DOI 10.1007/s10958-022-05682-6
Journal of Mathematical Sciences, Vol. 260, No. 2, January, 2022

ON ADDITIVE BINARY PROBLEMS WITH SEMIPRIME NUMBERS OF A SPECIFIC FORM

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UDC 511.345

Abstract. The paper is devoted to methods of solution of binary additive problems with semiprime numbers, which form sufficiently "rare" subsequences of the natural series. Additional conditions are imposed on these numbers; the main condition is belonging to so-called Vinogradov intervals. We solve two problems that are analogs to the Titchmarsh divisor problem; namely, based on the Vinogradov method of trigonometric sums, we obtain asymptotic formulas for the number of solutions to Diophantine equations with semiprime numbers of a specific form.

***Keywords and phrases*:** binary additive problem, trigonometric sum, prime number, semiprime number, short interval.

***AMS Subject Classification*:** 11D75

1. Introduction. An important branch of the additive number theory is devoted to problems on prime numbers from so-called *Vinogradov* (*short*) intervals:

$$\big[(2m)^c,\ (2m+1)^c\big), \quad m \in \mathbb{N}, \quad c \in (1, 2]. \tag{1}$$

If we denote by $\{a\}$ the fractional part of a number a, then the fact that a prime number belongs to the interval (1) is equivalent to the condition

$$\left\{\frac{1}{2}p^{1/c}\right\} < \frac{1}{2}.$$

This branch of number theory is related to works of I. M. Vinogradov [14] and S. A. Gritsenko [4]. Additive problems with prime numbers from intervals of the form (1) were considered, for example, in [1, 3, 5, 6]. Problems examined in these papers either were ternary or admitted solution by the ternary scheme inapplicable in the case of additive binary problems with prime numbers from *Vinogradov* intervals. Solutions of such problems are based on an equipollent analog of the Bombieri–Vinogradov theorem (see [2, 13]), which has not yet been obtained. We mention D. Tolev's theorem (see [12] which asserts that if the inequalities $0 < \lambda < 1/4$, $0 < \theta < 1/4 - \lambda$, and $A > 0$ are fulfilled, then

$$\sum_{k \le x^\theta} \max_{y \le x} \max_{\gcd(a,k)=1} \left| \psi_\lambda(y; k, a) - \frac{y^{1-\lambda}}{\varphi(k)(1-\lambda)} \right| \le \frac{x^{1-\lambda}}{\ln^A x},$$

where

$$\psi_\lambda(y; k, a) = \sum_{\substack{n \le y \\ n \equiv a \pmod k \\ \{\sqrt{n}\} < 1/n^\lambda}} \Lambda(n).$$

In this theorem, the range of variation of the parameter k is less than $x^{1/4}$, whereas in the classical Bombieri–Vinogradov theorem it is close to $x^{1/2}$. Therefore, one cannot apply Tolev's theorem to binary additive problems with prime numbers from intervals of the form (1). For a particular case, this boundary was approximated to $x^{1/3}$ in [7], but this was insufficient.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 166, Proceedings of the IV International Scientific Conference "Actual Problems of Applied Mathematics," Kabardino-Balkar Republic, Nalchik, Elbrus Region, May 22–26, 2018. Part II, 2019.

At the present time, one can solve binary additive problems with *semiprime* numbers from (1), i.e., numbers of the form p_1p_2 or $p_1p_2^a$. They form sufficiently rare sequences in the set of natural numbers. For $a \geq 2$, the latter sequence is "close" to the sequence of prime numbers. In this paper, we consider two binary additive problems with these numbers from Vinogradov intervals, which can be regarded as analogs of the Titchmarsh divisor problem. These problems are solved by I. M. Vinogradov's method of trigonometric sums. Methods applicable to problems discussed in this paper can also be used for other binary problems with semiprime numbers from Vinogradov intervals.

2. Notation and auxiliary lemmas. We will use the following notation:

p, p_1, p_2 are prime numbers (primes);

$\pi(x) = \sum\limits_{p\leq x} 1$ is the number of primes that do not exceed x;

$\pi(x,a,k) = \sum\limits_{\substack{p\leq x,\\ p\equiv a \pmod k}} 1$;

$\tau(m)$ is the number of natural divisors of a number m;

$\mu(m)$ is the value of the Möbius function at m;

$\varphi(m)$ is the value of the Euler function at m;

$\Lambda(n)$ is the value of the von Mangoldt function at m;

$\operatorname{Li} x = \int\limits_2^x \frac{du}{\ln u}$;

$\{x\}$ is the fractional part of a number x;

$\gcd(a,b)$ is the greatest common divisor of numbers a and b;

$[a,b]$ is the least common multiple of numbers a and b;

$f(x) \sim g(x)$ means that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$;

$A \asymp B$ means that there exist c_1 and c_1 such that $c_1 B \leq A \leq c_2 B$.

