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STRATUM OF FREENESS FOR DEFORMATIONS OF SINGULARITIES

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Abstract. The purpose of this note is to study the problem of classification of quasihomogeneous Saito free divisors making use of deformation theory of varieties with $\mathbb{G}_m$-action. In particular, we describe an approach for computation of the stratum of freeness for deformations of quasicones over quasismooth varieties. We also discuss some useful applications in a more general context including methods of computation of this stratum for deformations of hypersurface singularities, the compactification of modular spaces, etc.

Keywords: logarithmic vector fields, free divisors, free deformations, compactification, stratum of freeness.

Introduction

The purpose of this note is to study the problem of classification of quasihomogeneous Saito free divisors making use of deformation theory of varieties with $\mathbb{G}_m$-action. In particular, we describe an approach for computation of free deformations of quasicones over quasismooth varieties. We also discuss some useful applications in a more general context. Among other things we show that from quite a general point of view in the theory of isolated singularities Saito free divisors play a role of stable curves of the classical theory of compactification of the moduli space for curves of given genus.

In the first section a brief survey of classification methods and technique is given. In the next three sections basic notions and results from the deformation theory of varieties with $\mathbb{G}_m$-action are described. Then we discuss the notion of Saito singularities, their basic properties and relations with problems of classification of non-isolated hypersurface singularities and compactification of flat families, deformations of isolated singularities. Some computational examples, problems and applications are also considered including computation of freeness locus for deformations of certain simple and unimodal singularities.

The paper clarifies the construction of a series examples of Saito free divisors in 3-dimensional case recently produced by J.Sekiguchi (see [19], [20], [21]). In contrast with our methods his approach is based essentially on the classification of Lie algebras formed by logarithmic vector fields tangent to a hypersurface.

1 Methods of classification, enumeration and deformations

Many problems concerning classification and enumeration of singularities are closely related with different aspects of the modern singularity theory. In general, one may consider different equivalence relations between singularities such as the right equivalence (change of coordinates in the source of defining mapping), contact equivalence (change of coordinates and multiplication

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with a unit, that is, preserving the isomorphism class of the corresponding germ), and others. Anyway the initial stage of the solving of classification problems is a description of simple singularities. As a rule, one can write out a finite or at least perceptible list of normal forms or similar data. However a scheme of classification of more complicated isolated singularities seems to be a rather nontrivial and hard problem since such phenomenon as moduli or parametric families may occur. Furthermore, essential and serious difficulties arise in the theory of non-isolated singularities. Among different approaches to further classification it seems methods of the deformation theory are very fruitful and useful. The following observation is also very important: some types of non-isolated singularities appear as degenerate fibers in parametric families or deformations of isolated singularities, other ones (for example, divisors with normal crossing) are natural ingredients of compactifications of algebraic varieties in the Hodge theory and in related questions, and so on.

Historically, the theory of singularities originated in studies of quasihomogeneous functions with isolated critical points or, in other terms, hypersurfaces with isolated singularities given by quasihomogeneous functions. Unfortunately, in contrast with the theory of isolated singularities the type of homogeneity in non-isolated case does not determine neither topological nor analytical structure of singularities. Moreover, there are types of homogeneity associated with non-isolated singularities that can not be realized as types of isolated singularities at all.

In fact, there is a natural approach to the problem of classification of all objects of some given type. If one can organize them into families which are, in some extended sense, "continuous and then determine how nearby objects are related, their basic properties become more understandable and clearly expressed. The idea of continuous families or, in other words *deformations*, of abstract objects goes back at least to B.Riemann who found that the isomorphism classes of Riemann surfaces of genus $g > 0$ form a single continuous, almost everywhere analytic, complex family. Its complex dimension, called by B.Riemann "the number of moduli is given by the Riemann-Roch theorem and is equal to 1 for $g = 1$ and $3g - 3$ for $g > 1$.

There are few different ways of looking at the problem of classification or enumeration; they occur naturally in the context of the deformation theory. Thus in order to classify non-isolated singularities one can endeavor to create a list of them ordered by some rules or numerical invariants: Milnor numbers, types of homogeneity, weights of variables or vector fields, etc. It is also very important to choose a suitable representation for members of the list. In the standard theory one usually takes generators of the defining ideal, functions or polynomials, in other terms, normal forms of singularities (see [9], [1]). However in the non-isolated case any classification depends on types of singular loci of singularities themselves. So it is necessary to analyze singular loci. Further, it is possible to classify all pairs of singular hypersurfaces and their singular loci. Another way is to obtain a classification of local algebras associated with singularities, Lie algebras of differentiations (see [21]), and so on.

