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GENERALIZED POTENTIALS OF DOUBLE LAYER FOR SECOND ORDER ELLIPTIC SYSTEMS

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Abstract. Second order elliptic systems on the plane are considered. The notion of generalized potentials of double layer for these systems is introduced.

Keywords: second order elliptic systems, lame system, potentials of double layer, Dirichlet problem.

1 Second order elliptic systems

Let us consider the elliptic system of second order

$$\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad u = (u_1, \dots, u_l), \quad x_1 = x,\ x_2 = y,$$

with constant and only leading coefficients $a_{ij} \in \mathbb{R}^{l\times l}$. In view of the elliptic condition

$$\det\left(\sum a_{ij}\lambda_i\lambda_j\right) \neq 0, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

the characteristical polynomial

$$\chi(z) = \det p(z), \quad p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$$

has no real roots. Let σ_+ denote a set of all these roots in the upper half-plane.

Let $D \subseteq \mathbb{C}^2$ be a finite domain with a smooth boundary $\Gamma = \partial D$. As it's well known the Dirichlet problem

$$u\big|_\Gamma = f$$

isn't always Fredholm. The first example of this type belongs to A. V. Bitsadze[1]. He noticed that the homogeneous Dirichlet problem for elliptic system with coefficients ($l = 2$)

$$a_{11} = -a_{22} = 1, \quad a_{12} = a_{21} = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$$

in the unite circle has infinitely linear independent solutions.

Later A. V. Bitsadze introduced the notion of the so-called weakly connected elliptic systems for which the Dirichlet problem is Fredholm. According to modern elliptic theory this requirement simply implies that the corresponding Shapiro- Lopatinski condition holds[2]. It's convenient to formulate this condition in the following way.

The elliptic system is weakly connected iff

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$$\det\left[\int_{\mathbb{R}} p^{-1}(\lambda)d\lambda\right] \neq 0.$$

The Bitsadze example stimulated the definitions of the various classes of elliptic systems for which the Dirichlet problem is Fredholm. The most important of them was the notion of strong elliptic system introduced by M. I. Vishik[3]. They are defined by the condition of positive definiteness of the matrix

$$\sum_{i,j=1}^{2} a_{ij}\lambda_i\lambda_j > 0$$

for all $\lambda,\lambda_2 \in \mathbb{R}$, $|\lambda_1| + |\lambda_2| \neq 0$.

In this case the matrix $p^{-1}(\lambda)$ is also positive definite, so these systems are really weakly connected. More restrict condition was introduced earlier by C. Somigliano[4] and is expressed in the form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0.$$

The intermediate position between these definitions occupies the notion of the strengthened elliptic system [5]. By definition this system have to be elliptic and the matrix $a \geq 0$. Note that another classification of elliptic systems in the case $l = 2$ is given by Lin Wei[6] and Wu Ci-Quian[7].

2 Generalized potentials of double layer

Let the elliptic system be weakly connected. As it will be said earlier then the Dirichlet problem is Fredholm. More exactly the following result is valid [8]. Here and below $C^{+0}(E)$ implies the Holder class $\cup_{\mu>0} C^{\mu}(E)$.

Let $\Gamma = \partial D$ be Lyapunov contour i.e. its inner normal $n(t) = n_1(t) + in_2(t) \in C^{+0}(\Gamma)$ and let $f \in C^{+0}(\Gamma)$. Then homogeneous Dirichlet problem has a finite number linear independent solutions $u_1, \ldots, u_n \in C^{+0}(\overline{D})$ and there exist a real vector- valued linear independent functions $g_1, \ldots, g_n \in C^{+0}(\Gamma)$ such that nonhomogeneous Dirichlet problem is solvable in $C^{+0}(\overline{D})$ iff

$$(f, g_i) = 0,\ 1 \leq i \leq n,$$

where

$$(f, g) = \int_{\Gamma} f(t)g(t)|dt|.$$

The case of strengthened elliptic system is remarkable as $n = 0$ for these systems. In other words *the Dirichlet problems for a strengthened elliptic system is uniquely solved.*