We will need the following auxiliary assertions.

Lemma 1 (Brun–Titchmarsh theorem; see [8])**.** *For natural numbers a and k satisfying the conditions $(a,k) = 1$ and $k \leq x$, the following relation holds*:

$$\pi(x,a,k) = \sum_{\substack{p\leq x,\\ p\equiv a \pmod k}} 1 < \frac{(2+\eta)x}{\varphi(k)\ln\left(\frac{2x}{k}\right)},$$

where $\eta > 0$ and $x > x_0(\eta)$.

Lemma 2. *Let $X \geq 2$; then*

$$\sum_{m\leq X} \frac{1}{\varphi(m)} = c_0 \ln X + O(1).$$

Lemma 3 (see [16])**.** *For $N > 2$ and a positive integer l, we have*

$$\sum_{0<m\leq N} (\tau(m))^l \ll N(\ln N)^{2l-1},$$

where

$$c_0 = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{r\varphi(r)}.$$

Lemma 4 (Bombieri–Vinogradov theorem). *For any $A > 0$, there exists B such that*

$$\sum_{k\le\frac{\sqrt{x}}{\ln^B x}} \max_{\substack{l\equiv l_0(\bmod k)\\(l,k)=1}} \left|\pi(x,l,k)-\frac{\mathrm{Li}(x)}{\varphi(k)}\right| = O\left(\frac{x}{\ln^A x}\right).$$

Lemma 5 (see [15]). *Let r be a positive integer, α and β be real numbers, and $0 < \Delta < 0.25$ be such that $\Delta \le \beta-\alpha \le 1-\Delta$. Then there exists a periodic function $\psi(x)$ with period 1 possessing the following properties*:

(1) *$\psi(x) = 1$ in the interval $\alpha + 0.5\Delta \le x \le \beta - 0.5\Delta$;*
(2) *$0 \le \psi(x) \le 1$ in the intervals $\alpha - 0.5\Delta \le x \le \alpha + 0.5\Delta$ and $\beta - 0.5\Delta \le x \le \beta + 0.5\Delta$;*
(3) *$\psi(x) = 0$ in the interval $\beta + 0.5\Delta \le x \le 1 + \alpha - 0.5\Delta$;*
(4) *$\psi(x)$ can be expanded into the Fourier series of the form*

$$\psi(x) = \beta-\alpha+\sum_{m=1}^{\infty}\left(g_m e^{2\pi imx}+h_m e^{-2\pi imx}\right),$$

where g_m and h_m depend only on m, α, β, and Δ and, moreover,

$$|g_m| \le \frac{1}{\pi m},\quad |h_m| \le \frac{1}{\pi m},\quad |g_m| \le \beta-\alpha,\quad |h_m| \le \beta-\alpha,$$
$$|g_m| < \frac{1}{\pi m}\left(\frac{r}{\pi m\Delta}\right)^r,\quad |h_m| < \frac{1}{\pi m}\left(\frac{r}{\pi m\Delta}\right)^r.$$

Lemma 6. *Let x be a large number, $D \le x^{1-\alpha}$, where*

$$0<\alpha<\frac{1}{2},\quad (l,D)=1,\quad x_1<x,\quad x-x_1>x^{1-\alpha/2}.$$

Then

$$\sum_{x_1\le Dm+l\le x}\left(\tau(Dm+l)\right)^k = O\left(\frac{x-x_1}{D}(\ln x)^{a(k)}\right),$$

where $a(k)$ is a constant depending only on k.

The well-known Titchmarsh divisor problem is an example of binary additive problems. This problem consists of the search for an asymptotic formula for the number of solutions of the equation $p-1 = xy$, $p \le n$; it was stated and solved by E. Titchmarsh in 1930 (see [11]) under the assumption that the extended Riemann hypothesis is valid. In the 1960s, Yu. V. Linnik using his dispersion method obtained an asymptotic formula for this problem without additional conditions (see [10]).

Since binary additive problems with prime numbers from short intervals cannot yet be solved, the search for analogs of the Titchmarsh divisor problem with semisimple numbers from (1) is of great interest now.

3. Analog of the Titchmarsh divisor problem for semiprime numbers of the form p_1p_2 from short intervals. We consider the equation

$$p_1p_2 - xy = 1,\quad p_1p_2 \le n, \tag{2}$$

with the variables x, y, p_1, and p_2, where x and y are natural numbers and the prime numbers p_1 and p_2 satisfy the additional conditions $p_1 > \exp(\sqrt{\ln n})$ and $p_2 > \exp(\sqrt{\ln n})$.