2 Singularities with $\mathbb{G}_m$-action

Let k be a field of characteristic zero, and let $P_w = k[z_1, \dots, z_n]$ be the polynomial algebra graded as follows: $\deg(z_i) = w_i \in \mathbb{Z}_+$, $i = 1, \dots, n$. Then $\mathbb{P}(w) = \mathrm{Proj}(P_w)$ is called the weighted projective space of type $w = (w_1, \dots, w_n)$. For brevity, let us set $P = P(1, \dots, 1)$ and, similarly, $\mathbb{P} = \mathbb{P}(1, \dots, 1)$. It should be underlined that weighted projective spaces are toric varieties.

Let X be a closed subscheme of a weighted projective space $\mathbb{P}(w)$. Denote by $U = \mathbb{A}^n - \{0\} = \mathrm{Spec}(P) - \mathfrak{m}$, $\mathfrak{m} = (z_1, \ldots, z_n)$, and let $p\colon U \to \mathbb{P}(w)$ be the canonical projection. The scheme closure C_X of $p^{-1}(X) \subset \mathbb{A}^n$ is called the affine quasicone over X. The point $0 \in C_X$ is called the vertex of C_X.

Let $\mathbb{G}_m$ be the group k^* of units of the ground field k under multiplication. An affine algebraic variety V over k is called the quasicone if there is an effective action of $\mathbb{G}_m$ on V such that the intersection of the closures of all orbits is a closed point, the vertex of the quasicone. It is well-known that any affine quasicone is a quasicone. Conversely, any quasicone without embedded components in its vertex is an *affine* quasicone C_X for some $X \subset \mathbb{P}(w)$ (cf. [10]).

Denote by J the defining ideal of X in $\mathbb{P}(w)$ and by $I \subset P$ the ideal of C_X in $\mathbb{A}^n$. In the ordinary projective case (that is, $w = (1, \ldots, 1)$) we have

$$I \cong H^0(U, p^*(J) \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_U) \cong \oplus_{\nu \in \mathbb{Z}} H^0(U, p^*(J) \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_U(\nu)).$$

In particular, the ideal I is homogeneous. By analogy, it is possible to prove that for an arbitrary quasicone V there is a closed embedding $\iota\colon V \to \mathbb{A}^n$ such that the ideal of $\iota(V)$ is generated by weighted homogeneous polynomials with integer positive weights (called, for brevity, homogeneous elements of the graded ring $P(w)$).

Closed subvarieties in weighted projective spaces are called *affine* isomorphic if their affine quasicones are isomorphic; they are called *projectively* isomorphic if their quasicones are $\mathbb{G}_m$-isomorphic.

3 Deformations of weighted hypersurfaces and quasicones

A closed subscheme $X_0 \subset \mathbb{P}(w)$ is called quasismooth if the corresponding affine quasicone $Y_0 = C_{X_0}$ is smooth outside its vertex, that is, Y_0 has an isolated singularity at the vertex. In particular, a quasismooth variety as well as its affine quasicone are reduced schemes.

It is well-known that in quasismooth case the $\mathbb{G}_m$-action on Y_0 can be extended to the minimal versal deformation. Moreover, the space of infinitesimal deformations of the first order $T^1(Y_0)$ is endowed with a natural grading $T^1 = \oplus_{\nu \in \mathbb{Z}} T^1(Y_0)_\nu$. Of course, T^1 is finite-dimensional over the ground field, its dimension is called the Tjurina number of the singularity; it is denoted by τ.

Below we restrict ourselves to the case when X_0 is a quasismooth hypersurface. Thus the ideal I of C_{X_0} is generated by a homogeneous element $f \in P(w)$, that is, by a quasihomogeneous polynomial in P, so that $g \in \mathbb{G}_m$ takes f to $g^d f$ for some $d \in \mathbb{Z}_+$ called the degree or weight of f. The collection $(d; w_1, \ldots, w_n)$ is called the type of weighted singularity X_0 or its quasicone Y_0; it is denoted by π. Choose also a homogeneous (monomial) basis $\sigma_k \in T^1(Y_0)_{\nu_k}$, $\sigma_k \in P$, $k = 1, \ldots, \tau$, and set $R = k[[t_1, \ldots, t_\tau]]$, $S = \mathrm{Spec}(R)$. Then the minimal versal deformation $\phi\colon Y \to S$ of the singularity Y_0 is defined by the element $F = f + t_1\sigma_1 + \ldots + t_\tau\sigma_\tau \in R[z_1, \ldots, z_n]$, so that t_k has weight $-\nu_k$ under the natural action of $\mathbb{G}_m$.

