Transmutation Operators Boundary Value Problems

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Abstract Transmutation operators method is used to solve and study boundary value problems. In this paper several ways to obtain transformation operators are considered: the finite integral transforms, Neumann series, the Fourier transforms, and reflection techniques. The finite integral transform technique leads to solution in the form of a composition of the Fourier sine transform and inverse finite integral transform. The Neumann series technique implies decomposition of the solution in power series of the shift operator. The Fourier transform technique provides transition to the Fourier images and comparison with the model boundary value problem. Reflection technique involves a consistent approach to the solution as a reflection from the borders. In all cases, the solution of the boundary value problem is obtained as an expansion in the solutions of the model boundary value problem. In some cases, the sum of a series can be calculated in elementary functions. New formulas have been found for solving the Dirichlet problem in a three-dimensional layer.

Keywords Transmutation operators · Boundary value problems · Integral transforms · Laplace equation · Poisson operator

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1 Introduction

The aim of this article is to develop the theory of transmutation operators and apply it to solving boundary value problems for the Laplace equation in domains with plane symmetry. The classical transmutation operators are introduced by

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K. Weierstrass, S. D. Poisson N. Y. Sonin and are used in mathematical physics [2–4, 6–8, 10, 14, 17, 18]. S.M. Sitnik [17] describes the general definition of the transmutation operator, see Definition 1 in [15].

Definition 1 An operator J is called the transmutation operator if for operators A, B the following condition holds

$$JA = BJ.$$

If the solution $y = B^{-1}x$ of the model problem $By = x$ is known, then the solution of the new problem$Az = x$ can be found using the transmutation operator J by the formula $z = J^{-1}B^{-1}Jx$. If we select

$$A = \frac{d^2}{dx^2}, B = B_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x},$$

B_α—the Bessel operator, then the transmutation operator $J = P_0$ is the Poisson operator [17]

$$P_0 [f (x)] = \frac{2}{\pi} \int_0^1 \frac{f (\varepsilon x)}{\sqrt{1-\varepsilon^2}} d\varepsilon.$$

The transmutation operator has the form $P_0 = H^{-1}F_c$, here H is the Hankel transform, and F_c is the Fourier cosine transform.

In the article, we clarify the concept of a transmutation operator in order to solve boundary value problems for potential theory. For this, we consider two boundary value problems for the Laplace equation

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x, -\infty < y < \infty; \\ \Gamma u (0, y) = g (y); \end{cases} \quad \begin{cases} \tilde{u''}_{xx} + \tilde{u''}_{yy} = 0, 0 < x, -\infty < y < \infty; \\ \tilde{\Gamma} \tilde{u} (0, y) = g (y). \end{cases}$$

Below in Definition 2 we define the transmutation operator associated with boundary conditions. The transmutation operator establishes an isomorphism of these boundary value problems.

Definition 2 Let two boundary operators $\tilde{\Gamma}$, Γ be given. An operator J is called the transmutation operator if the following conditions hold:

(1) the transmutation operator J and operator $\frac{d^2}{dx^2}$ are permutable,
(2) $\tilde{\Gamma} J = \Gamma$.

In contrast to the general case [17], Definition 2 introduces special transmutation operators that take into account boundary conditions.The introduced operators are permutable with the Laplace operator, they transform the harmonic function into a harmonic function and change the type of boundary conditions.For example, the Dirichlet problem in a semi-plane is transformed into a boundary value problem with non-local boundary conditions. The transmutation operators introduced in the article (see Definition 2) establish a functional connection between the different boundary-value problems of the potential theory. Moreover, the properties of the solution of a new boundary value problem are determined by the properties of the solution of the model boundary value problem. The transmutation operator allows us to obtain the solution of a boundary value problem in the form of Neumann series, more convenient when implemented on a computer.The members of the Neumann series are powers of the shift operator, therefore, the calculations are cyclical.In addition, the usage of transmutation operators allows us to clarify the structure of potential field and present it as a sum of field reflections from domain boundary. Further, in Sect. 2 we present four ways to construct the transmutation operators: The finite integral transforms technique, Reflection method, the Fourier transform technique, Neumann series technique. The main results and conclusions are formulated in Sects. 3 and 4.

2 Materials and Methods

2.1 The Finite Integral Transforms Technique

The transmutation operators technique is based on the study of a pair of Sturm-Liouville problems. The transmutation operator establishes an isomorphism of the singular and regular Sturm-Liouville problems [5, 13]. For the most important cases in applications, an explicit expression for the transmutation operators is found.