We denote by $T(n)$ the number of solutions of Eq. (2). Obviously,

$$T(n) = \sum_{\substack{p_1p_2\le n\\ p_1>\exp(\sqrt{\ln n})\\ p_2>\exp(\sqrt{\ln n})}} \tau(p_1p_2-1). \tag{3}$$

We denote by $T_1(n)$ the number of solutions of Eq. (2) with semiprime numbers p_1p_2 from the intervals (1). Obviously,

$$T_1(n) = \sum_{\substack{p_1p_2 \le n \\ p_1 > \exp(\sqrt{\ln n}) \\ p_2 > \exp(\sqrt{\ln n}) \\ \{\frac{1}{2}(p_1p_2)^{1/c}\} < \frac{1}{2}}} \tau(p_1p_2 - 1). \tag{4}$$

Theorem 1. *If c is an arbitrary number from the semi-interval $(1, 2]$ and $T(n)$ and $T_1(n)$ are defined by the relations* (3) *and* (4), *respectively, then the following equality holds*:

$$T_1(n) = \frac{1}{2}T(n) + O(n \ln\ln\ln n), \tag{5}$$

where

$$T(n) \sim c_0 n \ln\ln n, \quad c_0 = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{r\varphi(r)}.$$

Proof. We split the proof of Theorem 1 into several steps.

1. First, we consider the sum $T_1(n)$ and estimate the error, which appears if we restrict the range of the variable x in Eq. (2). For this, we transform the sum by using the following auxiliary periodic function with period 1:

$$\varrho(y) = \begin{cases} 1 & \text{if } 0 \le y < 1/2, \\ 0 & \text{if } 1/2 \le y < 1. \end{cases} \tag{6}$$

We have

$$T_1(n) = \sum_{\substack{p_1p_2 \le n \\ p_1 > \exp(\sqrt{\ln n}) \\ p_2 > \exp(\sqrt{\ln n})}} \tau(p_1p_2 - 1)\varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right) = \sum_{\substack{p_1p_2 \le n,\ p_1p_2 - xy = 1 \\ p_1 > \exp(\sqrt{\ln n}) \\ p_2 > \exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right)$$

$$= \sum_{\substack{p_1p_2 \le n,\ p_1p_2 - xy = 1,\ x \le \sqrt{n} \\ p_1 > \exp(\sqrt{\ln n}),\ p_2 > \exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right) + \sum_{\substack{p_1p_2 \le n,\ p_1p_2 - xy = 1,\ x > \sqrt{n} \\ p_1 > \exp(\sqrt{\ln n}),\ p_2 > \exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right).$$

Let $P = n^{1/(\ln\ln n)^2}$; then

$$T_1(n) = 2 \sum_{\substack{p_1p_2 \le n,\ p_1p_2 - xy = 1 \\ x \le \sqrt{n}P^{-10} \\ p_1 > \exp(\sqrt{\ln n}),\ p_2 > \exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right) - R_2(n) + 2R_1(n), \tag{7}$$

where

$$R_1(n) = \sum_{\substack{p_1p_2 - xy = 1,\ p_1p_2 \le n,\ \sqrt{n}P^{-10} < x \le \sqrt{n} \\ p_1 > \exp(\sqrt{\ln n}),\ p_2 > \exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right),$$

$$R_2(n) = \sum_{\substack{p_1p_2 - xy = 1,\ x \le \sqrt{n},\ y \le \sqrt{n} \\ p_1 > \exp(\sqrt{\ln n}),\ p_2 > \exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right).$$

For estimating remainders, we use Lemma 1. Finally, we obtain

$$T_1(n) = 2 \sum_{\substack{p_1p_2\le n,\ p_1p_2-xy=1,\ x\le\sqrt{n}P^{-10}\\ p_1>\exp(\sqrt{\ln n}),\ p_2>\exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right) + O\left(\frac{n}{\ln\ln n}\right). \tag{8}$$

2. Now we restrict the range of one of the primes p_1 or p_2. We transform the sum in the right-hand side of Eq. (8) as follows:

$$\sum_{\substack{p_1p_2\le n,\ p_1p_2-xy=1,\ x\le\sqrt{n}P^{-10}\\ p_1>\exp(\sqrt{\ln n}),\ p_2>\exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right)$$

$$= 2 \sum_{\substack{p_1p_2\le n,\ p_1p_2-xy=1,\ x\le\sqrt{n}P^{-10}\\ \exp(\sqrt{\ln n})<p_1\le P,\ p_2>\exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right) + 2R_3(n) + R_4(n), \tag{9}$$

where

$$R_3(n) = \sum_{\substack{p_1p_2\le n,\ p_1p_2-xy=1,\ x\le\sqrt{n}P^{-10}\\ P<p_1\le\sqrt{n},\ p_2>\exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right),$$

$$R_4(n) = \sum_{\substack{p_1p_2-xy=1,\ p_1\le\sqrt{n},\ p_2\le\sqrt{n}\\ p_1>\exp(\sqrt{\ln n}),\ p_2>\exp(\sqrt{\ln n})\\ x\le\sqrt{n}P^{-10}}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right).$$

