d(\tilde{x})$ is called a kernel of the operator. Such operators and related ones are called discrete convolutions and were studied from different points of view in a lot of papers (see, for example, [2, 9, 10, 12–17]).

This paper is devoted to more general operators and equations related to the special canonical domain $D = \mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > 0\}$ although there are some first results for other canonical domains, for example $D = C^a_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > a|x'|, a > 0\}$ [19–21]. We will develop a certain discrete theory similar [3, 4].

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2. DIGITAL PSEUDO-DIFFERENTIAL OPERATORS

In this section we give some auxiliary data and definitions for our studies. Let $u_d(\tilde{x})$ be a function of a discrete variable $\tilde{x} \in h\mathbb{Z}^m, h > 0$. The discrete Fourier transform F_d of the function u_d is called the following series

$$(F_d u_d)(\xi) \equiv \tilde{u}(\xi) \equiv \sum_{\tilde{x} \in h\mathbb{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbb{T}^m, \quad \hbar \equiv h^{-1},$$

if the series converges. Evidently the function $\tilde{u}(\xi)$ is defined on $\mathbb{R}^m$, ant it is a periodic function with basic cube of periods $\hbar\mathbb{T}^m$; such functions we call periodic functions.

The Fourier transform is an isomorphism between $L_2(h\mathbb{Z}^m)$ and $L_2(\hbar\mathbb{T}^m)$, moreover

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} e^{-i\tilde{x}\cdot\xi} \tilde{u}(\xi) d\xi, \quad \tilde{x} \in h\mathbf{Z}^m.$$

Using divided differences and their Fourier transforms we introduce discrete functional spaces.

Definition 1. Discrete Sobolev–Slobodetskii space $H^s(h\mathbb{Z}^m), s \in \mathbb{R}$, consists of functions for which the following norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^m} (1 + |\hat{\zeta}^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}$$

is finite,

$$\hat{\zeta}^2 \equiv \hbar^2 \sum_{k=1}^{m} (e^{ih\xi_k} - 1)^2, \quad \hat{\zeta}_k = \hbar(e^{ih\xi_k} - 1)^2, \quad k = 1, 2, \ldots, m..$$

Definition 2. The discrete space $H^s(D_d)$ consists of functions from $H^s(h\mathbb{Z}^m)$ for which their supports belong to $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by the norm of $H^s(h\mathbb{Z}^m)$. The space $H^s_+(D_d)$ consists of functions of a discrete variable defined in D_d which admit continuation on the whole $H^s(h\mathbb{Z}^m)$. The norm in such a space is given by the formula $||u_d||^+_s = \inf ||\ell u_d||_s$, where infimum is taken over all continuations ℓ.

We will denote by $\widetilde{H}^s(D_d), \widetilde{H}^s(h\mathbb{Z}^m \setminus D_d)$ images of the spaces $H^s(D_d), H^s(h\mathbb{Z}^m \setminus D_d)$ under discrete Fourier transform F_d. Similar functional spaces were introduced and studied in the paper [11], there are a lot of their useful properties.

Let $\widetilde{A}_d(\xi)$ be a measurable periodic function with basic cube of periods $\hbar\mathbb{T}^m$. The function $\widetilde{A}_d(\xi)$ is called a symbol of digital pseudo-differential operator A_d, which is defined by the formula

$$(A_d u_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \sum_{\tilde{y} \in h\mathbb{Z}^m} \int\limits_{\hbar\mathbb{T}^m} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \widetilde{A}_d(\xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in h\mathbb{Z}^m.$$

The symbol $\widetilde{A}_d(\xi)$ is called an elliptic symbol if

$$ess \inf_{\xi \in \hbar\mathbb{T}^m} \widetilde{A}_d(\xi)| > 0.$$

We denoted by E_α the class of periodic symbols satisfying the condition

$$c_1(1 + |\hat{\zeta}^2|)^{\frac{\alpha}{2}} \leq |A_d(\xi)| \leq c_2(1 + |\hat{\zeta}^2|)^{\frac{\alpha}{2}} \tag{1}$$

with constants c_1, c_2 non-depending on h.

Remark 1. We use this definition taking into account in future limit transfer from discrete structure to continue one, and $|\hat{\zeta}^2| \sim |\xi|^2, h \to 0$.

Theorem 1. *A digital pseudo-differential operator with symbol $\tilde{A}_d(\xi) \in E_\alpha$ is a linear bounded operator $A_d : H^s(h\mathbb{Z}^m) \to H^{s-\alpha}(h\mathbb{Z}^m)$ with a norm non-depending on h.*

Each such operator generates the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{2}$$

and we will seek the solution $u_d \in H^s(D_d)$ for the given right-hand side $v_d \in H^s_+(D_d)$ and given operator A_d with symbol $\tilde{A}_d(\xi) \in E_\alpha$.

3. DISCRETE EQUATIONS IN A HALF-SPACE

In this section we describe an auxiliary technique for studying solvability of the equation (2) for the special case $D=\mathbb{R}^m_+$.

We will remind here the classical Hilbert transform and its connections with boundary properties of analytic functions [5–7] and will describe same properties of its periodic analogue.