2.1.1 Sturm–Liouville Problem with Dirichlet Boundary Conditions

Let's consider the Sturm–Liouville problem on finding nontrivial solutions on the interval $(0, \pi)$

$$\begin{cases} y'' + \lambda^2 y = 0, \\ y(0) = 0,\ y(\pi) = 0. \end{cases}$$

The eigenvalues have the form $\lambda_k = k, k = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $y_k(x) = \sin kx, k = 1, 2, 3, \ldots$ Let the function $y = f(x)$ be defined on the segment $[0, \pi]$ and $\hat{f}(k)$ be its the Fourier integral transform

$$\hat{f}(k) = \int_0^\pi \sin kx f(x)\, dx. \tag{1}$$

Then the function$y = f(x)$ can be represented

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{f}(k) \sin kx. \tag{2}$$

For the function $y = F(x)$ on the interval $[0, \infty)$ we consider the Fourier sin transforms on the real semi-axis, direct:

$$\hat{F}(\lambda) = \int_0^\infty \sin \lambda x F(x)\, dx,$$

inverse:

$$F(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \hat{F}(\lambda)\, d\lambda.$$

Let the function $y = f(x)$ on the interval $[0, \pi]$ corresponds to the function $\hat{F}(\lambda)$ by formula (2):

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{F}(k) \sin kx, \ x \in [0, \pi].$$

The mapping $J : F \to f$ is a transmutation operator

$$J[F](x) \equiv f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{F}(k) \sin kx.$$

Let the function $F(x)$ be sufficiently smooth and decreases sufficiently rapidly at infinity so that all arising integrals and series converge. We will transform the function $\hat{F}(k)$:

$$\hat{F}(k) = \int_0^\infty \sin kx F(x)\, dx =$$

$$\begin{aligned}
&= \textstyle\sum_{j=0}^{\infty} \left(\int_{2\pi j}^{2\pi j+\pi} \sin kx F(x)\, dx + \int_{2\pi j+\pi}^{2\pi j+2\pi} \sin kx F(x)\, dx \right) = \\
&= \textstyle\sum_{j=0}^{\infty} \int_0^{\pi} \sin kx F(x + 2\pi j)\, dx + \int_{\pi}^{2\pi} \sin kx F(x + 2\pi j)\, dx = \\
&= \textstyle\sum_{j=0}^{\infty} \int_0^{\pi} \sin kx F(x + 2\pi j)\, dx - \int_0^{\pi} \sin kx F(2\pi - x + 2\pi j)\, dx = \\
&= \textstyle\sum_{j=0}^{\infty} \int_0^{\pi} \sin kx \left(F(x + 2\pi j) - F(2\pi - x + 2\pi j)\right) dx = \\
&= \textstyle\int_0^{\pi} \sin kx \sum_{j=0}^{\infty} \left(F(x + 2\pi j) - F(2\pi - x + 2\pi j)\right) dx.
\end{aligned}$$

We find the original $y = f(x)$ by formula (2). The transmutation operator J has the form:

$$\begin{aligned}
J[F](x) = f(x) = \tfrac{2}{\pi} \textstyle\sum_{k=1}^{\infty} \hat{F}(k) \sin kx = \\
= \textstyle\sum_{j=0}^{\infty} \left(F(x + 2\pi j) - F(2\pi - x + 2\pi j)\right).
\end{aligned} \tag{3}$$

To apply the transmutation operator (3), we consider the Dirichlet problem for the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x < \pi, -\infty < y < \infty; \\ u\left(0, y\right) = g\left(y\right), u\left(\pi, y\right) = 0, \end{cases} \tag{4}$$

and the Dirichlet problem for the semi-plane

$$\begin{cases} \tilde{u}''_{xx} + \tilde{u}''_{yy} = 0, 0 < x, -\infty < y < \infty; \\ \tilde{u}\left(0, y\right) = g\left(y\right). \end{cases} \tag{5}$$

Using the transmutation operator (3), we establish relation of problems (4) and (5)

$$u\left(x, y\right) = J\left[\tilde{u}\left(x, y\right)\right] = \sum_{j=0}^{\infty} \left(\tilde{u}\left(x + 2\pi j, y\right) - \tilde{u}\left(2\pi - x + 2\pi j, y\right)\right). \tag{6}$$

Based on Poisson's formula for a semi-plane

$$\tilde{u}\left(x, y\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + \left(y - \eta\right)^2} g\left(\eta\right) d\eta,$$

and on identity from [12], we get

$$\sum_{j=0}^{\infty} \left(\frac{x + 2\pi j}{\left(x + 2\pi j\right)^2 + \left(y - \eta\right)^2} - \frac{2\pi - x + 2\pi j}{\left(2\pi - x + 2\pi j\right)^2 + \left(y - \eta\right)^2} \right) = \frac{1}{2} \frac{\sin x}{ch\left(y - \eta\right) - \cos x}. \tag{7}$$

Formula (7) is established for solving the Dirichlet problem in the strip [13]

$$u\left(x, y\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin x}{ch\left(y - \eta\right) - \cos x} g\left(\eta\right) d\eta.$$

2.1.2 Sturm–Liouville Problem with Neumann Boundary Conditions

Sturm–Liouville problem with Neumann boundary conditions is to find non-trivial solutions on the interval $(0, \pi)$

$$\begin{cases} y'' + \lambda^2 y = 0, \\ y'\left(0\right) = 0, y'\left(\pi\right) = 0. \end{cases}$$