To estimate the remained $R_3(n)$, we use the fact that $|\varrho(n)| \le 1$ and Lemmas 1 and 3. We have

$$R_3(n) \ll n \ln\ln\ln n.$$

For estimating $R_4(n)$, we partition the summation interval with respect to p_1:

$$R_4(n) \le \sum_{\substack{p_1p_2-xy=1\\ p_1\le P,\ p_2\le\sqrt{n}\\ x\le\sqrt{n}P^{-10}}} 1 + \sum_{\substack{p_1p_2-xy=1\\ P<p_1\le\sqrt{n},\ p_2\le\sqrt{n}\\ x\le\sqrt{n}P^{-10}}} 1 = r' + r''.$$

The second sum in the right-hand side of the inequality obtained can be estimated exactly as for $R_3(n)$; we have

$$r'' \ll n \ln\ln\ln n.$$

Applying Lemma 2, we obtain

$$r' \ll \sum_{m\le P\sqrt{n}} \tau(m) \ll P\sqrt{n}\ln P\sqrt{n} \ll \frac{n\ln n}{(\ln\ln n)^2} \ll n\ln\ln\ln n.$$

Therefore, $R_4(n) \ll n\ln\ln\ln n$. Substituting the estimates for $R_3(n)$ and $R_4(n)$ into (9), we obtain from (8)

$$T_1(n) = 4\widetilde{T}_1(n) + O(n\ln\ln\ln n), \tag{10}$$

where

$$\widetilde{T}_1(n) = \sum_{\substack{p_1p_2\le n,\ p_1p_2-xy=1,\ x\le\sqrt{n}P^{-10}\\ \exp(\sqrt{\ln n})<p_1\le P,\ p_2>\exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right).$$

Arguing similarly, we arrive at the formula

$$T(n) = 4\widetilde{T}(n) + O(n \ln\ln\ln n), \tag{11}$$

where

$$\widetilde{T}(n) = \sum_{\substack{p_1p_2 - xy = 1,\ x \le \sqrt{n}P^{-10},\ p_1p_2 \le n \\ \exp(\sqrt{\ln n}) < p_1 < P,\ p_2 > \exp(\sqrt{\ln n})}} 1$$

3. Now we obtain an asymptotic formula for the sum $\widetilde{T}(n)$. Verify that the leading term of this formula has order $n \ln\ln n$. Note that if the numbers p_1 and x are not coprime ($\gcd(p_1, x) \ne 1$), then the congruence $p_1p_2 \equiv 1 \pmod{x}$ has no solutions and terms of the sum $\widetilde{T}(n)$ corresponding to such p_1 are equal to zero. Therefore, in the sequel we assume that summation is performed only over p_1 that are coprime with x. Therefore,

$$\widetilde{T}(n) = \sum_{x \le \sqrt{n}P^{-10}} \sum_{\substack{\exp(\sqrt{\ln n}) < p_1 < P \\ \widetilde{T}(n)}} \sum_{\substack{p_2 \le \frac{n}{p_1} \\ p_2 \equiv p_1^* (\bmod x)}} 1,$$

where p_1^* is a solution of the congruence $p_1 y \equiv 1 \pmod{x}$.

Changing the order of summation and using the notation from Lemma 1 in the interior sum, we obtain

$$\widetilde{T}(n) = \sum_{\exp(\sqrt{\ln n}) < p_1 < P} \sum_{\substack{x \le \sqrt{n}P^{-10} \\ \gcd(p_1, x) = 1}} \pi\left(\frac{n}{p_1}, p_1^*, x\right).$$

We represent $\widetilde{T}(n)$ in the form

$$\widetilde{T}(n) = \sum_{\exp(\sqrt{\ln n}) < p_1 < P} \sum_{\substack{x \le \sqrt{n}P^{-10} \\ \gcd(x, p_1) = 1}} \frac{\operatorname{Li}\left(\frac{n}{p_1}\right)}{\varphi(x)} + r(n), \tag{12}$$

where

$$r(n) \le \sum_{\exp(\sqrt{\ln n}) < p_1 < P} \sum_{x \le \sqrt{n}P^{-10}} \left(\pi\left(\frac{n}{p_1}, p^*, x\right) - \frac{\operatorname{Li}\left(\frac{n}{p_1}\right)}{\varphi(x)}\right).$$

Applying the Bombieri–Vinogradov theorem (Lemma 4), we have

$$\sum_{x \le \sqrt{n}P^{-10}} \left| \pi\left(\frac{n}{p_1}, p^*, x\right) - \frac{\operatorname{Li}\left(\frac{n}{p_1}\right)}{\varphi(x)} \right| \ll \frac{n}{p_1 \ln^A n},$$

where $A > 0$. Therefore,

$$r(n) \le \frac{n}{\ln^A n} \sum_{\exp(\sqrt{\ln n}) < p_1 < P} \frac{1}{p_1} \ll \frac{n \ln\ln n}{\ln^A n}. \tag{13}$$