The classical Hilbert transform is defined by the following one-dimensional singular integral

$$(Hu)(x)=p.v.\int\limits_{-\infty}^{+\infty}\frac{u(y)dy}{x-y},\quad x\in\mathbb{R}.$$

This transform plays key role under studying solvability of model elliptic pseudo-differential equations in a multidimensional half-space $\mathbb{R}^m_+=\{x\in\mathbb{R}^m: x=(x_1,\cdots,x_m), x_m>0\}$. Its periodic analogue is the following

$$(H^{per}u)(x)=\frac{1}{2\pi i}p.v.\int\limits_{-\pi}^{\pi}\cot\frac{x-y}{2}u(y)dy,\quad x\in[-\pi,\pi].$$

It was shown [14] that this periodic singular integral appears under studying discrete equations in the discrete half-space $\mathbb{Z}^m_+=\mathbb{Z}^m\cap\mathbb{R}^m_+$, also such integrals appear under summation of Fourier series [8].

Let us denote by P_+, P_- projection operators on $D_d, h\mathbb{Z}^m\setminus D_d$ respectively. To apply the discrete Fourier transform F_d to the equation (2) we need to know what are the operators F_dP_+, F_dP_-. It was done in papers [12, 14], and here we will briefly describe these constructions.

One can define a discrete analogue of the Schwartz space $S(h\mathbf{Z}^m)$ (see for example [11]) and introduce for such functions the following operators which are generated by periodic analogue of the Hilbert transform, $\xi=(\xi',\xi_m)$,

$$(H^{per}_{\xi'}\tilde{u}_d)(\xi)=\frac{h}{2\pi i}p.v.\int\limits_{-\hbar\pi}^{\hbar\pi}\cot\frac{h(\xi_m-\eta_m)}{2}\tilde{u}_d(\xi',\eta_m)d\eta_m,\quad \xi'\in\hbar\mathbb{T}^{m-1},$$

$$P^{per}_{\xi'}=1/2(I+H^{per}_{\xi'}),\quad Q^{per}_{\xi'}=1/2(I-H^{per}_{\xi'}).$$

Lemma 1. *We have the following relations*

$$F_dP_+=P^{per}_{\xi'}F,\quad F_dP_-=Q^{per}_{\xi'}F.$$

Lemma 1 implies that a solvability of the equation (2) is closely related to a solvability of one-dimensional singular integral equation with the periodic Hilbert transform and a parameter $\xi'\in\hbar\mathbb{T}^{m-1}$. Such an equation can be solved with the help of so called periodic Riemann problem [14] which is formulated in the following way.

Let us denote by $\Pi_\pm$ the upper and lower half-strips in a complex plane $\mathbb{C}$,

$$\Pi_\pm=\{z\in\mathbb{C}: z=t+is, t\in[-\pi,\pi],\ \pm s>0\}.$$

Finding two functions $\Phi^\pm(t), t\in[-\pi,\pi]$ (from appropriate functional spaces), which admit an analytical continuation into $\Pi_\pm$ and satisfy the linear relation

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t),\tag{3}$$

where $G(t), g(t)$ are given functions on $[-\pi,\pi], G(-\pi)=G(\pi), g(-\pi)=g(\pi)$. If $G(t)\equiv 1$ then the problem (3) is called a jump problem.

Lemma 2. *For $|s|<1/2$, the operators $P^{per}_{\xi'}, Q^{per}_{\xi'}$ are bounded projectors $P^{per}_{\xi'}:\widetilde{H}^s(h\mathbb{Z}^m)\to\widetilde{H}^s(D_d)$, $Q^{per}_{\xi'}:\widetilde{H}^s(h\mathbb{Z}^m)\to\widetilde{H}^s(h\mathbb{Z}^m\setminus D_d)$, and a jump problem has unique solution $\Phi^+\in\widetilde{H}^s(D_d)$, $\Phi^-\in\widetilde{H}^s(h\mathbb{Z}^m\setminus D_d)$ for arbitrary $g\in\widetilde{H}^s(h\mathbb{Z}^m)$,*

$$\Phi^+=P^{per}_{\xi'}g,\quad \Phi^-=-Q^{per}_{\xi'}g.$$

To study the general Riemann boundary value problem (3) we will use the following concept.

Definition 3. Periodic factorization of an elliptic symbol $\tilde{A}_d(\xi) \in E_\alpha$ is called its representation in the form $\tilde{A}_d(\xi) = \tilde{A}_{d,+}(\xi) \cdot \tilde{A}_{d,-}(\xi)$, where the factors $\tilde{A}_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$|\tilde{A}^{\pm 1}_{d,+}(\xi)| \leq c_1(1 + |\hat{\zeta}^2_\tau|)^{\pm \frac{\text{æ}}{2}}, \quad |\tilde{A}^{\pm 1}_{d,-}(\xi)| \leq c_2(1 + |\hat{\zeta}^2_\tau|)^{\pm \frac{\alpha - \text{æ}}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2_\tau \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number æ $\in \mathbb{R}$ is called an index of periodic factorization.