The eigenvalues have the form $\lambda_k = k, k = 0, 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $y_k(x) = \cos kx, k = 0, 1, 2, 3, \ldots$ Let a function $y = f(x)$ be given on a segment $[0, \pi]$ and $\hat{f}(k)$ be its the finite Fourier transform

$$\hat{f}(k) = \int_0^{\pi} \cos kx f(x)\, dx. \tag{8}$$

Then the conversion formula has the form:

$$f(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \hat{f}(k) \cos kx. \tag{9}$$

Let the function $F(x)$ be defined on the real semi-axis, and $\hat{F}(\lambda)$ be its Fourier cosine transform:

$$\hat{F}(\lambda) = \int_0^{\infty} \cos \lambda x F(x)\, dx.$$

As a result, we get the transmutation operator $J : F \to f$:

$$J[F](x) \equiv f(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \hat{F}(k) \cos kx, x \in [0, \pi]. \tag{10}$$

Simplify the function $\hat{F}(k)$

$$\begin{aligned} &\hat{F}(k) = \int_0^{\infty} \sin kx F(x)\, dx = \\ &= \int_0^{\pi} \cos kx \sum_{j=0}^{\infty} (F(x + 2\pi j) + F(2\pi - x + 2\pi j))\, dx, \end{aligned}$$

and back to (10):

$$\begin{aligned} &J[F](x) = f(x) = \tfrac{2}{\pi} \sum_{k=0}^{\infty} \hat{F}(k) \cos kx = \\ &= \sum_{j=0}^{\infty} (F(x + 2\pi j) + F(2\pi - x + 2\pi j)). \end{aligned} \tag{11}$$

Formula (11) defines the required transmutation operator. We will apply it to the Neumann problem in the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x < \pi, -\infty < y < \infty; \\ u'(0, y) = g(y), u'(\pi, y) = 0, \end{cases} \tag{12}$$

Let a function $U(x, y)$ be the solution of Neumann problem for a semi-plane

$$\begin{cases} U''_{xx} + U''_{yy} = 0, 0 < x, -\infty < y < \infty; \\ U'(0, y) = g(y). \end{cases} \tag{13}$$

By using (11), we obtain a new formula for solving problem (12):

$$u\left(x,y\right)=J\left[U\left(x,y\right)\right]=\sum_{j=0}^{\infty}\left(U\left(x+2\pi j,y\right)-U\left(2\pi-x+2\pi j,y\right)\right). \qquad (14)$$

By integrating identity (7), we get

$$\sum_{j=0}^{\infty}\left(\frac{1}{2}\ln\frac{(x+2\pi j)^2+(y-\eta)^2}{(2\pi j)^2}+\frac{1}{2}\ln\frac{(2\pi-x+2\pi j)^2+(y-\eta)^2}{(2\pi+2\pi j)^2}\right)=$$

$$=\frac{1}{2}\ln\left(ch\left(y-\eta\right)-\cos x\right).$$

As a result, we obtain a solution to the Neumann problem in the strip:

$$u\left(x,y\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left(ch\left(y-\eta\right)-\cos x\right)g\left(\eta\right)d\eta.$$

2.1.3 Sturm–Liouville Mixed Boundary Value Problem

The Sturm–Liouville problem about finding non-trivial solutions on the interval $[0,\pi]$

$$\begin{cases} y''+\lambda^2 y=0, \\ y\left(0\right)=0,\ y'\left(\pi\right)=0. \end{cases}$$

has eigenvalues $\lambda_k=k$, $k=1,2,3,\ldots$ and corresponding eigenfunctions

$$y_k\left(x\right)=\sin\left(\left(k-\frac{1}{2}\right)x\right),\ k=1,2,3,\ldots$$

Let the function $y=f\left(x\right)$ be given on segment $[0,\pi]$ and $\hat{f}\left(k\right)$ be its finite Fourier transform on segment $[0,\pi]$

$$\hat{f}\left(k\right)=\int_0^{\pi}\sin\left(k-\frac{1}{2}\right)xf\left(x\right)dx,$$

then

$$f\left(x\right)=\frac{2}{\pi}\sum_{k=1}^{\infty}\hat{f}\left(k\right)\sin\left(k-\frac{1}{2}\right)x.$$

Let the function $y = F(x)$ be given on the interval $[0, \infty)$ and $\hat{F}(\lambda)$ be its Fourier sine transform

$$\hat{F}(\lambda) = \int_0^\infty \sin \lambda x F(x)\, dx,$$

then

$$F(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \hat{F}(\lambda)\, d\lambda.$$

The function $y = f(x)$ on the interval $[0, \pi]$ corresponds to the function $F(x)$ by the rule:

$$f(x) = J[F](x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{F}(k) \sin\left(k - \frac{1}{2}\right) x \tag{15}$$

The transmutation operator J is given by formula (15). Formula (15) can be simplified:

$$J[F(x)] = f(x) = \sum_{j=0}^{\infty} (-1)^j \left(F(x + 2\pi j) + F(2\pi - x + 2\pi j)\right). \tag{16}$$