After a simple transformation of the first term in the right-hand side of (12) we obtain

$$\sum_{\exp(\sqrt{\ln n}) < p_1 < P} \sum_{\substack{x \le \sqrt{n}P^{-10} \\ \gcd(x, p_1) = 1}} \frac{\operatorname{Li}\left(\frac{n}{p_1}\right)}{\varphi(x)} = s_1(n) + O(s_2(n)), \tag{14}$$

where

$$s_1(n) = n \sum_{\exp(\sqrt{\ln n})<p_1<P} \frac{1}{p_1 \ln \frac{n}{p_1}} \sum_{\substack{x\le\sqrt{n}P^{-10}\\ \gcd(x,p_1)=1}} \frac{1}{\varphi(x)},$$

$$s_2(n) = \frac{n}{\ln^2 n} \sum_{\exp(\sqrt{\ln n})<p_1<P} \frac{1}{p_1} \sum_{x\le\sqrt{n}P^{-10}} \frac{1}{\varphi(x)}.$$

Now we obtain an asymptotic formula for $s_1(n)$. First, we calculate the interior sum; for this, we represent it in the difference form:

$$\sum_{\substack{x\le\sqrt{n}P^{-10}\\ (x,p_1)=1}} \frac{1}{\varphi(x)} = \sum_{x\le\sqrt{n}P^{-10}} \frac{1}{\varphi(x)} - \sum_{\substack{x\le\sqrt{n}P^{-10}\\ p_1|x}} \frac{1}{\varphi(x)}.$$

Lemma 2 implies that

$$\sum_{x\le\sqrt{n}P^{-10}} \frac{1}{\varphi(x)} = c_0 \ln\sqrt{n}P^{-10} + O(1) = \frac{c_0}{2}\ln n + O\left(\frac{\ln n}{(\ln\ln n)^2}\right),$$

where

$$c_0 = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{r\varphi(r)}.$$

Since the Euler function φ satisfies the inequality $\varphi(a\cdot b) \ge \varphi(a)\cdot\varphi(b)$, we have

$$\sum_{\substack{x\le\sqrt{n}P^{-10}\\ p_1|x}} \frac{1}{\varphi(x)} = \sum_{x_1\le\sqrt{n}/P^{10}p_1} \frac{1}{\varphi(x_1p_1)} \le \sum_{x_1\le\sqrt{n}/P^{10}p_1} \frac{1}{\varphi(x_1)\varphi(p_1)}.$$

Taking into account the condition $p_1 > \exp(\sqrt{\ln n})$, the formula $\varphi(p_1) = p_1 - 1$, and Lemma 2, we get

$$\sum_{\substack{x\le\sqrt{n}P^{-10}\\ p_1|x}} \frac{1}{\varphi(x)} \le \frac{1}{p_1-1} \sum_{x_1\le\sqrt{n}P^{-10}} \frac{1}{\varphi(x_1)} = O(1).$$

Therefore,

$$\sum_{\substack{x\le\le\sqrt{n}P^{-10}\\ \gcd(x,p_1)=1}} \frac{1}{\varphi(x)} = \frac{c_0}{2}\ln n + O\left(\frac{\ln n}{(\ln\ln n)^2}\right).$$

Now we calculate the sum

$$\sum_{\exp(\sqrt{\ln n})<p_1<P} \frac{1}{p_1 \ln \frac{n}{p_1}}.$$

Applying the Abel transform and the formula

$$\sum_{p\le n} \frac{1}{p} = C + \ln\ln n + O\left(\frac{1}{\ln n}\right),$$

which is valid for $n > 2$, we have

$$\sum_{\exp(\sqrt{\ln n})<p_1<P} \frac{1}{p_1 \ln \frac{n}{p_1}} = \frac{\ln\ln n}{2\ln n} + O\left(\frac{1}{(\ln n)^{3/2}}\right). \tag{15}$$

Therefore, taking into account the estimate for the interior sum we have

$$s_1(n) = \frac{c_0}{4} n \ln\ln n + O\left(\frac{n}{\sqrt{\ln n}}\right).$$

For estimating $s_2(n)$, we apply Lemma 2 and the formula for the sum $\sum\limits_{p\le n} \frac{1}{p}$. We obtain

$$s_2(n) \ll \frac{n}{\ln^2 n} \cdot \ln\ln n \cdot \ln n = \frac{n \ln\ln n}{\ln n}.$$