For some simple cases one can use the topological formula

$$\text{æ} = \frac{1}{2\pi} \int_{-\hbar\pi}^{\hbar\pi} d \arg \tilde{A}_d(\cdot, \xi_m),$$

where $\tilde{A}_d(\cdot, \xi_m)$ means that $\xi' \in \hbar\mathbb{T}^{m-1}$ is fixed, and the integral is the integral in Stieltjes sense. It means that we need to calculate divided by 2π variation of the argument of the symbol $\tilde{A}_d(\xi)$ when ξ_m varies from $-\hbar\pi$ to $\hbar\pi$ under fixed ξ'.

Example 1. Let $\tilde{A}_d(\xi) = k^2 + \hat{\xi}^2, k \in \mathbb{R}$, such that the condition (1) is satisfied, in other words A_d is the discrete Laplacian plus $k^2 I$. The variation of an argument mentioned above can be calculated immediately, and it equals to 1.

4. SOLVABILITY

As we will see the index of factorization very influences on the solvability picture of the equation (3).

Theorem 2. *If the elliptic symbol $\tilde{A}_d(\xi) \in E_\alpha$ admits periodic factorization with index æ so that $|\text{æ} - s| < 1/2$ then the the equation (2) has unique solution in the space $H^s(D_d)$ for arbitrary right-hand side $v_d \in H^{s-\alpha}(D_d)$.*

Proof. Let ℓv_d be an arbitrary continuation of v_d on the whole $h\mathbb{Z}^m$ so that $\ell v_d \in H^{s-\alpha}(h\mathbb{Z}^m)$. Let $w_d(\tilde{x}) = (\ell v_d)(\tilde{x}) - (A_d u_d)(\tilde{x})$ and rewrite $(A_d u_d)(\tilde{x}) + w_d(\tilde{x}) = (\ell v_d)(\tilde{x})$. Further applying the discrete Fourier transform F_d and using the periodic factorization we write

$$\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) + \tilde{A}^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) = \tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi).$$

According to Theorem 1 we have $\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-\text{æ}}(h\mathbb{Z}^m)$, $\tilde{A}^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-\alpha+\alpha-\text{æ}}(h\mathbb{Z}^m)$ and analogously $\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi) \in \widetilde{H}^{s-\text{æ}}(h\mathbb{Z}^m)$. Moreover, really $\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-\text{æ}}(D_d)$ in view of a holomorphic property, and accurate considerations with supports of $\tilde{A}_{d,-}(\xi)$ and $\tilde{w}_d(\xi)$ show that in fact $\tilde{A}^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-\text{æ}}(h\mathbb{Z}^m \setminus D_d)$.

Thus we obtain a variant of the jump problem for the space $\widetilde{H}^{s-\text{æ}}(h\mathbb{Z}^m)$ which can be solved by Lemma 2. According to this lemma we have

$$\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) = P^{per}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi))$$

or finally

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)P^{per}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)).$$

It finishes the proof. □

Remark 2. It is easy to see that the solution does not depend on choice of continuation ℓv_d.

Here we consider more complicated case when the condition $|æ - s| < 1/2$ does not hold. There are two possibilities in this situation, and we consider one case which leads to typical boundary value problems. The following result is obtained in [18].

Theorem 3. *Let $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. Then a general solution of the equation* (2) *in Fourier images has the following form*

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi) X_n(\xi) P^{per}_{\xi'}(X_n^{-1}(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ is an arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \cdots, m$, satisfying the condition (1), *$c_k(\xi'), j = 0, 1, \cdots, n-1$, are arbitrary functions from $H_{s_k}(h\mathbb{T}^{m-1}), s_k = s - æ + k - 1/2$.*

Theorem 3 implies that if we want to have a unique solution in the case $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$, we need some additional conditions to determine uniquely unknown functions $c_k(\xi'), k = 0, 1, \cdots, n-1$. This case we will discuss in the next section.

Corollary 1. *Let $æ - s = n + \delta, \in \mathbb{N}, |\delta| < 1/2,\ v_d \equiv 0$. A general solution of the equation* (2) *has the following form*

$$\tilde{u}_d(\xi', \xi_m) = \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m. \tag{4}$$

5. BOUNDARY VALUE PROBLEMS

This section is a direct continuation of the previous one and gives a statement of simple boundary value problem for the equation (2). We start from a formula for general solution for the equation (2) including unknown functions $c_k(\xi'), k = 0, 1. \cdots, n-1$. For simplicity we consider a homogeneous equation (2) and the formula (4) although all results will be valid for inhomogeneous case without additional special requirements.