The main result of this talk is the following: *if $f \in C(\Gamma)$ satisfies the orthogonality conditions then the Dirichlet problem is solvable in the class $C(\overline{D})$.*

Our approach is based on using generalized potentials of double layer for the elliptic system. From the weakly connected property it follows the following lemma: *there exists the unique matrix $J \in \mathbb{C}^{l\times l}$ such that*

$$a_{11} + (a_{12} + a_{21})J + a_{22}J^2 = 0,$$

$$\sigma(J) = \sigma_+, \quad \det(\mathrm{Im}\, J) \neq 0.$$

Recall that σ_+ denotes a set of all roots in the upper half-plane of the characteristical polynomial $\chi(z) = \det p(z)$, $p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$. The matrix J is called a characteristical matrix of the elliptic system. If it is diagonal then the system reduces to l scalar equations. More exactly the exists an invertible matrix c such that all matrixes ca_{ij} are diagonal. So we may suggest that J is not diagonal.

Let us put

$$Q(t,\xi) = \frac{n_1(t)\xi_1 + n_2(t)\xi_2}{|\xi|^2} H(\xi),$$

$$H(\xi) = \mathrm{Im}\,[(-\xi_2 1 + \xi_1 J)(\xi_1 1 + \xi_2 J)^{-1}],$$

where 1 implies the unit matrix and n is the unit vector of inner normal. *Then the integral*

$$(P\varphi)(z) = \frac{1}{\pi}\int_\Gamma Q(t, t-z)\varphi(t)|dt|, \quad z \in D,$$

describes solutions of the elliptic system. Note that for $H = 1$ this integral corresponds to the classical potentials of double layer for Laplace equation. The following theorem shows that $P\varphi$ plays an analogous role for the elliptic system.

The integral operator P is bounded $C(\Gamma) \to C(\overline{D})$ and

$$(P\varphi)^+(t_0) = \varphi(t_0) + \int_\Gamma Q(t, t-t_0)\varphi(t)|dt|, \quad t_0 \in \Gamma.$$

Let $K\varphi$ imply the integral on the right hand side. Under assumptions $n(t) \in C^{+0}(\Gamma)$ the kernel $k(t_0,t) = (t-t_0)Q(t,t-t_0)$ belongs to $C^{+0}(\Gamma\times\Gamma)$ and $k(t,t) \equiv 0$. So the operator K is compact in $C(\Gamma)$.

Theorem. *The exist a finite-dimensional space $X \subseteq C^{+0}(\overline{D})$ of solutions of the elliptic system and a space $Y \subseteq C^{+0}(\Gamma)$ of the same dimension such that each solution $u \in C(\overline{D})$ of the elliptic system is uniquely represented in the form*

$$u = P\varphi + u_0, \quad u_0 \in X,$$

where $\varphi \in C(\Gamma)$ satisfies the orthogonality condition $(\varphi, g) = 0$, $g \in Y$.

If the system is strengthened elliptic then in this representation $X = 0$, $Y = 0$.

The theorem shows that the Dirichlet problem is equivalent to the following system of Fredholm integral equations:

$$\varphi + K\varphi + \sum_1^m \lambda_i u_i = f,$$

$$(\varphi, g_i) = 0, \quad i = 1,\dots,m,$$

where $u_1,\dots,u_m$ and $g_1,\dots,g_m$ are basises of X and Y respectively.

In the case $l = 2$ the matrix $H(\xi)$ can be described explicitly. In this case there are only two possibility for σ_+ when (i) $\sigma_+ = \{\nu_1,\nu_2\}$, $\nu_1 \neq \nu_2$, and (ii) $\sigma_+ = \{\nu\}$. So the exists an invertible matrix $b \in \mathbb{C}^{2\times 2}$ such that

$$(i)\; bJb^{-1} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad (ii)\; bJb^{-1} = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}.$$

The case $bJb^{-1} = \nu$ is excluded as the matrix J is not diagonal. Note that the matrixes

$$E_1 = b\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b^{-1}, \quad E_2 = b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b^{-1},$$

don't depend on the choice of b.