Substituting the asymptotic formula for $s_1(n)$ into (14) and taking into account the estimate for $s_2(n)$, we get

$$\sum_{\exp(\sqrt{\ln n})<p_1<P} \ \sum_{\substack{x\le\sqrt{n}P^{-10} \\ \gcd(x,p_1)=1}} \frac{\mathrm{Li}\left(\frac{n}{p_1}\right)}{\varphi(x)} = \frac{c_0}{4} n \ln\ln n + O\left(\frac{n}{\sqrt{\ln n}}\right).$$

Using this formula and the estimates for $r(n)$ from (12), we arrive at the following asymptotic formula:

$$\widetilde{T}(n) = \frac{c_0}{4} n \ln\ln n + O\left(\frac{n}{\sqrt{\ln n}}\right), \quad c_0 = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{r\varphi(r)}. \tag{16}$$

4. We obtain an asymptotic formula for

$$\widetilde{T}_1(n) = \sum_{\substack{p_1p_2\le n,\ p_1p_2-xy=1,\ x\le\sqrt{n}P^{-10} \\ \exp(\sqrt{\ln n})<p_1\le P,\ p_2>\exp(\sqrt{\ln n})}} \varrho\left(\frac{1}{2}(p_1p_2)^{1/c}\right).$$

4.1. We use Lemma 5 and take the parameters r, Δ, α, and β in two ways. First, we define these parameters as follows: $r = [\ln n]$, $\Delta = 1/\ln^2(n)$, where $n > [e^2]$, $n \in \mathbb{N}$, $\alpha = \Delta/2$, and $\beta = (1-\Delta)/2$. It is easy to verify that these numbers satisfy the conditions of Lemma 5. We denote by $\varrho_1(x)$ the function $\psi(x)$ whose existence follows from Lemma 5.

Next, for the same r and Δ, we set $\alpha = -\Delta/2$ and $\beta = (1+\Delta)/2$ and denote the corresponding function by $\varrho_2(x)$. Then Lemma 5 implies that $\varrho_1(x) \le \chi(x) \le \varrho_2(x)$ and

$$\widetilde{T}_{11}(n) \le \widetilde{T}_1(n) \le \widetilde{T}_{12}(n), \tag{17}$$

where

$$\widetilde{T}_{1i}(n) = \sum_{\substack{p_1p_2-xy=1,\ p_1p_2\le n,\ x\le\sqrt{n}P^{-10} \\ \exp(\sqrt{\ln n})<p_1<P,\ p_2>\exp(\sqrt{\ln n})}} \varrho_i\left(\frac{1}{2}(p_1p_2)^{1/c}\right), \quad i = 1, 2.$$

If we obtain asymptotic formulas for $\widetilde{T}_{11}(n)$ and $\widetilde{T}_{12}(n)$ with the same leading terms, then the inequality (17) implies that a formula with the same leading terms ia also valid for $\widetilde{T}_1(n)$.

Expand the functions $\varrho_1(x)$ and $\varrho_2(x)$ into Fourier series:

$$\varrho_1\left(\frac{1}{2}(p_1p_2)^{1/c}\right) = \frac{1}{2} - \Delta + \sum_{|m|\ge1} g_1(m) e^{\pi i m (p_1p_2)^{1/c}},$$

$$\varrho_2\left(\frac{1}{2}(p_1p_2)^{1/c}\right) = \frac{1}{2} + \Delta + \sum_{|m|\ge1} g_2(m) e^{\pi i m (p_1p_2)^{1/c}},$$

where $|g_i(m)| \le 1/\pi|m|$ for $i = 1, 2$. This and (11) imply that

$$4\widetilde{T}_{1i}(n) = \left(\frac{1}{2} + O(\Delta)\right) T(n) + O(\ln\ln\ln n) + 4 \sum_{0<|m|<\Delta^{-1}\ln n} g_i(m) v_m(n), \tag{18}$$

where

$$v_m(n) = \sum_{\substack{p_1p_2-xy=1,\ p_1p_2\le n,\ x\le\sqrt{n}P^{-10}\\ \exp(\sqrt{\ln n})<p_1<P,\ p_2>\exp(\sqrt{\ln n})}} e^{\pi i m (p_1p_2)^{1/c}}.$$

Therefore, the same formula is also valid for $4\widetilde{T}_1(n)$:

$$4\widetilde{T}_1(n) = \left(\frac{1}{2}+O(\Delta)\right)T(n) + O(\ln\ln\ln n) + 4\sum_{0<|m|<\Delta^{-1}\ln n} g(m)v_m(n).$$

Substituting the equality obtained into (10), we obtain

$$T_1(n) = \left(\frac{1}{2}+O(\Delta)\right)T(n) + O(\ln\ln\ln n) + 4\sum_{0<|m|<\Delta^{-1}\ln n} g(m)v_m(n).$$

Therefore,

$$T_1(n) = \frac{1}{2}c_0 n\ln\ln n + O(\Delta n\ln\ln n) + O(n\ln\ln\ln n) + 4\sum_{0<|m|<\Delta^{-1}\ln n} g(m)v_m(n). \qquad (19)$$

Now we must estimate the last sum in (19) and obtain an asymptotic formula for $T_1(n)$.