Let us introduce the following boundary conditions

$$(B_{d,j}u_d)(\tilde{x}', 0) = b_{d,j}(\tilde{x}'), \quad j = 0, 1. \cdots, n-1, \tag{5}$$

where $B_{d,j}$ is digital pseudo-differential operators of order $\beta_j \in \mathbb{R}$ with symbols $\widetilde{B}_{d,j}(\xi) \in E_{\beta_j}$

$$(B_{d,j}u_d)(\tilde{x}) = \frac{1}{(2\pi)^m}\int_{\hbar\mathbf{T}^m}\sum_{\tilde{y}\in h\mathbf{Z}^m} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\widetilde{B}_j(\xi)\tilde{u}_d(\xi)d\xi.$$

One can rewrite boundary conditions (5) in Fourier images

$$\int_{-h^{-1}\pi}^{h^{-1}\pi} \widetilde{B}_{d,j}(\xi', \xi_m)\tilde{u}_d(\xi', \xi_m)d\xi_m = \tilde{b}_{d,j}(\xi'), \quad j = 0, 1. \cdots, n-1, \tag{6}$$

so that according to properties of digital pseudo-differential operators (Theorem 1) and trace properties [11] we need to require $b_{d,j}(\tilde{x}') \in H^{s-\beta_j-1/2}(h\mathbb{Z}^{m-1})$.

Let us denote

$$s_{jk}(\xi') = \int_{-\hbar\pi}^{\hbar\pi} \tilde{A}^{-1}_{d,+}(\xi)\widetilde{B}_{d,j}(\xi', \xi_m)\hat{\zeta}^k_m d\xi_m.$$

Now we can formulate the following result.

Theorem 4. *If $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$, then the boundary value problem* (2), (5) *has a unique solution in the space $H^s(D_d)$ for arbitrary $b_{d,j} \in H^{s-\beta_j-1/2}(h\mathbb{Z}^{m-1}), j = 0, \cdots, n-1$,* iff

$$\det(s_{kj}(\xi'))_{k,j=0}^{n-1} \neq 0, \quad \forall\, \xi' \in \mathbb{T}^{m-1}. \tag{7}$$

A priori estimate holds

$$||u_d||_s \leq c\sum_{j=0}^{n-1}[b_{d,j}]_{s-\beta_j-1/2},$$

where c *does not depend on* h*, and* $[\cdot]_s$ *denotes* H^s*-norm in the space* $H^s(h\mathbb{Z}^{m-1})$.

Proof. Substituting the general solution of the equation (2) into boundary conditions (6) we have

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}_{d,+}^{-1}(\xi)\widetilde{B}_{d,j}(\xi',\xi_m)\sum_{k=0}^{n-1}c_k(\xi')\hat{\zeta}_m^k d\xi_m = \tilde{b}_{d,j}(\xi'), \quad j=0,1.\cdots,n-1,$$

and further

$$\sum_{k=0}^{n-1}c_k(\xi')\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}_{d,+}^{-1}(\xi)\widetilde{B}_{d,j}(\xi',\xi_m)\hat{\zeta}_m^k d\xi_m = \tilde{b}_{d,j}(\xi'), \quad j=0,1.\cdots,n-1,$$

Thus, we obtain the following system of linear algebraic equations

$$\sum_{k=0}^{n-1}s_{jk}(\xi')c_k(\xi') = \tilde{b}_{d,j}(\xi'), \quad j=0,1.\cdots,n-1, \tag{8}$$

with respect to unknown functions $c_k(\xi'), k=0,1.\cdots,n-1$. The condition (7) is necessary and sufficient for a unique solvability of inhomogeneous system.

A priori estimates can be easily obtained using properties of pseudo-differential operators and appropriate properties of discrete H^s-spaces. □

The condition (7) is a variant of Shapiro–Lopatinskii condition [1].

6. A COMPARISON BETWEEN DISCRETE AND CONTINUOUS

The continuous analogue of the considered discrete boundary value problem

$$\begin{cases}(A_d u_d)(\tilde{x}) = 0,\\ B_{d,j}u_d|_{\tilde{x}_m=0} = b_{d,j}(\tilde{x}'),\end{cases} \tag{9}$$

$j=0,1,...,n-1$, is the following

$$\begin{cases}(Au)(x) = 0,\\ B_j u|_{x_m=0} = b_j(x'),\end{cases} \tag{10}$$

$j=0,1,...,n-1$, where A is a pseudo-differential operator with the symbol $\tilde{A}(\xi)$ satisfying the condition

$$c_1(1+|\xi|)^\alpha \leq |\tilde{A}(\xi)| \leq c_2(1+|\xi|)^\alpha,$$

$B_j, j=0,1,...,n-1$, are also pseudo-differential operators with symbols $\tilde{B}_j(\xi)$ satisfying similar condition

$$c_3(1+|\xi|)^{\beta_j} \leq |\tilde{B}_j(\xi)| \leq c_4(1+|\xi|)^{\beta_j}.$$

6.1. Solving the Problem (10)

Here we will remind some constructions from [1].