Now one can projectivize the fibers of the versal deformation without projectivization of its base substituting $t_i z_0^{-\nu_i}$ for t_i in F (cf. [15], (13.3)). Denote the polynomial so obtained by $\overline{F}$. Then the quotient ring $A = R[z_0, \ldots, z_n]/(\overline{F})$ is a graded k-algebra in $z_0, \ldots, z_n$ alone with $z_1, \ldots, z_n$ having the same weights w_i as before and z_0 having weight 1. In the case when $-\nu_i \geq 0$ for all $i = 1, \ldots, \tau$, that is, C_{X_0} has negative grading in the sense of [15] the morphism $\varphi\colon \overline{Y} = \mathrm{Proj}(A) \to S$ is flat and proper with fibers reduced projective curves (cf. [loc. cite], (13.4)). All the fibers of the flat morphism φ (as well as ϕ) a given $\mathbb{G}_m$-orbit of S are isomorphic. Of course, φ is not, in general, versal or minimal versal.

Remark. One can often take the weight of new variable z_0 equals not only to 1, but to any positive (or even non-positive) integer. In general it gives different projectivizations and families of singularities.

Let U be the open subset of S consisting of all the points $s \in S$ such that fibers Y_s of the deformation $\phi\colon Y \to S$ are smooth. Then U is $\mathbb{G}_m$-invariant as well as its complement $D = S \setminus U$, called the discriminant of the deformation ϕ. It is well-known that for the minimal versal deformation of a hypersurface isolated singularity the discriminant D is reduced and defined by a principal ideal in the base ring R. So one can choose its generator as a quasihomogeneous polynomial $h \in R$ without multiple factors. When ϕ is not minimal versal then D may have multiple components. On the other hand, points of U correspond namely to fibers of the flat mapping φ, which are reduced smooth projective curves, while all the fibers φ over D are singular.

Of course, this construction can be applied to deformations of Y_0 associated with arbitrary collections of monomials σ_i of $T^1(Y_0)$. When the family ϕ is the minimal versal deformation then U as well as D are non-empty sets. In general, U may be empty while D is always non-empty because the discriminant contains the origin.

4 Weighted plane curves

It is well-known that a quasismooth hypersurface X_0 of type $\pi = (d; w_0, w_1, w_2)$ in a weighted projective plane is a smooth projective curve (see [10]); its affine quasicone Y_0 has an isolated singularity at the origin given by a weighted-homogeneous polynomial of degree d. Set $c = d - w_0 - w_1 - w_2$, so that $-c$ is equal to the weight of a generator of Grothendieck dualizing module in virtue of the canonical isomorphism $\omega_{Y_0} \cong \mathcal{O}_{Y_0}(dz/df)$. In fact, c is nothing but the degree of canonical class of the projective curve X_0. Then any weighted plane curve with $c < 0$ is affine isomorphic to one of curves with simple elliptic singularities of types A_k, D_k, E_6, E_7, E_8. A weighted plane curve with $c = 0$ is projectively isomorphic to one of three types of weighted curves with parabolic singularities P_8, X_9, J_{10}. Furthermore, there are only 31 non-isomorphic weighted plane curves with $c = 1$ (see Table (5.5) in [2]). Moreover, it is possible to prove that for any fixed c there is only a finite number of collections $(d; w_0, w_1, w_2)$ for which there exists a smooth weighted plane curve of the same type π (cf. [10]).