4.2. An estimate of the trigonometric sum $v_m(n)$ was obtained in [17]. We present a sketch of the proof. First, we partition the interval $\big(\exp(\sqrt{\ln n}),\ P\big]$ of summation by p_1 into intervals $p_1\in(P_1/2,P_1]$, where $P_1\le P$, $P_1/2>e^{\sqrt{\ln n}}>1$, and consider the sums corresponding to these intervals:

$$S(m,P_1) = \sum_{\substack{p_1p_2-xy=1\\ x\le\sqrt{n}P^{-10}}}\ \sum_{\frac{P_1}{2}<p_1\le P_1}\ \sum_{p_1p_2\le n} e^{\pi i m (p_1p_2)^{1/c}},$$

where $P_1\in\big(\exp(\sqrt{\ln n}),\ P\big]$. Estimating this sum from above and applying the Cauchy inequality, we obtain

$$|S(m,P_1)|^2 \ll \frac{n}{P_1}\sum_{\substack{P_1/2<p_1\le P_1\\ P_1/2<p_1'\le P_1}}\ \sum_{n_2\le\min\left(\frac{n}{p_1},\frac{n}{p_1'}\right)} \tau'(p_1n_2-1)\tau'(p_1'n_2-1)e^{\pi i m (p_1^{1/c}-(p_1')^{1/c})n_2^{1/c}},$$

where

$$\tau'(k) = \sum_{\substack{xy=k\\ x\le\sqrt{n}P^{-10}}} 1.$$

Extracting the diagonal terms ($p_1=p_1'$), we have

$$|S(m,P_1)|^2 \ll \frac{n}{P_1}\sum_{P_1/2<p_1\le P_1}\ \sum_{n_2\le 2n/P_1}\tau^2(p_1n_2-1)$$
$$+\frac{n}{P_1}\sum_{\frac{P_1}{2}<p_1\le P_1}\ \sum_{\substack{P_1/2<p_1\le P_1\\ p_1\ne p_1'}}\left|\sum_{n_2\le\min\left(\frac{n}{p_1},\frac{n}{p_1'}\right)}\tau'(p_1n_2-1)\tau'(p_1'n_2-1)e^{\pi i m(p_1^{1/c}-(p_1')^{1/c})n_2^{1/c}}\right|.$$

Using Lemma 6 for estimating the first sum, we obtain the inequality

$$|S(m,P_1)|^2 \ll n^2\exp\left(-\frac{1}{2}\sqrt{\ln n}\right) + \frac{n}{P_1}\sum_{P_1/2<p_1\le P_1}\ \sum_{\substack{P_1/2<p_1'\le P_1\\ p_1\ne p_1'}}|S(m,P_1,p_1,p_1')|, \qquad (20)$$

where

$$S(m,P_1,p_1,p_1')=\sum_{n_2\le n_3}\tau'(p_1n_2-1)\tau'(p_1'n_2-1)e^{\pi im(p_1^{1/c}-(p_1')^{1/c})n_2^{1/c}},\quad n_3=\min\left(\frac{n}{p_1},\ \frac{n}{p_1'}\right).$$

Obviously,

$$S(m,P_1,p_1,p_1')=\sum_{x\le\sqrt{n}P^{-10}}\ \sum_{x_1\le\sqrt{n}P^{-10}}\ \sum_{\substack{n_2\le n_3,\\ n_2\equiv p_1^*(\bmod x)\\ n_2\equiv p_1'^*(\bmod x)}} e^{\pi im(p_1^{1/c}-(p_1')^{1/c})n_2^{1/c}},$$

where $p_1p_1^*\equiv 1 \pmod{x}$ and $p_1'p_1'^*\equiv 1 \pmod{x}$. Also we see that if the interior sum in the last equality is not empty, then n_2 belongs to one of residue classes modulo $[x_1,x]$. Therefore,

$$S(m,P_1,p_1,p_1')=\sum_{\delta\le\sqrt{n}P^{-10}}\ \sum_{x\le\sqrt{n}P^{-10}}\ \sum_{\substack{x_1\le\sqrt{n}P^{-10}\\ (x,x_1)=\delta}}\ \sum_{m_2\le m_3} e^{\pi im(p_1^{1/c}-(p_1')^{1/c})(\eta+xx_1m_2/\delta)^{1/c}},$$

where $m_3=\dfrac{n_3-\eta}{[x_1,x]}$ and $0\le\eta\le[x_1,x]$. Note that the sum with respect to m_2 is "sufficiently long." For example, if $\delta=1$, then $m_3\gg P^{19}$.