A solution of the problem (10) is constructed in the following way [1]. Using factorization for the symbol $\tilde{A}(\xi)$ with the index æ such that æ $- s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$,

$$\tilde{A}(\xi) = \tilde{A}_+(\xi) \cdot \tilde{A}_-(\xi)$$

we can write a general solution for the homogeneous pseudo-differential equation

$$\tilde{u}(\xi) = \tilde{A}_+^{-1}(\xi) \sum_{k=0}^{n-1} \tilde{C}_k(\xi')(i\xi_m)^k. \tag{11}$$

Then we use our boundary operators B_j and with properties of the Fourier transform we obtain the following relations

$$\int_{-\infty}^{\infty} \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m) \sum_{k=0}^{n-1} \tilde{C}_k(\xi')(i\xi_m)^k d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1. \cdots, n-1,$$

If we will introduce notations

$$S_{jk}(\xi') = \int_{-\infty}^{\infty} \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)(i\xi_m)^k d\xi_m,$$

we obtain a system of linear algebraic equations with respect to unknown functions $\tilde{C}_k(\xi'), k = 0, 1, ..., n-1$,

$$\sum_{k=0}^{n-1} S_{jk}(\xi')\tilde{C}_k(\xi') = \tilde{b}_j(\xi'), \quad j = 0, 1. \cdots, n-1. \tag{12}$$

The condition

$$ess \inf_{\xi' \in \mathbb{R}^{m-1}} |\det(S_{jk}(\xi'))| > 0$$

is necessary and sufficient condition for unique solvability of the boundary value problem (10).

Since the solutions u_d and u are fully determined by solutions c_k and C_k we need to compare the systems (8) and (12).

6.2. A Comparison

Lemma 3. *Let A and B be non-degenerated matrices and* $||A - C||_{\mathbb{R}6n \to \mathbb{R}^n} \sim \varepsilon$ *for enough small* $\varepsilon > 0$. *Then* $||A^{-1} - C^{-1}||_{\mathbb{R}6n \to \mathbb{R}^n} \sim \varepsilon$.

Proof. Indeed, the following property can be easily verified.

$$C^{-1} - A^{-1} = C^{-1}(A - C)A^{-1},$$

so that

$$||A^{-1} - C^{-1}||_{\mathbb{R}^n \to \mathbb{R}^n} \leq ||C^{-1}||_{\mathbb{R}^n \to \mathbb{R}^n} \, ||A - C||_{\mathbb{R}^n \to \mathbb{R}^n} \, ||A^{-1}||_{\mathbb{R}^n \to \mathbb{R}^n},$$

and the assertion is proved. □

Lemma 4. *If we have two* $n \times n$*-systems of linear algebraic equations*

$$Ax = b, \quad CX = B$$

such that $\det A \neq 0$ *and for enough small* $\varepsilon > 0$ $||A - C||_{\mathbb{R}^n \to \mathbb{R}^n} < \varepsilon, ||B - b||_{\mathbb{R}^n} < \varepsilon$ *then we have the following estimate for unique solutions x and X*

$$||X - x||_{\mathbb{R}^n} < const \, \varepsilon.$$

Proof. First, the matrix C is also invertible, it follows from our assumptions. Let us represent

$$X - x = C^{-1}B - A^{-1}b = C^{-1}B - C^{-1}b + C^{-1}b - A^{-1}b = C^{-1}(B-b) + (C^{-1} - A^{-1})b$$

Therefore, we have

$$||X - x||_{\mathbb{R}^n} \le ||C^{-1}||_{\mathbb{R}^n \to \mathbb{R}^n} ||B - b||_{\mathbb{R}^n} + ||C^{-1} - A^{-1}||_{\mathbb{R}^n \to \mathbb{R}^n} ||b||_{\mathbb{R}^n},$$

from which we obtain the required estimate. □

We will use these lemmas taking into account that we have some parameter ξ'. Now we want to estimate $|\tilde{\mathbf{B}}(\xi') - \tilde{\mathbf{b}}(\xi')|$ for $\xi' \in \hbar\mathbb{T}^{m-1}$, where

$$\tilde{\mathbf{B}}(\xi') = (\tilde{b}_0(\xi'), \tilde{b}_1(\xi'), ..., \tilde{b}_{n-1}(\xi'))^T,$$

$$\tilde{\mathbf{b}}(\xi') = (\tilde{b}_{d,0}(\xi'), \tilde{b}_{d,1}(\xi'), ..., \tilde{b}_{d,n-1}(\xi'))^T.$$

Also, we will estimate the matrix difference $(s_{jk})_{j,k=0}^{n-1} - (S_{jk})_{j,k=0}^{n-1}$. For this purpose we need a special choice of approximate constructions.

6.3. Discrete Approximations; a Special Choice

Given boundary value problem (10) we construct the operator A_d, boundary operators $B_{d,j}, j = 0, 1, ..., n-1$ and the right-hand side

$$\tilde{\mathbf{b}}(\xi') = (\tilde{b}_{d,0}(\xi'), \tilde{b}_{d,1}(\xi'), ..., \tilde{b}_{d,n-1}(\xi'))^T.$$

in the following way.

For $b_j(x'), j = 0, 1, ..., n-1$ we take its Fourier transform $\tilde{b}_j(\xi')$, its restriction on $\hbar\mathbb{T}^{m-1}$, then we take the periodic continuation on $\mathbb{R}^{m-1}$ and then inverse discrete Fourier transform. Thus, we obtain a discrete function $b_{d,j}(\tilde{x};)$ defined on $h\mathbb{Z}^{m-1}$.

For the symbol $\tilde{B}_j(\xi', \xi_m)$ we use similar method. We take its restriction on $\hbar\mathbb{T}^m$, then we periodically continue it on the whole $\mathbb{R}^m$, and such obtained symbol we call $\tilde{B}_{d,j}(\xi', \xi_m)$.