We will apply the constructed transmutation operator (16) for the mixed boundary value problem in the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x < \pi, -\infty < y < \infty; \\ u(0, y) = g(y), u'(\pi, y) = 0, \end{cases} \tag{17}$$

and Dirichlet problem for the semi-plane (5). By using (16), we obtain a new formula for solving problem (17)

$$u(x, y) = J\left[\tilde{u}(x, y)\right] = \sum_{j=0}^{\infty} (-1)^j \left(\tilde{u}(x + 2\pi j, y) + \tilde{u}(2\pi - x + 2\pi j, y)\right). \tag{18}$$

Based on the identity of [12]

$$\sum_{j=0}^{\infty} (-1)^j \left(\frac{x + 2\pi j}{(x + 2\pi j)^2 + (y - \eta)^2} + \frac{2\pi - x + 2\pi j}{(2\pi - x + 2\pi j)^2 + (y - \eta)^2}\right) = \frac{\sin \frac{x}{2} ch \frac{y-\eta}{2}}{ch(y - \eta) - \cos x},$$

we get a new formula for solving a mixed boundaries [15] value problem in the strip [12]

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{x}{2} ch \frac{y-\eta}{2}}{ch(y - \eta) - \cos x} g(\eta)\, d\eta.$$

2.1.4 Sturm–Liouville Problem with Dirichlet Boundary Conditions on Composite Real Semi-Axis

Let's consider the Sturm–Liouville singular problem about finding nontrivial solutions on composite real semi-axis $E_{1+} = (0, l) \cup (l, \infty)$,

$$\lambda^2 y_j + a_j^2 y''_{jxx} = 0, \quad x \in E_{1+}, \, j = 1, 2; \tag{19}$$

with boundary conditions

$$y_1(0) = 0, \, |y_2(x)| < \infty \tag{20}$$

and inner boundary conditions

$$y_1(l) = y_2(l), \quad \lambda_1 y'_1(l) = \lambda_2 y'_2(l). \tag{21}$$

The eigenvalues of problem (19)–(21) are the interval $(0, \infty)$, and eigenfunctions are, [16]

$$y_1(x, \lambda) = Jm\left[\left(e^{i\lambda \frac{x}{a_1}} - \frac{k-1}{k+1} e^{i\lambda \frac{2l-x}{a_1}}\right)\left(1 - \frac{k-1}{k+1} e^{i\lambda \frac{2l}{a_1}}\right)^{-1}\right], 0 < x < l,$$

$$y_2(x, \lambda) = \frac{2}{k+1} Jm\left[e^{i\lambda \frac{x-l}{a_2}} e^{i\lambda \frac{l}{a_1}}\left(1 - \frac{k-1}{k+1} e^{i\lambda \frac{2l}{a_1}}\right)^{-1}\right], l < x, k = \frac{\lambda_2}{\lambda_1}\frac{a_1}{a_2}.$$

Formulas can be represented as

$$y_1(x, \lambda) = \sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(\sin\left(\frac{x+2lj}{a_1}\right) - \frac{k-1}{k+1}\sin\left(\frac{2l-x+2lj}{a_1}\right)\right), 0 < x < l,$$

$$y_2(x, \lambda) = \frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j \sin\left(\frac{x-l}{a_2} + \frac{l+2lj}{a_1}\right), l < x. \tag{22}$$

The decomposition theorem on eigenfunctions is valid

$$f_1(x) = \frac{2}{\pi}\int_0^{\infty} y_1(x, \lambda)\, F(\lambda)\, d\lambda, 0 < x < l;$$

$$f_2(x) = \frac{2}{\pi}\int_0^{\infty} y_2(x, \lambda)\, F(\lambda)\, d\lambda, l < x. \tag{23}$$

where $F(\lambda)$ is the spectral function. Let the function $y = \tilde{f}(x)$ be define on the real semi-axis, and the function $F(\lambda)$ be its Fourier sine transform

$$F(\lambda) = \int_0^\infty \sin(\lambda\xi)\, \tilde{f}(\xi)\, d\xi.$$

The transmutation operator J is defined by formulas (23), i.e. $J : \tilde{f} \to f$,

$$f(x) = f_1(x)\,(\theta(l - x) \cdot \theta(x)) + f_2(x)\,\theta(x - l).$$

We obtain transformation operator from (22):

$$f_1(x) = \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1}\right)^j \left(\tilde{f}\left(\frac{x + 2lj}{a_1}\right) - \frac{k-1}{k+1}\tilde{f}\left(\frac{2l - x + 2lj}{a_1}\right)\right), 0 < x < l;$$

$$f_2(x) = \frac{2}{k+1}\sum_{j=0}^{\infty} \left(\frac{k-1}{k+1}\right)^j \tilde{f}\left(\frac{x-l}{a_2} + \frac{l + 2lj}{a_1}\right), l < x; \tag{24}$$

2.2 *Reflection Method*

In this section a transmutation operator is constructed as infinite sum of reflections from the domain boundaries. As a result, the solution of the basic boundary value problem is obtained on the base of the model boundary value problem.

