Some Recent Developments in the Transmutation Operator Approach

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Abstract This is a brief overview of some recent developments in the transmutation operator approach to practical solution of mathematical physics problems. It introduces basic notions and results of transmutation theory, and gives a brief historical survey with some important references. Mainly applications to linear ordinary and partial differential equations and to related boundary value and spectral problems are discussed.

Linear second order differential equations arise in innumerable models and problems of mathematics, physics, engineering, chemistry, biology and even social sciences. While linear ordinary differential equations of first order are easily solved, and the method of their solution is taught to students even of specialities not particularly close to mathematics, the situation of linear ordinary second order differential equations with variable coefficients is pretty much different. No general method for their solution in a closed form is known. On one hand this resembles the situation that had been occurring throughout centuries that separated the full understanding by the antique mathematicians of the algebraic quadratic equations from the epoch of N. Tartaglia, G. Cardano and L. Ferrari when finally algebraic equations of third and fourth orders succumbed to the efforts of mathematicians. On the other hand, the problem of a closed form solution of linear ordinary second order differential equations with variable coefficients is not even contemplated among the most important mathematical problems (of the century or millennium), perhaps because it is not expected to be solved ever.

One of the approaches used at all times is to reduce the difficult problem to a simpler one. Since linear second order equations with constant coefficients admit

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such a closed-form solution, a natural idea is to relate solutions of the equation with constant coefficients to solutions of the equation with variable coefficients via an operator which is called a transmutation operator. Consider the second order linear differential expression

$$L := -\frac{d^2}{dx^2} + q(x) \tag{1}$$

with q being an L_2-function defined on a finite interval. The equation

$$Ly(x) = \lambda y(x), \quad \lambda \in \mathbb{C} \tag{2}$$

is called the one-dimensional Schrödinger equation or very often the Sturm–Liouville equation, taking into account that a large variety of linear ordinary second order equations reduce to this form by a Liouville transformation.

A transmutation operator is sought to relate L to the simplest linear second order expression $B := -\frac{d^2}{dx^2}$ by the formula

$$LT = TB.$$

If T is linear and invertible its knowledge allows one to solve (2) at least formally. Indeed, one can look for a solution of (2) in the form $y = Tv$, where v is a solution of the equation $Bv = \lambda v$ (whose general solution is of course $v(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$). Then $Ly = LTv = TBv = \lambda Tv = \lambda y$, thus y is a solution of (2).

This idea in the theory of linear differential equations appeared in 1938 in the work [18] by J. Delsarte and later on it was developed in [1, 8–11, 19, 29, 47–51, 60, 62] and in many other publications. In particular, for Eq. (2) with the Sturm–Liouville operator (1) in [53] it was proved that such an operator T exists and even possesses some wonderful properties. Namely, it can be realized in the form of a linear Volterra integral operator of second kind with a continuous integral kernel. Hence T is invertible and its inverse T^{-1} admits the same form of a linear Volterra integral operator of second kind. Additionally, such T can be chosen to preserve the initial conditions fulfilled by the solutions. In [53] also applications to generalized positive definite functions were proposed. Similarly in [46] such transmutations were constructed on semi-axis with applications to inverse and scattering problems. Also transmutations for the Bessel operator

$$B_c := \frac{d^2}{dx^2} + \frac{c}{x}\frac{d}{dx} \tag{3}$$

of Sonine and Poisson types were introduced into the theory (cf. [8–11, 29, 33, 49, 62] together with transmutations for the permuted Bessel operator

$$L := \frac{d^2}{dx^2} + \frac{c}{x}\frac{d}{dx} + q(x) \tag{4}$$

which were widely applied, cf. [62–64]. A new class of Buschman-Erdélyi transmutations was studied in [29, 59, 61, 62]. For applications to special radial Schrödinger equation and construction of Jost solutions cf. [26, 27]. A general method for constructing transmutations from basic integral transforms called Integral Transform Composition Method (ITCM) was developed in [22, 28, 29, 62]. Transmutations for problems with Stark potentials were considered in [25] and with quantum oscillator potential in [52]. Interesting problems in transmutation theory in connection with fractional powers of Bessel operators were studied in [58]. In papers of E. Shishkina transmutations were applied to Euler–Poisson–Darboux equations [24, 57] and to the potential theory [54–56]. Applications of transmutations to problems in mechanics were considered in [68]. Connections of transmutation theory and generalized analytic functions were studied in [3, 35, 67]. Starting from the paper of V. Stashevskaya [64] a line of studying transmutations based on Paley–Wiener theory was developed in [13, 65, 66]. Applications of Sonine and Poisson type transmutations to pseudo differential and PDE equations were considered in [29, 33]. Applications to hyper-Bessel equations based on Obreshkov transform were studied in [20, 34]. Special representations of transmutation kernels via Bessel function series were developed in [14].