Finally, the periodic symbol $\tilde{A}_d(\xi)$ is constructed from two factors $\tilde{A}_{d,+}(\xi), A_{d,-}(\xi)$ which are created like $\tilde{B}_{d,j}(\xi', \xi_m)$.

We will suppose in this section that such discrete constructions are done and we consider the discrete boundary value problem (9) with such data.

Lemma 5. *We have the following estimate for $\beta_j < s + \delta - 1, j = 0, 1, ..., n-1$,*

$$|s_{jk}(\xi') - S_{jk}(\xi')| \le const\, h$$

for all $\xi' \in \hbar\mathbb{T}^{m-1}$.

Proof. Let us consider the difference

$$s_{jk}(\xi') - S_{jk}(\xi') = \int_{-\hbar\pi}^{\hbar\pi} \tilde{A}_{d,+}^{-1}(\xi)\widetilde{B}_{d,j}(\xi', \xi_m)\hat{\zeta}_m^k d\xi_m - \int_{-\infty}^{\infty} \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)(i\xi_m)^k d\xi_m$$

$$= \int_{-\hbar\pi}^{\hbar\pi} \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)\left(\hat{\zeta}_m^k - (i\xi_m)^k\right)d\xi_m + \left(\int_{-\infty}^{-\hbar\pi} + \int_{\hbar\pi}^{\infty}\right)\tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)(i\xi_m)^k d\xi_m,$$

according to our choice. We have

$$|\tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)| \le const(1 + |\xi|)^{\beta_j - æ}$$

in view of our assumptions on $A_+(\xi)\tilde{B}_j(\xi)$. Now we will prove the following estimate

$$\left|\hat{\zeta}_m^k - (i\xi_m)^k\right| \le k^2 h|\xi_m|^{k+1}, \quad k \in \mathbb{N}. \tag{13}$$

Indeed, $\zeta_m = \hbar(e^{i\xi_m h} - 1)$ and using the Taylor series $e^{i\xi_m h} = \sum\limits_{k=0}^{+\infty} \frac{(i\xi_m h)^k}{k!}$ we can obtain

$$\hbar(e^{i\xi_m h} - 1) = \sum_{k=1}^{+\infty} \frac{i^k \xi_m^k h^{k-1}}{k!} = i\xi_m \sum_{k=1}^{+\infty} \frac{i^{k-1}\xi_m^{k-1} h^{k-1}}{k!} = i\xi_m e^{i\xi_m h}.$$

Thus, $\hat{\zeta}_m^k = (i\xi_m)^k e^{ik\xi_m h}$ and

$$\hat{\zeta}_m^k - (i\xi_m)^k = (i\xi_m)^k (e^{ik\xi_m h} - 1) = (i\xi_m)^k \frac{e^{ik\xi_m h} - 1}{kh} kh = (i\xi_m)^k kh(ik\xi_m)e^{ik\xi_m h}.$$

After these calculations the inequality (13) is obtained immediately.

Now we estimate

$$\left|\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)\left(\hat{\zeta}_m^k - \xi_m^k\right) d\xi_m\right| \leq const\ h \int\limits_0^{+\infty} (1 + |\xi'| + |\xi_m|)^{\beta_j - æ}|\xi_m|^{k+1} d\xi_m$$

$$\leq const\ h \int\limits_0^{+\infty} (1 + |\xi'| + |\xi_m|)^{\beta_j - æ + n} d\xi_m \leq const\ h$$

under the condition $\beta_j < s + \delta - 1,\ j = 0, 1, ..., n - 1.$ One can remind that $æ - s = n + \delta$, and then $\beta_j - æ + n < -1.$

Let us consider other integrals

$$\left|\left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{\infty}\right) \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi', \xi_m)\xi_m^k d\xi_m\right| \leq const \int\limits_{\hbar\pi}^{\infty} (1 + |\xi'| + |\xi_m|)^{\beta_j - æ + n - 1} d\xi_m$$

$$\leq const(1 + |\xi'| + \hbar\pi)^{\beta_j - æ + n} \leq const\ \hbar^{\beta_j - s - \delta} \leq const\ h,$$

since $\beta_j - s - \delta < -1, j = 0, 1, ..., n - 1.$ □

Remark 2. By the way, Lemma 5 guarantees that the condition

$$ess \inf_{\xi' \in \mathbb{R}^{m-1}} |\det(S_{jk})(\xi')| > 0$$

implies the condition

$$ess \inf_{\xi' \in \hbar\mathbb{T}^{m-1}} |\det(s_{jk})(\xi')| > 0$$

for enough small h.

Lemma 6. *Let ess* $\inf_{\xi' \in \mathbb{R}^{m-1}} |\det(S_{jk})(\xi')| > 0$. *If* $\tilde{c}_k(\xi'), \tilde{C}_k(\xi')$ *are solutions of systems (8), (12) respectively then the following estimates*

$$|\tilde{C}_k(\xi')| \leq const \sum_{k=0}^{n-1} |\tilde{b}_k(\xi')|, \quad \xi' \in \mathbb{R}^{m-1},$$

$$|\tilde{c}_k(\xi') - \tilde{C}_k(\xi')| \leq const\ h \sum_{k=0}^{n-1} |\tilde{b}_k(\xi')|, \quad \xi' \in \hbar\mathbb{T}^{m-1},$$

hold.