Let us illustrate the construction from the previous section by concrete examples of quasicones over isolated singularities. First, let Y_0 be a simple A_1-singularity. Set $z_1 = y$, $z_2 = z$. Then Y_0 is defined by $f = y^2 + z^2$, and $T^1(Y_0) = k\langle 1 \rangle$ is generated by the unit of k. Let us consider the principal deformation $F = y^2 + z^2 + t$, $t \in k$. The discriminant of the deformation consists of one point, the origin, its defining equation is $t = 0$. There are many different ways to projectivize fibers. For example, one gets two flat mappings φ associated with polynomials $\overline{F}_1 = tx + y^2 + z^2$ and $\overline{F}_2 = tx^2 + y^2 + z^2$, where $x = z_0$ has weight equals to 2 and 1, respectively; and so on. In the first case quasicones over fibers $\overline{Y}_t$ for all $t \neq 0$ are isomorphic to a hyperplane. While in the second case all the fibers $\overline{Y}_t$ of the deformation φ are isomorphic to a smooth rational projective curve, the projectivization of the Milnor fiber of the deformation. In this case the quasicones over $\overline{Y}_t$ are isomorphic to an ordinary cone having a normal singularity at the origin. In both cases the quasicones over the fiber $\overline{Y}_0$ are isomorphic to a two-dimensional linear non-isolated singularity of type A_∞.

Similarly, in the case of an A_2-singularity we have $f = y^2 + z^3$, the space $T^1 = k\langle 1, z\rangle$ is generated by two monomials. As before one can consider two different projectivizations: $\overline{F}_1 = y^2 + z^3 + t_2x^2z + t_1x^6$, and $\overline{F}_2 = y^2 + z^3 + t_2xz + t_1x^2$, where $t_1, t_2 \in k$. In both cases the discriminant D is defined as zero set of the function $h = 27t_1^2 + 4t_2^3$. Further, all the fibers of φ over U are smooth curves and their quasicones have isolated singularity at the origin, they are normal varieties. Fibers over the discriminant D are singular, their quasicones are two-dimensional affine hypersurfaces with non-isolated singularities.

5 Saito free divisors and non-isolated Saito singularities

Let S be the germ of a complex manifold of dimension m, and let $D \subset S$ be a reduced hypersurface defined by $h \in \mathcal{O}_S$. Following K.Saito [16], we denote the $\mathcal{O}_S$-module of vector fields *logarithmic* along $D \subset S$ by $\mathrm{Der}_S(\log D)$. This module consists of germs of holomorphic vector fields $V \in \mathrm{Der}(\mathcal{O}_S)$ on S such that $V(h)$ belongs to the principal ideal $(h)\cdot\mathcal{O}_S$. The hypersurface $D \subset S$ is called *Saito free* divisor if the module of germs of logarithmic vector fields $\mathrm{Der}_S(\log D)$ is a free $\mathcal{O}_S$-module (cf. [11]).

It should be remarked that the singular locus of a Saito divisor has codimension one; in other words, this hypersurface has non-normal singularities (see [5]). The following statement is due to K.Saito [16], it gives a criterion of freeness for reduced hypersurfaces.

Proposition 1 (Saito's Criterion) *The $\mathcal{O}_{S,0}$-module $\mathrm{Der}_{S,0}(\log D)$ is free if and only if there are m germs of logarithmic vector fields $V^0, \ldots, V^{m-1} \in \mathrm{Der}_{S,0}(\log D)$ such that the determinant of the $m \times m$-matrix $\mathcal{V} = \| v_{ij}\|$ whose entries are the coefficients of V^i, $i = 0, \ldots, m-1$, is equal to αh, where α is a unit. These vector fields form a basis of $\mathrm{Der}_{S,0}(\log D)$.*

For example, $\mathrm{Der}_S(\log D)$ as well as its $\mathcal{O}_S$-dual $\Omega^1_S(\log D)$, the module of logarithmic differential forms with poles along D, are locally free if D is a hyperplane, a plane curve or a divisor with *strict* normal crossings. In the latter case a defining equation of D can be written as $h = z_1 \cdots z_k = 0$, where $k \leq m$. It is not difficult to verify that $\Omega^1_S(\log D) \cong \Omega^1_S[\log D]$, where

$$\Omega^1_S[\log D] = \mathcal{O}_S\left\langle \frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_m \right\rangle.$$

Further, the discriminants of the minimal versal deformations of isolated hypersurface or complete intersection singularities are Saito free divisors (see [16], [5]).

In the local situation the germ of a Saito free divisor D is called the Saito singularity. However, it is often more convenient to exclude trivial cases of hyperplanes or smooth hypersurfaces since they have no singularities at all. The following statement [5] can be considered as an improvement of Saito's Criterion.