2.2.1 Non-local Boundary Value Problem on the Strip

Let the function $\tilde{u}(x, y)$ be a solution of the Dirichlet model problem (5) and let the function $u(x, y)$ be a solution of boundary value problem with non-local boundary conditions for the Laplace equation in the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, \\ u(0, y) = f(y), \\ u'(0, y) = -u'(l, y). \end{cases} \tag{25}$$

We will apply the method of successive reflections from the boundaries $x = 0$ and $x = l$. As a zero-order approximation, we choose the solution of model problem (5), i.e. $u_0(x, y) = \tilde{u}(x, y)$. We will look for the first-order approximation in the form

$$u_1(x, y) = \tilde{u}(x, y) + v_0(x, y),$$

here $v_0(x, y)$ is a harmonic function in the right semi-plane

$$\tilde{u}'(0, y) - \tilde{u}'(l, y) = -v_0'(0, y) + v_0'(l, y).$$

Then $v_0(x, y) = \tilde{u}(l - x, y)$. So, the first-order approximation is

$$u_1(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y).$$

Repeating the algorithm we find the second-order approximation $u_2(x, y)$ and a sequence of approximations

$$u_2(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y) - \tilde{u}(l + x, y).$$

$$u_3(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y) - \tilde{u}(l + x, y) + \tilde{u}(2l + x, y).$$

$$u_4(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y) - \tilde{u}(l + x, y) + \tilde{u}(2l + x, y) - \tilde{u}(2l - x, y).$$

$$\dots$$

$$u_{2n}(x, y) = u_{2n-2}(x, y) + (-1)^n (\tilde{u}(x + nl, y) - \tilde{u}(-x + nl, y)).$$

$$u_{2n-1}(x, y) = u_{2n-2}(x, y) + (-1)^n \tilde{u}(x + nl, y).$$

As a limit we obtain the exact solution to problem (25)

$$u(x, y) = \tilde{u}(x, y) + \sum_{j=1}^{\infty} (-1)^j (\tilde{u}(x + lj, y) - \tilde{u}(-x + lj, y)).$$

2.2.2 Boundary Value Problem with Inner Boundary Conditions in a Strip

Let's consider the Dirichlet problem for the Laplace equation in the strip:

$$S_1 = \{(x, y) : x \in (0, l) \cup (l, L), y \in (-\infty, \infty)\}$$

$$\begin{aligned} u''_{1xx} + u''_{1yy} &= 0, 0 < x < l, -\infty < y < \infty, \\ u''_{2xx} + u''_{2yy} &= 0, l < x < L, -\infty < y < \infty \end{aligned}$$

with boundary conditions

$$y_1(0) = 0, |y_2(x)| < \infty$$

$$\begin{aligned} u_1(0, y) &= f(y), -\infty < y < \infty; \\ u_2(L, y) &= 0, -\infty < y < \infty \end{aligned} \tag{26}$$

and inner boundary conditions on the straight line $x = l$

$$u_1(l, y) = u_2(l, y), -\infty < y < \infty;$$
$$\lambda_1 u_1'(l, y) = \lambda_2 u_2'(l, y), -\infty < y < \infty.$$

The solution to problem (26) will be found by the reflection method. The zero-order approximation will be the solution of the model problem (4), i.e.

$$u_1^0(x, y) = \tilde{u}_0(x, y), 0 < x < l, \quad u_2^0(x, y) = \tilde{u}_0(x, y), l < x < L.$$

First- order approximation has the form

$$u_1^1(x, y) = \tilde{u}_0(x, y) + \tfrac{1-k}{1+k}\tilde{u}_0(2l - x, y), 0 < x < l;$$
$$u_2^1(x, y) = \tfrac{2}{1+k}\tilde{u}_0(x, y), l < x < L, k = \tfrac{\lambda_2}{\lambda_1}.$$

Let the function $\tilde{u}_1(x, y)$ be a solution of the model problem (4) with the boundary condition $\tilde{u}_1(0, y) = \tilde{u}_0(2l, y)$, then the second-order approximation will be

$$u_1^1(x, y) = u_1^0(x, y) + \tfrac{k-1}{k+1}\left(\tilde{u}_1(x, y) - \tfrac{k-1}{k+1}\tilde{u}_1(2l - x, y)\right), 0 < x < l;$$
$$u_2^1(x, y) = u_2^0(x, y) + \tfrac{2}{k+1}\tilde{u}_1(x, y), l < x < L.$$

If $u_1^n(x, y), u_2^n(x, y)$ are an approximations of order n, then the $(n + 1)$—order approximations are

$$u_1^{n+1}(x, y) = u_1^n(x, y) + \tfrac{k-1}{k+1}\left(\tilde{u}_{n+1}(x, y) - \tfrac{k-1}{k+1}\tilde{u}_{n+1}(2l - x, y)\right), 0 < x < l;$$
$$u_2^{n+1}(x, y) = u_2^n(x, y) + \tfrac{2}{k+1}\tilde{u}_{n+1}(x, y), l < x < L,$$

where $u_{n+1}(x, y)$ is the solution of model problem (4) with the boundary condition

$$\tilde{u}_{n+1}(0, y) = \tilde{u}_n(2l, y).$$

If $n \to \infty$ we get

$$u_1(x, y) = \sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(\tilde{u}_j(x, y) - \frac{k-1}{k+1}\tilde{u}_j(x, y)\right), 0 < x < l; \tag{27}$$

$$u_2(x, y) = \frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\tilde{u}_j(x, y), l < x. \tag{28}$$