Proof. Indeed, let us denote $\tilde{\mathbf{C}}(\xi') = (\tilde{C}_1(\xi'), ..., \tilde{C}_{n-1}(\xi'))^T, \tilde{\mathbf{c}}(\xi') = (\tilde{c}_1(\xi'), ..., \tilde{c}_{n-1}(\xi'))^T$. Then

$$\tilde{\mathbf{C}}(\xi') = (S_{jk})^{-1}(\xi')\tilde{\mathbf{B}}(\xi').$$

For the element $S_{jk}(\xi')$ we have

$$|S_{jk}(\xi')| = \left|\int\limits_{-\infty}^{\infty} \tilde{A}_+^{-1}(\xi)\tilde{B}_j(\xi',\xi_m)\xi_m^k d\xi_m\right| \leq const \int\limits_{-\infty}^{\infty} (1+|\xi|)^{\beta_j-æ}|\xi_m|^k d\xi_m$$

$$\leq const \int\limits_{-\infty}^{\infty} (1+|\xi|)^{\beta_j-æ+n-1} d\xi_m = const\,(1+|\xi'|)^{\beta_j-æ+n} \leq const,$$

since $\beta_j - æ + n < -1$. Thus, all elements S_{jk} are bounded. Then

$$|\tilde{C}_k(\xi')| \leq ||\tilde{\mathbf{C}}(\xi')||_{\mathbb{R}^n} = \sum_{k=0}^{n-1} |\tilde{C}_k(\xi')| \leq ||(S_{jk})^{-1}(\xi')||_{\mathbb{R}^n\to\mathbb{R}^n}\, ||\tilde{\mathbf{B}}(\xi')||_{\mathbb{R}^n}$$

$$\leq const\, ||\tilde{\mathbf{B}}(\xi')||_{\mathbb{R}^n} = const \sum_{k=0}^{n-1} |\tilde{b}_k(\xi')|.$$

Further, using Lemmas 3,4,5 and our choice for the vector $\mathbf{b}$ we conclude for $\xi' \in \hbar\mathbb{T}^{m-1}$

$$|\tilde{c}_k(\xi') - \tilde{C}_k(\xi')| \leq ||\tilde{\mathbf{c}}(\xi') - \tilde{\mathbf{C}}(\xi')||_{\mathbb{R}^n} = ||(s_{jk})^{-1}(\xi')\tilde{\mathbf{b}} - (S_{jk})^{-1}(\xi')\tilde{\mathbf{B}}||_{\mathbb{R}^n}$$

$$\leq ||(s_{jk})^{-1}(\xi') - (S_{jk})^{-1}(\xi')||_{\mathbb{R}^n\to\mathbb{R}^n}\, ||\tilde{\mathbf{b}}||_{\mathbb{R}^n} \leq\ const\, ||(s_{jk})(\xi') - (S_{jk})(\xi')||_{\mathbb{R}^n\to\mathbb{R}^n}\, ||\tilde{\mathbf{b}}||_{\mathbb{R}^n}$$

$$\leq\ const\ h||\tilde{\mathbf{b}}||_{\mathbb{R}^n}$$

according to the fact that $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{B}}$ are the same in $\hbar\mathbb{T}^{m-1}$. □

Theorem 5. *Let æ be index of factorization of the symbol $\tilde{A}(\xi)$ such that $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2, 1/2 < s < \beta_j < s + \delta - 1, s > \frac{m+2}{2} + (\delta - \beta_j), j = 0, 1, \dots, n-1,$ and*

$$ess \inf_{\xi'\in\mathbb{R}^{m-1}} |\det(S_{jk})(\xi')| > 0.$$

A comparison between discrete and continuous solution of problems (9) and (10) respectively is given by the estimate

$$|u_d(\tilde{x}) - u(\tilde{x})| \leq\ const\ h \sum_{j=0}^{n-1} ||b_j||_{\beta_j},$$

for enough small h, where const does not depend on h.

Proof. Now we are ready to compare the solutions (4) and (11). For $\xi \in \hbar\mathbb{T}^m$ we have

$$\tilde{u}_d(\xi) - \tilde{u}(\xi) = \tilde{A}_{d,+}^{-1}(\xi) \sum_{k=0}^{n-1} \tilde{c}_k(\xi')\hat{\zeta}_m^k - \tilde{A}_+^{-1}(\xi) \sum_{k=0}^{n-1} \tilde{C}_k(\xi')\xi_m^k$$

$$= \tilde{A}_+^{-1}(\xi) \left(\sum_{k=0}^{n-1} (\tilde{c}_k(\xi') - \tilde{C}_k(\xi'))\hat{\zeta}_m^k + \sum_{k=0}^{n-1} \tilde{C}_k(\xi')(\hat{\zeta}_m^k - \xi_m^k) \right)$$