An important property of the Volterra-type transmutation operator related to (1) or (4) consists in the fact that the coefficient q often called the potential, can be easily found whenever the integral kernel of the transmutation operator T happens to be known. This together with other attractive properties converted the transmutation operators into one of the main theoretical tools of spectral theory and especially of the theory of inverse spectral problems developed in the works of V. A. Marchenko, I. M. Gel'fand and B. M. Levitan and of many other mathematicians. During that classical period in transmutation theory many famous problems were studied with the aid of this technique, among them: the inverse problem by a spectral function data via the Marchenko equation, the inverse scattering problem by a scattering data via the Gelfand–Levitan equation, Gelfand–Levitan trace formulas and many other. We refer to the books [1, 8–10, 23, 47, 48, 50, 51, 69] presenting this important and extremely beautiful piece of modern mathematics.

Attempts to convert the transmutation operators of this kind into practical tools for solving different problems of mathematical physics have been made for decades. Many applications to problems of mathematical physics were considered in [8–11]. We mention a series of publications of R. Gilbert with coauthors (referring to [2] and references therein) in which transmutation operators were used for solving acoustic wave propagation problems in inhomogeneous media, the work of D. Colton (see [15]) in which with the aid of transmutation operators complete systems of solutions for parabolic PDEs with variable coefficients were introduced and applied to solution of initial-boundary value problems. In those works the integral kernels of the transmutation operators were computed numerically by the successive approximation method whose implementation complicates since the iterations involve two-dimensional integrals. In [4] the transmutation operator kernel was approximated by a partial sum of its trigonometric series, however based on this method solution of linear ODEs does not seem practical.

In a series of recent publications [5–7, 12, 30, 32, 41, 44, 45] the idea from [2] and [15] to obtain complete systems of solutions of PDEs with variable coefficients as images of complete systems of solutions of PDEs with constant coefficients under the action of an appropriate transmutation operator was further developed based on the observation known since the work of M. K. Fage (see the book [21]) and called in [7] the mapping property of the transmutation operator, which indicates what are the images of integer nonnegative powers of the independent variable under the action of the transmutation operator. They result to be so-called formal powers arising from spectral parameter power series (SPPS) representations of solutions of linear ODEs (see [31, 38]) and for their computation an efficient recurrent integration procedure is developed. Thus, some complete systems of solutions for classes of PDEs can be constructed without knowledge of the transmutation operator itself but simply computing the formal powers.

Another advancement in the efficient construction of the integral transmutation kernels was reported in [39, 40, 42, 43] for transmutation operators with boundary conditions in the origin (according to the terminology used by B. M. Levitan), and in [17, 37] for the transmutation operators with boundary conditions at infinity. Based on the proposed representations for the integral transmutation kernels new practical and efficient methods were developed for solution of forward [17, 39, 40, 43] and inverse spectral and scattering problems [16, 36, 37]. In the case of the forward problems large sets of spectral data can be computed with a nondeteriorating accuracy due to the possibility of convenient uniform estimates for the approximate solutions. Meanwhile the approach developed for solving the inverse problems leads to a direct reduction of the problem to a corresponding system of linear algebraic equations. This new and promising area of the transmutation operator theory and applications is still in its beginning, attracting attention of researchers from different applied fields.

In general, the diversity of the topics associated with the transmutation operator theory and, in particular, of those considered in the present volume reveals the importance of the transmutation operators in a large number of fields as well as their intrinsic interconnections and applications.