In this terms we have:

$$(i)\ H(\xi) = (\mathrm{Im}^2\nu_2) g(\xi,\nu_2) + \mathrm{Im}[(\nu_1 - \nu_2) g(\xi,\nu_1) g(\xi,\nu_2) E_1],$$

$$(ii)\ H(\xi) = (\mathrm{Im}^2\nu) g(\xi,\nu) + \mathrm{Im}[g^2(\xi,\nu) E_2],$$

where $g(\xi,\nu) = |\xi|(\xi_1 + \nu\xi_2)^{-1}$.

3 Applications to the plane elasticity

The plane elastic medium is characterized by the displacement vector $u = (u_1, u_2)$ and by stress and deformation tensors

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2 \end{pmatrix},$$

where

$$\varepsilon_i = \frac{\partial u_i}{\partial x_i}, \quad i = 1, 2, \quad 2\varepsilon_3 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}.$$

They are connected by Hooke law i.e. by linear relation

$$\tilde{\sigma} = \alpha\tilde{\varepsilon}, \quad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 \\ \alpha_4 & \alpha_2 & \alpha_6 \\ \alpha_5 & \alpha_6 & \alpha_3 \end{pmatrix} > 0,$$

where $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, 2\varepsilon_3)$.

If the external forces are absent then the equilibrium equations have the form

$$\frac{\partial \sigma_{(1)}}{\partial x_1} + \frac{\partial \sigma_{(2)}}{\partial x_2} = 0,$$

where $\sigma_{(j)}$ means $j-$column of the matrix σ. Using the Hooke law we receive the Lame system

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0$$

for the replacement vector u with the coefficients a_{ij}, defined by the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_6 & \alpha_6 & \alpha_4 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_4 & \alpha_5 & \alpha_5 & \alpha_2 \end{pmatrix}.$$

This system is strengthened elliptic and rang $a = 3$.

The elastic medium is called orthotropic if $\alpha_5 = \alpha_6 = 0, \quad \alpha_3 + \alpha_4 \neq 0$, and isotropic if $\alpha_5 = \alpha_6 = 0, \quad \alpha_1 = \alpha_2 = 2\alpha_3 + \alpha_4$. We can also point out the special case $\alpha_5 = \alpha_6 = 0, \quad \alpha_3 + \alpha_4 = 0$. In this case the Lame system reduces to scalar equations

$$\alpha_1 \frac{\partial^2 u_1}{\partial x^2} + \alpha_3 \frac{\partial^2 u_1}{\partial y^2} = 0, \quad \alpha_3 \frac{\partial^2 u_2}{\partial x^2} + \alpha_2 \frac{\partial^2 u_2}{\partial y^2} = 0.$$

So this case we put away below.

Let us consider the characteristic polynomial of Lame system

$$p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2 = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix},$$

where $p_1(z) = \alpha_1 + 2\alpha_6 z + \alpha_3 z^2$, $p_2(z) = \alpha_3 + 2\alpha_5 z + \alpha_2 z^2$, $p_3(z) = \alpha_6 + (\alpha_3 + \alpha_4)z + \alpha_5 z^2$.

In the case (i) *we can put*

$$E_1 = \frac{1}{p_2(\nu_2)p_3(\nu_1) - p_2(\nu_1)p_3(\nu_2)} \begin{pmatrix} -p_2(\nu_1)p_3(\nu_2) & -p_2(\nu_1)p_2(\nu_2) \\ -p_3(\nu_1)p_3(\nu_2) & p_2(\nu_2)p_3(\nu_1) \end{pmatrix},$$

if one of the following conditions $(*)$

$$\alpha_3^2 < \alpha_1\alpha_2, \ \alpha_5^2 < \alpha_2\alpha_3, \ \alpha_2\alpha_6 = \alpha_3\alpha_5, \ \alpha_2(\alpha_3 + \alpha_4) = 2\alpha_5^2,$$

disturbs and

$$E_1 = \frac{1}{p_1(\nu_1)p_3(\nu_2) - p_1(\nu_2)p_3(\nu_2)} \begin{pmatrix} -p_1(\nu_2)p_3(\nu_1) & -p_3(\nu_1)p_3(\nu_2) \\ -p_1(\nu_1)p_1(\nu_2) & p_1(\nu_1)p_3(\nu_2) \end{pmatrix},$$

if one of the following conditions $(**)$

$$\alpha_3^2 < \alpha_1\alpha_2, \ \alpha_6^2 < \alpha_1\alpha_3, \ \alpha_1\alpha_5 = \alpha_3\alpha_6, \ \alpha_1(\alpha_3 + \alpha_4) = 2\alpha_6^2$$

disturbs.