Using Lemmas 5,6 and the inequality (13) we obtain

$$|\tilde{u}_d(\xi) - \tilde{u}(\xi)| \leq const\,(1+|\xi|)^{-æ} \left(h \sum_{k=0}^{n-1} |\xi_m|^k |\tilde{b}_k(\xi')| + h \sum_{k=0}^{n-1} |\xi_m|^{k+1} |\tilde{b}_k(\xi')| \right)$$

$$\leq\ const\,(1+|\xi|)^{-æ} h \sum_{k=0}^{n-1} |\xi_m|^{k+1} |\tilde{b}_k(\xi')|.$$

Further, we compare the inverse Fourier transforms in discrete points.

$$u_d(\tilde{x}) - u(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\hbar\mathbb{T}^m} e^{-i\tilde{x}\,\xi}\tilde{u}_d(\xi)d\xi - \frac{1}{(2\pi)^m}\int\limits_{\mathbb{R}^m} e^{-i\tilde{x}\,\xi}\tilde{u}(\xi)d\xi$$

$$= \frac{1}{(2\pi)^m}\int\limits_{\hbar\mathbb{T}^m} e^{-i\tilde{x}\,\xi}(\tilde{u}_d(\xi) - \tilde{u}(\xi))d\xi + \frac{1}{(2\pi)^m}\int\limits_{\mathbb{R}^m\setminus\hbar\mathbb{T}^m} e^{-i\tilde{x}\,\xi}\tilde{u}(\xi)d\xi.$$

Now we estimate

$$|u_d(\tilde{x}) - u(\tilde{x})| \leq \ const\left(\int\limits_{\hbar\mathbb{T}^m} |\tilde{u}_d(\xi) - \tilde{u}(\xi)|d\xi + \int\limits_{\mathbb{R}^m\setminus\hbar\mathbb{T}^m} |\tilde{u}(\xi)|d\xi\right)$$

$$\leq \ const\left(h\int\limits_{\hbar\mathbb{T}^m} (1+|\xi|)^{-æ}\sum_{k=0}^{n-1}|\xi_m|^{k+1}|\tilde{b}_k(\xi')|d\xi + \int\limits_{\mathbb{R}^m\setminus\hbar\mathbb{T}^m} (1+|\xi|)^{-æ}|\sum_{k=0}^{n-1}\tilde{C}_k(\xi')||\xi_m|^k d\xi\right)$$

$$\leq \ const\left(h\int\limits_{\mathbb{R}^m} (1+|\xi|)^{-æ+n}||\tilde{\mathbf{B}}(\xi')||d\xi + \int\limits_{\mathbb{R}^m\setminus\hbar\mathbb{T}^m} (1+|\xi|)^{-æ+n}||\tilde{\mathbf{B}}(\xi')||d\xi\right)$$

according to Lemma 6. Since

$$\int\limits_{-\infty}^{+\infty}(1+|\xi'|+|\xi_m|)^{-æ+n}d\xi_m \sim (1+|\xi'|)^{-æ+n+1},$$

$$\int\limits_{-\hbar\pi}^{+\infty}(1+|\xi'|+|\xi_m|)^{-æ+n}d\xi_m \sim (1+|\xi'|+\hbar)^{-æ+n+1},$$

we conclude

$$\int\limits_{\mathbb{R}^{m-1}} (1+|\xi'|)^{-æ+n+1}|\tilde{b}_j(\xi')|d\xi' \leq \ const \int\limits_{\mathbb{R}^{m-1}} (1+|\xi'|)^{-æ+n+1-\beta_j}|\tilde{b}_j(\xi')|(1+|\xi'|)^{\beta_j}d\xi' \leq$$

(we apply the Cauchy–Bunyakovsii inequality)

$$\leq const\left(\int\limits_{\mathbb{R}^{m-1}} (1+|\xi'|)^{2(-æ+n+1-\beta_j)}\right)^{1/2} ||b_j||_{\beta_j}$$

$$= const||b_j||_{\beta_j}\left(\int\limits_{\mathbb{S}^{m-2}}\int\limits_{0}^{+\infty} r^{m-2}(1+r)^{2(-æ+n+1-\beta_j)}drdS\right)^{1/2}$$

The integral over $(0,+\infty)$ exists only if $\gamma \equiv m-2+2(-æ+n+1-\beta_j) < -1$, and this condition takes the form $s > \frac{m+1}{2} + (\delta - \beta_j)$. The second integral

$$\int\limits_{\mathbb{R}^{m-1}\setminus\hbar\mathbb{T}^{m-1}} (1+|\xi'|+\hbar)^{-æ+n+1}|\tilde{b}_j(\xi')|d\xi'$$

is estimated in the same way and we have the following upper bound $const\,||b_j||h^{-\gamma-1}$. Since $-\gamma-1>1$ under the condition $s > \frac{m+2}{2} + (\delta - \beta_j)$ we have common estimate

$$|u_d(\tilde{x}) - u(\tilde{x})| \leq \ const\, h\sum_{j=0}^{n-1}||b_j||_{\beta_j}$$

and the assertion is proved. □

CONCLUSION

These results show that a theory of discrete boundary value problems can be useful for finding discrete approximate solutions of continuous boundary value problems. The next step is studying finite approximations for considered discrete boundary value problems.

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