2.3 The Fourier Transform Technique

Let the function $u(x, y)$ be a solution of Laplace equation with periodicity boundary conditions in the strip

$$S = \{(x, y) : x \in (0, l), y \in (-\infty, \infty)\}$$

$$\begin{cases} u''_{xx} + u''_{yy} = 0, \\ u(0, y) - u(l, y) = f(y), \\ u'(0, y) - u'(l, y) = 0. \end{cases} \tag{29}$$

And let $F(\lambda)$ be the Fourier transform of function $f(y)$, i.e.

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda\eta} f(\eta)\, d\eta,$$

then the solution to problem (27) takes form

$$u(x, y) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-|\lambda|x} - e^{-|\lambda|(l-x)}}{1 - e^{-|\lambda|l}} e^{i\lambda y} F(\lambda)\, d\lambda.$$

Expand the kernel in a series of powers $e^{-|\lambda|l}$

$$\frac{e^{-|\lambda|x} - e^{-|\lambda|(l-x)}}{1 - e^{-|\lambda|l}} = \sum_{k=0}^{\infty} \left(e^{-|\lambda|(x+lj)} - e^{-|\lambda|(l-x+lj)} \right).$$

Then we get

$$u(x, y) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-|\lambda|(x+lj)} e^{i\lambda y} F(\lambda)\, d\lambda - e^{-|\lambda|(l-x+lj)} e^{i\lambda y} F(\lambda)\, d\lambda \right).$$

The Inverse Fourier transform gives:

$$u(x, y) = \frac{1}{2} \sum_{j=0}^{\infty} \left(\tilde{u}(x + lj) - \tilde{u}(l - x + lj) \right).$$

Taking into account the formulae from [12]

$$\frac{1}{\pi} \sum_{j=0}^{\infty} \left(\frac{x + lj}{(x + lj)^2 + y^2} - \frac{l - x + lj}{(l - x + lj)^2 + y^2} \right) = \frac{1}{l} \frac{\sin \frac{2\pi x}{l}}{ch \frac{2\pi y}{l} - \cos \frac{2\pi x}{l}}.$$

we get a solution to the problem with boundary conditions of periodicity

$$u(x, y) = \frac{1}{2l} \int_{-\infty}^{\infty} \frac{\sin \frac{2\pi x}{l}}{ch \frac{2\pi(\eta - y)}{l} - \cos \frac{2\pi x}{l}} f(\eta)\, d\eta.$$

2.4 Neumann Series Technique

In the section the transmutation operator is searched as the Neumann series sum [9] of shift or generalized shift operators.

2.4.1 Solution of the Laplace Equation with Non-local Boundary Conditions in the Strip

Let the function $u(x, y)$ be a solution of the Laplace equation with non-local boundary conditions in the strip $S = \{(x, y) : x \in (0, l), y \in (-\infty, \infty)\}$

$$\begin{cases} u(0, y) = f(y), -\infty < y < \infty; \\ u'(0, y) = u'(l, y), -\infty < y < \infty. \end{cases} \tag{30}$$

The solution to problem (30) will be sought in the form

$$u(x, y) = A_1 \tilde{u}(x, y) + A_2 \tilde{u}(l - x, y),$$

where A_1, A_2 are unknown operators, $\tilde{u}$ is the solution of the model problem (5). We get the system of equations for operators A_1, A_2

$$\begin{cases} A_1 + A_2 = 0, \\ A_1 + A_2 T_l = I, \end{cases}$$

here T_l is the shift operator $T_l : u(x, y) \to u(x + l, y)$ and I is an identity operator. The solution to the system of operator equations is

$$A_1 = (I - T_l)^{-1}, \quad A_2 = -(I - T_l)^{-1}.$$

By using Neumann series

$$(I - T_l)^{-1} = \sum_{j=0}^{\infty} T_l^j,$$

we get

$$u\left(x,y\right)=\sum_{j=0}^{\infty}\left(\tilde{u}\left(x+lj,y\right)-\tilde{u}\left(l-x+lj,y\right)\right),0<x<l,-\infty<y<\infty. \tag{31}$$

Based on formula (31), we obtain the solution of the non-local problem (30)

$$u\left(x,y\right)=\frac{1}{l}\int_{-\infty}^{\infty}\frac{\sin\frac{2\pi x}{l}}{ch\frac{2\pi(\eta-y)}{l}-\cos\frac{2\pi x}{l}}f\left(\eta\right)d\eta. \tag{32}$$

Formula (32) is obtained for the first time.