Proposition 2 (Determinantal Criterion) *The $\mathcal{O}_{S,0}$-module $\mathrm{Der}_{S,0}(\log D)$ is free if and only if there are m germs of logarithmic vector fields $V^0, \ldots, V^{m-1}$, such that $V^i(h) = g_i \cdot h$, $g_i \in \mathcal{O}_S$, $i = 0, \ldots, m-1$, and such that maximal minors of the $m \times (m+1)$-matrix formed by the column $(g_0, \ldots, g_{m-1})^T$ and m columns of the matrix $\mathcal{V} = \| v_{ij}\|$ are equal to h and to its partial derivatives up to invertible factors from $\mathcal{O}^*_S$. These vector fields form a basis of $\mathrm{Der}_{S,0}(\log D)$.*

The next proposition [5] delivers a purely algebraic characterization of Saito free divisors and their singularities.

Proposition 3 (CM-Criterion) *Let $Z = \operatorname{Sing} D$ be the subgerm of D defined by the Jacobi ideal $J(h) = \sum(\partial h/\partial z_i)\mathcal{O}_D$ of the function h. Suppose that codim $(Z, D) = 1$. Then the following conditions are equivalent:*
1) D is a Saito singularity;
2) Z is a determinantal germ;
3) Z is a Cohen-Macaulay germ.

Recall (see [3]) that a singularity (D, x) is called Euler-homogeneous (or, equivalently, E-homogeneous) if there is a vector field $E \in \mathrm{Der}_S(\log D)$ such that $E(h) = gh$, where the function h is a local equation of D and $g \in \mathcal{O}_S^*$ is invertible. In particular, every weighted homogeneous singularity is E-homogeneous. Of course, the conversion is not true.

Let $\mathfrak{N} = \{V \in \mathrm{Der}_S(\log D) \colon V(h) = 0\}$ be a Lie subalgebra of $\mathrm{Der}_S(\log D)$. Elements of $\mathfrak{N}$ are called nilfields (see [4]) or trivial vector fields (see [6]). Then coefficients of an arbitrary vector field $V \in \mathfrak{N}$ define a relation or syzygy of the first order between partial derivatives of the function h and vice versa. All such relations generate an $\mathcal{O}_S$-module naturally isomorphic to the module $Z_1(dh)$ of 1-cycles of the usual Koszul complex $K_\bullet(dh)$ generated by the partial derivatives of h. In general, there are non-trivial relations between generators of $Z_1(dh)$ represented by syzygies of the second order. They generate $\mathcal{O}_S$-module $Z_2(dh)$, and so on. As a result we obtain the following statement which can be considered as a criterion of freeness for E-homogeneous singularity.

Proposition 4 (Syzygy Criterion) *Let D be an E-homogeneous Saito singularity. Then there is the following splitting into the direct sum of $\mathcal{O}_S$-modules:*

$$\mathrm{Der}_S(\log D) \cong \mathcal{O}_S\langle E\rangle \oplus \mathfrak{N},$$

where $\mathfrak{N} \cong Z_1(dh)$ is free. In particular, all syzygies $Z_i(dh)$, $i \geq 2$, of higher orders are trivial.

Proof. The condition of E-homogeneity gives us the relation $Eh = gh$, $g \in \mathcal{O}_S^*$. Take $V \in \mathrm{Der}_S(\log D)$. Then $V(h) = \theta h$, $\theta \in \mathcal{O}_S$. This implies $V - g^{-1}\theta E \in \mathfrak{N}$, that is, the splitting required. The freeness of $Z_1(dh)$ follows from definition of Saito singularity (cf. [22], [3]).

In other words, obstructions for freeness of a non-normal hypersurface singularity can be considered as a triviality condition of the second Koszul cohomology $Z_2(dh)$.

The above Determinantal Criterion can be reformulated in the E-homogeneous case as follows (see [3]).

Proposition 5 (E-determinantal Criterion) *Let D be an E-homogeneous singularity. Then D is a Saito singularity if and only if there exist $m-1$ germs of logarithmic vector fields $V^1, \ldots, V^{m-1} \in \mathrm{Der}_{S,0}(\log D)$, $V^i(h) = 0$, $i = 1, \ldots, m-1$, such that maximal minors of the $(m-1) \times m$-matrix $\mathcal{V} = \| v_{ij} \|$ are equal to partial derivatives of h up to sign. These $(m-1)$ vector fields form a basis of the free submodule $\mathfrak{N} \subset Der_{S,0}(\log D)$.*

There are also other criteria in terms of Lie algebra of logarithmic vector fields in three-dimensional weighted homogeneous case due to J.Sekiguchi (see [19], [20], [21]); they are based on the classification of Lie algebras formed by logarithmic vector fields tangent to a hypersurface, properties of the fundamental antiinvariants of finite reflection groups, etc.