References

1. Z.S. Agranovich, V.A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, London, 1963)
2. H. Begehr, R. Gilbert, *Transformations, Transmutations and Kernel Functions*, vol. 1–2 (Longman Scientific & Technical, Harlow, 1992)
3. S. Bergman, *Integral Operators in the Theory of Linear Partial Differential Equations* (Springer, Berlin, 1969)
4. A. Boumenir, The approximation of the transmutation kernel. J. Math. Phys. **47**, 013505 (2006)
5. H. Campos, R. Castillo, V.V. Kravchenko, Construction and application of Bergman-type reproducing kernels for boundary and eigenvalue problems in the plane, in *Complex Variables and Elliptic Equations*, vol. 57, Nos. 7–8 (Taylor & Francis, Didcot, 2012), pp. 787–824

6. H. Campos, V.V. Kravchenko, L.M. Méndez, Complete families of solutions for the Dirac equation: an application of bicomplex pseudoanalytic function theory and transmutation operators. Adv. Appl. Clifford Algebr. **22**(3), 577–594 (2012)
7. H. Campos, V.V. Kravchenko, S.M. Torba, Transmutations, L-bases and complete families of solutions of the stationary Schrödinger equation in the plane. J. Math. Anal. Appl. **389**(2), 1222–1238 (2012)
8. R.W. Carroll, Transmutation and operator differential equations, in *Mathematics Studies*, vol. 37 (Elsevier, Amsterdam, 1979)
9. R.W. Carroll, Transmutation, scattering theory and special functions, in *Mathematics Studies*, vol. 69 (Elsevier, Amsterdam, 1982)
10. R.W. Carroll, Transmutation theory and applications, in *Mathematics Studies*, vol. 117 (Elsevier, North-Holland, 1985)
11. R.W. Carroll, R.E. Showalter, *Singular and Degenerate Cauchy Problems* (Academic Press, New York, 1976)
12. R. Castillo, V.V. Kravchenko, R. Reséndiz, Solution of boundary value and eigenvalue problems for second order elliptic operators in the plane using pseudoanalytic formal powers. Math. Methods Appl. Sci. **34**(4), 455–468 (2011)
13. H. Chebli, Thèoréme de Paley—Winer associé à un opérateur différentiel singulier sur $(0, \infty)$. J. Math. Pures Appl. **58**, 1–19 (1979)
14. H. Chebli, A. Fitouhi, M.M. Hamza, Expansion in series of Bessel functions and transmutations for perturbed Bessel operators. J. Math. Anal. Appl. **181**(3), 789–802 (1994)
15. D.L. Colton, *Solution of Boundary Value Problems by the Method of Integral Operators* (Pitman, London, 1976)
16. B.B. Delgado, K.V. Khmelnytskaya, V.V. Kravchenko, The transmutation operator method for efficient solution of the inverse Sturm-Liouville problem on a half-line. Math. Methods Appl. Sci. **42**, 7359–7366 (2019). https://doi.org/10.1002/mma.5854
17. B.B. Delgado, K.V. Khmelnytskaya, V.V. Kravchenko, A representation for Jost solutions and an efficient method for solving the spectral problem on the half line. Math. Methods Appl. Sci. https://doi.org/10.1002/mma.5881
18. J. Delsarte, Sur certaines transformations fonctionnelles relatives aux équations linéaires aux dérivées partielles du second ordre. C. R. Acad. Sc. **206**, 178–182 (1938)
19. J. Delsarte, M.J.L. Lions, Transmutations d'operateurs differentieles dans le domaine complexe. Comment. Math. Helv. **32**, 113–128 (1957)
20. I. Dimovski, *Convolutional Calculus* (Kluwer Academic Publishers, Dordrecht, 1990)
21. M.K. Fage, N.I. Nagnibida, *The Problem of Equivalence of Ordinary Linear Differential Operators* (Nauka, Novosibirsk, 1987, in Russian)
22. A. Fitouhi, I. Jebabli, E.L. Shishkina, S.M. Sitnik, Applications of integral transforms composition method to wave-type singular differential equations and index shift transmutations. Electron. J. Differ. Equ. **2018**(130), 1–27 (2018)
23. G. Freiling, V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications* (Nova Science Publishers, Huntington, 2001)
24. Sh.T. Karimov, E.L. Shishkina, Some methods of solution to the Cauchy problem for a inhomogeneous equation of hyperbolic type with a Bessel operator. J. Phy.: Conf. Ser. **1203**(1), 1–12 (2019)
25. A.P. Katchalov, Ya.V. Kurylev, Transformation operator method for inverse scattering problem. J. Soviet Math. **57**(3), 3111–3122 (1991)
26. V.V. Katrakhov, S.M. Sitnik, A boundary-value problem for the steady-state Schrodinger equation with a singular potential. Soviet Math. Dokl. **30**(2), 468–470 (1984)
27. V.V. Katrakhov, S.M. Sitnik, Estimates of the Jost solution to a one-dimensional Schrodinger equation with a singular potential. Dokl. Math. **51**(1), 14–16 (1995)
28. V.V. Katrakhov, S.M. Sitnik, Composition method for constructing B-elliptic, B-hyperbolic, and B-parabolic transformation operators. Russ. Acad. Sci., Dokl. Math. **50**(1), 70–77 (1995)
29. V.V. Katrakhov, S.M. Sitnik, The transmutation method and boundary value problems for singular elliptic equations. Contemp. Math. Fundam. Dir. **64**(2), 211–426 (2018, in Russian)