2.4.2 Solution of the Laplace Equation with Generalized Non-local Boundary Conditions in a Strip

Let the function $u\left(x,y\right)$ be a solution of the Laplace equation in a strip $S=\{(x,y):x\in\left(0,l\right),y\in\left(-\infty,\infty\right)\}$ with non-local boundary conditions

$$\begin{cases} u\left(0,y\right)=f\left(y\right), \\ ku'\left(0,y\right)=u'\left(l,y\right),-1\leq k\leq 1. \end{cases} \tag{33}$$

We will seek a solution to the problem in the form

$$u\left(x,y\right)=A_1\tilde{u}\left(x,y\right)+A_2\tilde{u}\left(l-x,y\right).$$

From the boundary conditions (33) we have a system of equations

$$\begin{cases} kA_1-kA_2T_l-A_1T_l+A_2=0, \\ A_1+A_2T_l=I. \end{cases}$$

The formal solution to the system of equations has the form

$$\begin{aligned} A_1&=\left(I-kT_l\right)\left(I-2kT_l+T_l^2\right)^{-1}, \\ A_2&=\left(T_l-kI\right)\left(I-2kT_l+T_l^2\right)^{-1}. \end{aligned}$$

We apply formulas for the generating functions of Chebyshev polynomials [11] of first and second kind

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty}T_n(k)t^n&=\tfrac{1-tk}{1-2tk+t^2}; \\ \textstyle\sum_{n=0}^{\infty}U_n(k)t^n&=\tfrac{1}{1-2tk+t^2}. \end{aligned}$$

As a result, we get for operators A_1, A_2 [11]

$$\begin{aligned} A_1 &= (I - kT_l)\left(I - 2kT_l + T_l^2\right)^{-1} = \sum_{j=0}^{\infty} T_j\left(k\right) T_l^j, \\ A_2 &= (T_l - kI)\left(I - 2kT_l + T_l^2\right)^{-1} = \sum_{j=0}^{\infty} \left[-\tfrac{1}{k} T_j\left(k\right) + \tfrac{1-k^2}{k} U_j\left(k\right)\right] T_l^j. \end{aligned}$$

Thus, we have

$$u\left(x, y\right) = \sum_{j=0}^{\infty} T_j\left(k\right) \tilde{u}\left(x + lj, y\right) + \sum_{j=0}^{\infty} \left[-\frac{1}{k} T_j\left(k\right) + \frac{1-k^2}{k} U_j\left(k\right)\right] \tilde{u}\left(l - x + lj, y\right).$$

Using the recurrent relation [11], we obtain

$$T_{j+2}(k) = kT_{j+1}(k) - (1-k^2)U_j(k),$$

then

$$-\frac{1}{k} T_j\left(k\right) + \frac{1-k^2}{k} U_j\left(k\right) = -\frac{1}{k} T_j\left(k\right) + T_{j+1}\left(k\right) - \frac{1}{k} T_{j+2}\left(k\right).$$

The recurrent relation for Chebyshev polynomials of the first kind has the form

$$T_{j+2}(k) = 2kT_{j+1}(k) - T_j(k),$$

then

$$-\frac{1}{k} T_j\left(k\right) + \frac{1-k^2}{k} U_j\left(k\right) = -T_{j+1}\left(k\right).$$

As a result, we have the solution to the boundary value problem

$$u\left(x, y\right) = \tilde{u}\left(x, y\right) + \sum_{j=1}^{\infty} T_j\left(k\right) \left(\tilde{u}\left(x + lj, y\right) - \tilde{u}\left(-x + lj, y\right)\right).$$

3 Results

All proposed and developed methods from Sect. 2 are successfully applied to solving boundary value problems with non-classical boundary conditions. The proposed techniques allow us to find a formula, see (38), for solving the Dirichlet problem with inner boundary conditions for the semi-plane. We illustrate the proof of formula (38) by using the Neumann series expansion method. Formula (38) is a new result for the theory of potentials. To solve the Dirichlet problem with inner boundary conditions for the strip, the reflection method is most effective, the new result is represented in (39). Using the finite Fourier transforms method, a new result is obtained for the three-dimensional Dirichlet problem in a flat layer, see formula (41). We will apply the Neumann expansion method in solving problem of the Laplace equation in semi-plane $E_{1+} = \{ \quad (x, y) : y \in R, x \in (0, l) \cup (l, \infty) \}$

$$u''_{jyy} + a_j^2 u''_{jxx} = 0, \quad (x, y) \in E_{1+}, j = 1, 2; \tag{34}$$

with boundary condition

$$u_1 (0, y) = f (y) \tag{35}$$

and inner boundary conditions

$$u_1 (l, y) = u_2 (l, y), \quad \lambda_1 u'_{1x} (l, y) = \lambda_2 u'_{2x} (l, y). \tag{36}$$

We will seek a solution to problem (34)–(36) in the form

$$u_1 (x, y) = c_1 \tilde{u} \left(\frac{x}{a_1}, y \right) + c_2 \tilde{u} \left(\frac{2l - x}{a_1}, y \right), 0 < x < l;$$

$$u_2 (x, y) = c_3 \tilde{u} \left(\frac{x - l}{a_2} + \frac{l}{a_1}, y \right), l < x.$$