6 Locus and stratum of freeness

Now we are going to apply the construction of projectivization of fibers in order to show how Saito singularities appear in flat families. The following results are typical for the case $c < 0$, that is, for simple singularities. Thus, in notations of section 4 among different projectivizations for an A_1-singularity one can consider the flat mapping φ associated with polynomial $\overline{F}_1 = tx + y^2 + z^2$. In this case for all $t \neq 0$ the corresponding quasicones are isomorphic to a Saito free divisor (more exactly, to a hyperplane) due to Saito's criterion of freeness with the following data:

$$\mathcal{V} = \begin{pmatrix} 2x & y & z \\ y & -t/2 & 0 \\ z & 0 & -t/2 \end{pmatrix}, \quad \det(\mathcal{V}) = t(tx + y^2 + z^2)/2.$$

When $t = 0$ the corresponding quasicone is isomorphic to the direct product of x-line and a plane quadric, that is, it is a Saito singularity.

This example leads to the following definition. In notations of section 4 let us consider the deformation $\phi\colon Y \to S$ of the singularity Y_0 and its projectivization $\varphi^\nu\colon \overline{Y} \to S$ obtained with the help of the variable z_0 whose weight is equal to an integer $\nu \in \mathbb{Z}$.

Definition 1. Let us denote by $\mathcal{L}_\nu(Y_0)$ the subset of the discriminant $D \subset S$ of the deformation φ^ν consisting of all the points $s \in D$ such that the quasicones over the fibers $\overline{Y}_s$ have Saito singularities. Then $\mathcal{L}_\nu(Y_0)$ is called the *locus of freeness* of the deformation φ^ν.

Of course, in a similar manner the locus of freeness is defined in a more general situation, without the assumption on quasihomogeneity. Criteria from section 5 implies that the locus of freeness is, in fact, a closed set. In particular, one obtains that $\dim \mathcal{L}_\nu(A_1) = 0$ for all $\nu \neq 2$, since the locus of freeness consists of one point $\{0\}$. However, it is possible to show that in general the locus of freeness is not an equidimensional set, it may contain components of different dimensions. As a result one can pose the following question:

Problem 1. How one can compute the locus of freeness $\mathcal{L}_\nu(Y_0)$ for a given singularity ?

In fact, it is possible to give a natural description of the locus of freeness in terms of the so-called *flattening stratum* (cf. [8]). Thus, let us take an embedding $C_{\overline{Y}} \hookrightarrow \mathbb{C}^{n+1} \times S \to S$ associated with the deformation $\psi\colon C_{\overline{Y}} \to S$ induced by the mapping of projectivization $\varphi\colon \overline{Y} \to S$, obtained with the help of the variable z_0 whose weight is equal to an integer $\nu \in \mathbb{Z}$ in the notations of section 2. Set further $\Pi = \mathbb{C}^{n+1} \times S$. Then the defining ideal of $C = C_{\overline{Y}}$ is generated by one function, say $F \in \mathcal{O}_\Pi$.

Let $\Omega^1_{\Pi/S}$ be the $\mathcal{O}_\Pi$-module of relative Kähler differentials and let $\mathrm{Der}_{\mathbb{C}}(\Pi/S)$ be the module of relative vector fields on Π over S, $\mathrm{Der}_{\mathbb{C}}(\Pi/S) \cong \mathrm{Hom}_{\mathcal{O}_\Pi}(\Omega^1_{\Pi/S}, \mathcal{O}_\Pi)$.

Let us consider a coherent $\mathcal{O}_\Pi$-module $\mathrm{Der}_{\Pi/S}(\log C/S)$ of *relative logarithmic* vector fields consisting of the elements $v \in \mathrm{Der}_{\mathbb{C}}(\Pi/S)$ such that $v(F) \subseteq (F)\mathcal{O}_\Pi$. In fact, v induces vector fields on each fibers C_s tangent at their non-singular points. In other words, v induces the *vertical* vector field $\overline{v} \in T^0(C/S, \mathcal{O}_C)$ on the total space of the deformation ψ.

The above definition of the locus of freeness implies that $\mathcal{L}_\nu(Y_0) \subseteq S$ is the maximal locally closed subspace consisting of those points $s \in S$ such that the restriction $\mathrm{Der}_{\Pi/S}(\log C/S)\big|_{\mathbb{C}^{n+1}\times\{s\}}$ is a locally free $\mathcal{O}_{\mathbb{C}^{n+1}}$-module.