30. K.V. Khmelnytskaya, V.V. Kravchenko, S.M. Torba, S. Tremblay, Wave polynomials, transmutations and Cauchy's problem for the Klein-Gordon equation. J. Math. Anal. Appl. **399**, 191–212 (2013)
31. K.V. Khmelnytskaya, V.V. Kravchenko, H.C. Rosu, Eigenvalue problems, spectral parameter power series, and modern applications. Math. Methods Appl. Sci. **38**, 1945–1969 (2015)
32. K.V. Khmelnytskaya, V.V. Kravchenko, S.M. Torba, Modulated electromagnetic fields in inhomogeneous media, hyperbolic pseudoanalytic functions and transmutations. J. Math. Phys. **57**, 051503 (2016)
33. I.A. Kipriyanov, *Singular Elliptic Boundary-Value Problems* (Nauka Fizmatlit, Moscow, 1997, in Russian)
34. V. Kiryakova, *Generalized Fractional Calculus and Applications* (Longman, Harlow, 1994)
35. V.V. Kravchenko, *Applied Pseudoanalytic Function Theory* (Birkhäuser, Basel, 2009)
36. V.V. Kravchenko, On a method for solving the inverse Sturm–Liouville problem. J. Inverse Ill-Posed Probl. **27**, 401–407 (2019)
37. V.V. Kravchenko, On a method for solving the inverse scattering problem on the line. Math. Methods Appl. Sci. **42**, 1321–1327 (2019)
38. V.V. Kravchenko, R.M. Porter, Spectral parameter power series for Sturm-Liouville problems. Math. Methods Appl. Sci. **33**, 459–468 (2010)
39. V.V. Kravchenko, S.M. Torba, A Neumann series of Bessel functions representation for solutions of Sturm-Liouville equations. Calcolo **55**(11), 23 (2018)
40. V.V. Kravchenko, L.J. Navarro, S.M. Torba, Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions. Appl. Math Comput. **314**(1), 173–192 (2017)
41. V.V. Kravchenko, J.A. Otero, S.M. Torba, Analytic approximation of solutions of parabolic partial differential equations with variable coefficients. Adv. Math. Phys. **2017**, Article ID 2947275, 5 (2017)
42. V.V. Kravchenko, E.L. Shishkina, S.M. Torba, On a series representation for integral kernels of transmutation operators for perturbed Bessel equations. Math. Notes **104**(3–4), 530–544 (2018)
43. V.V. Kravchenko, S.M. Torba, R. Castillo-Pérez, A Neumann series of Bessel functions representation for solutions of perturbed Bessel equations. Appl. Anal. **97**(5), 677–704 (2018)
44. I.V. Kravchenko, V.V. Kravchenko, S.M. Torba, Solution of parabolic free boundary problems using transmuted heat polynomials. Math. Methods Appl. Sci. **42**, 5094–5105 (2019)
45. I.V. Kravchenko, V.V. Kravchenko, S.M. Torba, J.C. Dias, Pricing double barrier options on homogeneous diffusions: a Neumann series of Bessel functions representation. Int. J. Theor. Appl. Finance **22**, 1950030 (2019). https://doi.org/10.1142/S0219024919500304
46. B.Ya. Levin, Fourier and Laplace type transforms by means of solutions to the second order differential equations. Dokl. AN SSSR [Rep. Acad. Sci. USSR] **106**(2), 187–190 (1956, in Russian)
47. B.M. Levitan, *Inverse Sturm-Liouville Problems* (VSP, Zeist, 1987)
48. B.M. Levitan, I.S. Sargsjan, *Sturm-Liouville and Dirac Operators* (Springer, Dordrecht, Kluwer Academic Publishers, Amsterdam, 1991)
49. J.L. Lions, *Equations Differentielles Operationnelles et Problemes Aux Limites* (Springer, Berlin, 1961)
50. V.A. Marchenko, *Spectral Theory of Sturm-Liouville Operators* (Nauk Dumka, Kiev, 1972, in Russian)
51. V.A. Marchenko, *Sturm-Liouville Operators and Applications: Revised Edition* (AMS Chelsea Publishing, Providence, 2011)
52. G.M. Masmaliev, A.Kh. Khanmamedov, Transformation operators for Perturbed Harmonic Oscillators. Math. Notes **105**(5), 728–733 (2019)
53. A.Ya. Povzner, On differential equations of Sturm-Liouville type on a half line. Mat. Sb. **23**(1), 3–52 (1948, in Russian)
54. E.L. Shishkina, Inversion of the mixed Riesz hyperbolic B-potentials. Int. J. Appl. Math. **30**(6), 487–500 (2017)