In the case (ii) *we can put*

$$E_2 = \frac{1}{p_2'(\nu)p_3(\nu) - p_2(\nu)p_3'(\nu)} \begin{pmatrix} p_2(\nu)p_3(\nu) & p_2^2(\nu) \\ -p_3^2(\nu) & -p_2(\nu)p_3(\nu) \end{pmatrix}.$$

Note that fulfilments of both conditions $(*)$ and $(**)$ is equivalent to the special case $\alpha_5 = \alpha_6 = 0, \ \alpha_3 + \alpha_4 = 0$ when the Lame system is diagonal.

In the orthotropic case the polynomial p_j are simplify:

$$p_1(z) = \alpha_1 + \alpha_3 z^2, \ p_2(z) = \alpha_3 + \alpha_2 z^2, \ p_3(z) = (\alpha_3 + \alpha_4)z,$$

so in this case

$$E_1 = \frac{(\alpha_3 + \alpha_4)^{-1}}{\nu_1 p_2(\nu_2) - \nu_2 p_2(\nu_1)} \begin{pmatrix} -p_2(\nu_1)(\alpha_3 + \alpha_4)\nu_2 & -p_2(\nu_1)p_2(\nu_2) \\ -(\alpha_3 + \alpha_4)^2\nu_1\nu_2 & p_2(\nu_2)(\alpha_3 + \alpha_4)\nu_1 \end{pmatrix},$$

$$E_1 = \frac{(\alpha_3 + \alpha_4)^{-1}}{\nu_2 p_1(\nu_1) - \nu_1 p_1(\nu_2)} \begin{pmatrix} -p_1(\nu_2)(\alpha_3 + \alpha_4)\nu_1 & -(\alpha_3 + \alpha_4)^2\nu_1\nu_2 \\ -p_1(\nu_1)p_1(\nu_2) & p_1(\nu_1)(\alpha_3 + \alpha_4)\nu_2 \end{pmatrix},$$

respectively to (∗), (∗∗) and

$$E_2 = \frac{1}{\alpha_3\nu^2 - \alpha_2}\begin{pmatrix} \nu(\alpha_2+\alpha_3\nu^2) & (\alpha_3+\alpha_4)^{-1}(\alpha_2+\alpha_3\nu^2)^2 \\ -(\alpha_3+\alpha_4)\nu^2 & -\nu(\alpha_2+\alpha_3\nu^2) \end{pmatrix}.$$

Especially the simple picture we have in the orthotropic case when $\nu = i$ and $\alpha_1 = \alpha_2 = 2\alpha_3 + \alpha_4$. In this case

$$E_2 = -\frac{1}{æ}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \quad æ = \frac{\alpha_1+\alpha_3}{\alpha_1-\alpha_3},$$

and therefor

$$H(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{æ|\xi|^2}\begin{pmatrix} \xi_2^2-\xi_1^2 & 2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2-\xi_2^2 \end{pmatrix}.$$

Another function theoretical approaches for orthotropic Lame system were suggested by R P.Gilbert[9, 10].

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ОБОБЩЕННЫЙ ПОТЕНЦИАЛ ДВОЙНОГО СЛОЯ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ ВТОРОГО ПОРЯДКА

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Аннотация. Рассматриваются эллиптические слабо связанные (по терминологии А.В. Бицадзе) системы второго порядка с постоянными (и только старшими) коэффициентами. Для этих систем вводится понятие потенциалов двойного слоя, не связанное с фундаментальным решением. Оно позволяет редуциовать задачу Дирихле к эквивалентной системе интегральных уравнений Фредгольма на границе области.

Ключевые слова: эллиптические системы второго порядка, системы Ламэ, потенциал двойного слоя, задача Дирихле.