It should be remarked that the locus of freeness one can determine making use of a slightly modified algorithm for computation of *flattening stratum* (see [8]). The original algorithm generalizing Massey product computations was implemented in SINGULAR language [12]; it was used in computing of the modular stratum, a very interesting and important object in the deformation theory. As a result one can produce the following definition in a rather general context.

Definition 2. Let $\psi\colon X \to S$ be a flat deformation of a hypersurface singularity X_0, and let $X \hookrightarrow \Pi = \mathbb{C}^{n+1} \times S$ be the corresponding embedding of the total deformation space. Then the intersection of the discriminant $D \subset S$ of the deformation and the image of the projection of the flattening stratum associated with the sheaf of $\mathcal{O}_\Pi$-modules $\mathrm{Der}_{\Pi/S}(\log X/S)$ on the second factor of Π is called the *stratum of freeness* of the deformation ψ.

In fact, the computational procedure of the above mentioned algorithm gives us explicit equations for the stratum, that is, this stratum is endowed by a non-trivial *structure sheaf*. In general, the stratum of freeness contains singularities, it may be non-reduced, etc.

Thus, in the above notations for conic singularities the image of the projection on the second factor of Π of the flattening stratum associated with $\mathrm{Der}_{\Pi/S}(\log X/S)$ gives us the locus of freeness $\mathcal{L}_\nu(Y_0)$ for the deformation ψ of the quasicone $X_0 = C_{\overline{Y}_0}$; in fact, it is the reduced part of the stratum of freeness, its "sous-jacent."

For completeness it should be noted that there are other methods for computation of locus of freeness based on criteria from section 5. However, an experience shows that they require highly difficult calculations (cf. [8]).

7 Degeneration and compactification of deformations

Now let us discuss another situation when Saito free divisors and their non-isolated singularities closely related with properties of deformations of isolated singularities. In fact, there are many types of isolated critical points of functions or, equivalently, isolated singularities of hypersurfaces which can be deformed to non-isolated ones. The corresponding values of parameters of the family are defined by conditions of degeneracy, the associated fibers are called degenerate fibers of the deformation (cf. [9]). In general, it is very interesting to understand properties of degenerate fibers of flat families (cf. [15], (14.11)). We show below that sometimes such fibers are nothing but non-isolated Saito singularities. We call the corresponding family *free deformation* of the singularity.

On the other hand, any non-isolated hypersurface singularity can be deformed to isolated one: we can add some additional monomials to its defining equation which are defined by conditions of non-degeneracy for functions (some types of such conditions have been described by [16], [9], [1], [2], etc.). Of course, if a non-isolated singularity is weighted homogeneous and one wants to keep the grading up to a (multiple) common factor then the task is more complicated and occasionally becomes in an intractable problem.

Remark. It should be also underlined that in contrast with the theory of isolated singularities the type of homogeneity π in non-isolated case does not determine in general either topological or analytical structure of a singularity. Moreover, there are types of homogeneity that can not be realized as types of isolated singularities at all. However if we analyze non-isolated Saito singularities then the type of homogeneity together with weights of basis logarithmic vector fields determine basic properties of the associated local ring, the local cohomology of the De Rham complex, the structure of the Lie algebra of vector fields tangent to the hypersurface, Milnor numbers, and so on (see [4]).

Further when one constructs a deformation over an affine base space then an initial non-isolated singularity can be often included in a flat family as a fiber at infinity, the corresponding value of parameters defines a point at infinity of compactification of the base space of the family.

4) $x^3 + y^3 + z^3 - 3j^2xyz$.