From (34)–(36) we get the system of equations

$$\begin{cases} c_1 + c_2 T = I, \\ c_1 + c_2 = c_3, \\ c_1 - c_2 = k c_3, \end{cases} \tag{37}$$

where $k = \frac{\lambda_2}{\lambda_1} \frac{a_1}{a_2}$ and T is the shift operator $T \left[\tilde{u} (x, y) \right] = \tilde{u} \left(x + \frac{2l}{a_1} \right)$. The solution of the system of equations (37) is obtained as an expansion in a series of Neumann operators in powers of the operator $\frac{k-1}{k+1} \cdot T$

$$c_1 = \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1} \right)^j T^j, c_2 = -\frac{k-1}{k+1} \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1} \right)^j T^j, c_3 = \frac{2}{k+1} \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1} \right)^j T^j,$$

where T^j is the power of operator T i.e.

$$T^j \left[\tilde{u} (x, y) \right] = \tilde{u} \left(x + \frac{2lj}{a_1} \right), j = 0, 1, 2, \ldots$$

As a result, we obtain the formulas for solution to problem (34)–(36)

$$u_1(x,y)=\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(\tilde{u}\left(\frac{x+2lj}{a_1},y\right)-\frac{k-1}{k+1}\tilde{u}\left(\frac{2l-x+2lj}{a_1},y\right)\right),0<x<l;$$

$$u_2(x,y)=\frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\tilde{u}\left(\frac{x-l}{a_2}+\frac{l+2lj}{a_1},y\right),l<x. \tag{38}$$

The finite integral transforms method leads to formula (38) also. The reflection method is effective in the Dirichlet problem for the Laplace equation in the strip

$$S_1=\{(x,y):x\in(0,l)\cup(l,L),y\in(-\infty,\infty)\}.$$

Let $\tilde{u}(x,y)$be the solution of the model problem (4). The generalized shift operator T is defined by the rule $T\tilde{u}(0,y)=\tilde{u}(2l,y)$, then formulas (27)–(28) take the form

$$u_1(x,y)=\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(T^j\tilde{u}(x,y)-\frac{k-1}{k+1}T^j\tilde{u}(2l-x,y)\right),0<x<l;$$

$$u_2(x,y)=\frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^jT^j\tilde{u}(x,y),l<x. \tag{39}$$

The Fourier transform method and the Neumann series method are less effective, since solution of problem (34)–(36) is obtained in the form of multiple series and obtained formulas are difficult to apply in practice. The transmutation operators method has shown its effectiveness in solving model boundary value problems. Boundary value problems for the Laplace equation in the semi-plane and in the strip with inner boundary conditions can be investigated by the transmutation operators method. The method effectively works in the three-dimensional case. For example, consider the Dirichlet problem for the three-dimensional Laplace equation in the layer $0<x<\pi,-\infty<y_1,y_2<\infty$

$$\begin{cases}u''_{xx}+u''_{y_1y_1}+u''_{y_2y_2}=0,0<x<\pi,-\infty<y_1,y_2<\infty;\\ u(0,y_1,y_2)=g(y_1,y_2),u(\pi,y_1,y_2)=0,\end{cases} \tag{40}$$

We apply The finite Fourier integral transforms technique from Sect. 2.1 and we have

$$u(x,y_1,y_2)=\sum_{j=0}^{\infty}\left(\tilde{u}(x+2\pi j,y_1,y_2)-\tilde{u}(2\pi-x+2\pi j,y_1,y_2)\right),$$

there

$$\tilde{u}(x, y_1, y_2) = \frac{1}{2\pi^2} \int_0^{2\pi} \frac{x g(\eta_1, \eta_2)}{\left(x^2 + (y_1 - \eta_1)^2 + (y_2 - \eta_2)^2\right)^{\frac{3}{2}}} d\eta_1 d\eta_2.$$

To transform the formula for $u(x, y_1, y_2)$ we use the integral

$$\int_0^{2\pi} \frac{dt}{x + iy \sin t} = \frac{2\pi}{\sqrt{x^2 + y^2}}, x > 0,$$

it is obtained by the residue method, [1]. Find the derivative for the real part of the integral with respect to x, we get

$$-\frac{1}{2\pi} \frac{d}{dx} \left[\int_0^{2\pi} \frac{x dt}{x^2 + y^2 \sin^2 t} \right] = \frac{x}{\left(x^2 + y^2\right)^{\frac{3}{2}}}, x > 0.$$

From (7) we obtain the solution of the Dirichlet problem (40)

$$u(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{1 - \cos x ch(\sin t\,|y - \eta|)}{(ch(\sin t\,|y - \eta|) - \cos x)^2} dt g(\eta_1, \eta_2) d\eta_1 d\eta_2, \tag{41}$$

where $|y - \eta|^2 = |y_1 - \eta_1|^2 + |y_2 - \eta_2|^2$.

4 Conclusions

The universality of transmutation operators method gives the possibility of its application for any dimension problems with non-local boundary conditions. The method advantage is the easily implementation form on a computer due to the cyclical nature of the corresponding algorithm. Further, the transmutation operators method can be developed for boundary value problems with axial and central symmetry. The method can also be useful in the theory of integral transforms with discontinuous trigonometric kernels and for calculating integrals, summing series.

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