55. E.L. Shishkina, Solution of the singular Cauchy problem for a general inhomogeneous Euler–Poisson–Darboux equation. Carpathian J. Math. **2**, 255–267 (2018)
56. E.L. Shishkina, Properties of mixed hyperbolic B–potential. Progr. Fractional Differ. Appl. **4**(2), 83–98 (2018)
57. E.L. Shishkina, S.M. Sitnik, General form of the Euler–Poisson–Darboux equation and application of the transmutation method. Electron. J. Differ. Equ. **2017**(177), 1–20 (2017)
58. E.L. Shishkina, S.M. Sitnik, On fractional powers of Bessel operators. J. Inequalities Spec. Funct. (Special issue To honor Prof. Ivan Dimovski's contributions) **8**(1), 49–67 (2017)
59. S.M. Sitnik, Factorization and estimates of the norms of Buschman-Erdélyi operators in weighted Lebesgue spaces. Sov. Math. Dokl. **44**(2), 641–646 (1992)
60. S.M. Sitnik, Transmutations and applications: a survey, 141 (2010). http://arxiv.org/abs/1012.3741
61. S.M. Sitnik, A short survey of recent results on Buschman-Erdelyi transmutations. J. Inequalities Spec. Funct. (Special issue To honor Prof. Ivan Dimovski's contributions) **8**(1), 140–157 (2017)
62. S.M. Sitnik, E.L. Shishkina, *Method of Transmutations for Differential Equations with Bessel Operators*, (Fizmatlit, Moscow, 2019, in Russian)
63. A.S. Sokhin, On transforms of operators for equations with singularity of special form. Vestn. Khar'kov. un-ta [Bull. Kharkov Univ.] **113**, 36–42 (1974, in Russian)
64. V.V. Stashevskaya, On an inverse problem of spectral analysis for differential operator with singularity at zero. Uch. zap. Khar'kov. mat. ob-va [Sci. Notes Kharkov Math. Soc.] **25**(5), 49–86 (1957, in Russian)
65. Kh. Triméche, Transformation intégrale de Weil et thèoréme de Paley—Winer associés à un opérateur différentiel singulier sur $(0, \infty)$. J. Math. Pures Appl. **60**, 51–98 (1981)
66. Kh. Triméche, *Generalized Harmonic Analysis and Wavelet Packets* (Gordon and Breach, Amsterdam, 2001)
67. I.N. Vekua, *New Methods for Solving Elliptic Equations*. North-Holland Series in Applied Mathematics & Mechanics (North Holland, Amsterdam, 1967)
68. O. Yaremko, *Method of Transmutations in Problems of Mathematical Modelling* (Lambert Academic Publishing, Riga, 2012)
69. V.A. Yurko, *Introduction to the Theory of Inverse Spectral Problems* (Fizmatlit, Moscow, 2007, in Russian)