The degenerate fiber of the second family $\overline{F}_b$ with $b = -1/27$ is isomorphic to the fourth singularity. Here we observe an interesting phenomenon closely related with the theory of compactification of modular spaces or deformations (see details in [7]). In fact, a clear description of modular spaces is given by collections of charts or finite coverings, that is, by a collection of proper multivalent mappings $\varphi_V : V \to \mathcal{M}$, where $\mathcal{M}$ is the maximal modular deformation, $m = \dim \mathcal{M}$, and $V \subset \mathbb{C}^m$ are open subsets. It is well-known that P_8-singularity is *unimodular*, that is, $m = 1$ (cf. [loc. cite]). Set $V = \mathbb{C}^1$. Then the family $\mathcal{P}_8$ is given by the mapping $f : X \to V$ such that any fiber $X_a \subset \mathbb{C}^3$, $a \in V$, is given by the equation $x^3+y^3+z^3+axyz = 0$. Let us define a chart $\varphi_V : V \to \mathbb{CP}$ by $\varphi_V(a) = a^3$. The chart φ_V covers the Riemann sphere $\mathbb{CP}$, except for ∞. It may be covered by another chart. To see this, let us consider the second family $bx^3+y^3+z^3+xyz$, $b \in W = \mathbb{C}^1 \setminus \{0\}$. Then $\varphi_W : W \to \mathbb{CP}$, defined by $\varphi_W(b) = 1/b$, will be the desirable chart. Besides φ_W maps to ∞ the fiber of the family over $b = 0$. It is not difficult to verify that both charts are glued along $\varphi_V^{-1}(N)$ and $\varphi_W^{-1}(N)$, where $N = \varphi_V(V) \cap \varphi_W(W)$, to the sphere $\mathbb{CP}$. In fact, the gluing isomorphism is given by the formula $a^3 = 1/b$.

As a result the Riemann sphere $\mathbb{CP}$ is covered by two charts φ_V, φ_W, containing *non-isolated* singularities defined above. Thus, the following question arises:

Problem 4. (cf. [14], Problem 4.4). Does there exist a collection of charts corresponding to flat families of *isolated* singularities that covers all the points of $\mathbb{CP}$?

Considerations described above lead to a slightly different version of this problem:

Problem 4′. Does there exist such compactification of modular deformations by means of flat families of isolated and *non-isolated* singularities of certain type or types (e.g. Saito singularities).

The above computation gives us some information on the locus of freeness for deformations of the quasicone over a D_4-singularity. In fact, it is not difficult to verify that germs 2) - 4) satisfy Saito's Criterion in view of the following representation of the corresponding data by *symmetric* matrices:

$$\mathcal{V}_1 = \begin{pmatrix} x & y & z \\ y & z & x \\ z & x & y \end{pmatrix}, \quad \mathcal{V}_2 = \begin{pmatrix} x & y & z \\ y & z & x \\ z & x & y \end{pmatrix}, \quad \mathcal{V}_3 = \begin{pmatrix} x & y & z \\ y & j^2z & -jx \\ z & -jx & j^2y \end{pmatrix},$$

where $\det(\mathcal{V}_1) = -(x^3 + y^3 + z^3 - 3xyz)$, $\det(\mathcal{V}_2) = j(x^3 + y^3 + z^3 + 3jxyz)$, and $\det(\mathcal{V}_3) = -j^2(x^3 + y^3 + z^3 - 3j^2xyz)$. As a result, one obtains three non-isomorphic Saito singularities among quasicones associated with deformations of a simple isolated singularity D_4. Hence, the locus of freeness contains three points corresponding to germs $2) - 4)$. That is, $\dim \mathcal{L}_1(D_4) = 0$.

In conclusion let us fix a type of homogeneity $\pi = (d; w_0, \dots, w_n)$. Denote by $\mathcal{S}(\pi)$ the set of equivalence classes of Saito singularities of given type π modulo analytic isomorphisms. Thus, the above considerations imply that $\mathcal{S}(2; 2, 1, 1)$ contains a point corresponding to the deformation $\overline{F}_1 = tx + y^2 + z^2$ of an A_1-singularity, $\mathcal{S}(3; 1, 1, 1)$ contains at least three distinct points corresponding to germs $2) - 4)$ from the above, and so on. As a result one can pose the following question:

Problem 5. How one can describe the set $\mathcal{S}(\pi)$ for a given type of homogeneity ?

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СТРАТ СВОБОДНОСТИ ДЛЯ ДЕФОРМАЦИЙ ОСОБЕННОСТЕЙ

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Аннотация. Цель этой заметки – изучение проблемы классификации квазиоднородных свободных дивизоров Саито с помощью теории деформаций многообразий с $\mathbb{G}_m$-действием. В частности, мы описываем подход к вычислению страта свободности для деформаций квазиконусов над квазигладкими многообразиями. Мы также обсуждаем некоторые полезные приложения в более общем контексте, включая методы вычисления этого страта для деформаций особенностей гиперповерхностей, компактификации пространств модулей и т.д.

Ключевые слова: логарифмические дифференциальные формы, форма-вычет,регулярные мероморфные дифференциальные формы, кручение голоморфных дифференциалов.