Introduce the notation $\varkappa=m(p_1^{1/c}-(p_1')^{1/c})$ and

$$S(m_3,P_1,p_1,p_1',x,x_1,\delta)=\sum_{m_2\le m_3} e^{\pi i\kappa(\eta+xx_1m_2/\delta)^{1/c}};$$

then

$$S(m,P_1,p_1,p_1')=\sum_{\delta\le\sqrt{n}P^{-10}}\ \sum_{x\le\sqrt{n}P^{-10}}\ \sum_{\substack{x_1\le\sqrt{n}P^{-10}\\ (x,x_1)=\delta}} S(m_3,P_1,p_1,p_1',x,x_1,\delta).$$

We partition the range of m_3 into geometric progressions and consider the following two cases.

If $m_4\le m_3/P$, then, applying the trivial estimate, we obtain

$$S(m_3,P_1,p_1,p_1',x,x_1,\delta)\ll\frac{m_3}{P}\ln n,$$

which implies

$$S(m,P_1,p_1,p_1')\ll\frac{n\ln^4 n}{PP_1}.$$

Substituting this estimate into (20), we obtain

$$|S(m,P_1)|^2\ll n^2\exp\left(-\frac{1}{2}\sqrt{\ln n}\right).$$

Introduce the notation

$$\varkappa=m(p_1^{1/c}-(p_1')^{1/c}),\quad S(m_3,P_1,p_1,p_1',x,x_1,\delta)=\sum_{m_2\le m_3} e^{\pi i\kappa(\eta+xx_1m_2/\delta)^{1/c}};$$

then

$$S(m,P_1,p_1,p_1')=\sum_{\delta\le\sqrt{n}P^{-10}}\ \sum_{x\le\sqrt{n}P^{-10}}\ \sum_{\substack{x_1\le\sqrt{n}P^{-10}\\ (x,x_1)=\delta}} S(m_3,P_1,p_1,p_1',x,x_1,\delta).$$

In what follows, we assume that $m_4>m_3/P$. Consider the sum

$$S(m_4)=\sum_{m_4<m_2\le 2m_4} e^{\pi i\varkappa(\eta+xx_1m_2/\delta)^{1/c}}.$$

For the case where

$$0 < \frac{|\varkappa|\,(xx_1/\delta)^{1/c}}{m_4^{1-1/c}} < \frac{1}{10},$$

we have

$$S(m_4) \ll \left(\frac{xx_1}{\delta}\right)^{-1/c}\ \frac{m_4^{1-1/c}}{|\varkappa|}.$$

We consider the case where

$$\frac{|\varkappa|\,(xx_1/\delta)^{1/c}}{m_4^{1-1/c}} > \frac{1}{10}.$$

The sum $S(m_4)$ can be estimated by the Vinogradov method.

For the case where $xx_1/\delta > n^{0.99}$, we can use the Vinogradov scheme of estimating the zeta sum (see, e.g., [9]) involving the "mean theorem." We obtain the estimate

$$S(m_4) \ll m_4 \exp\left(-\gamma \frac{\ln n}{(\ln\ln n)^6}\right).$$

If $xx_1/\delta \le n^{0.99}$, then for estimating $S(m_4)$ we apply van der Corput's method; in this case, we obtain the estimate $S(m_4) \ll \sqrt{m_4}$. Finally we have

$$v_m(n) \ll n e^{-\frac{1}{4}\sqrt{\ln n}}.$$

Taking into account this estimate, we obtain from (18)

$$T_1(n) = \frac{1}{2}T(n) + O(n \ln\ln\ln n).$$

Since $T(n) \sim c_0 n \ln\ln n$, the formula obtained is an asymptotic formula. Theorem 1 is proved. □

4. Analog of the Titchmarsh divisor problem for semiprime numbers of the form $p_1p_2^a$ from short intervals. We deduce an asymptotic formula for the number of solutions of the equation $p_1p_2^a - xy = 1$, where $a \in \mathbb{N}$, $a \ge 2$, the numbers $p_1p_2^a$ belong to intervals of the form (1), $p_1p_2^a \le n$, and primes p_1 and p_2 satisfy additional conditions.

Theorem 2. *Let $n \ge n_0 > 0$ and $a \ge 2$ be natural numbers, $Q = \exp(\sqrt{\ln n}\,)$, $A_1 = [1, nQ^{-1}]$, $A_2 = [1, Q^{1/a}]$, and*

$$G(n) = \sum_{p_1 \in A_1} \sum_{\substack{p_2 \in A_2 \\ p_1p_2^a - xy = 1}} \sum_{x,y} 1, \quad G_1(n) = \sum_{p_1 \in A_1} \sum_{\substack{p_2 \in A_2 \\ p_1p_2^a - xy = 1 \\ \{\frac{1}{2}(p_1p_2^a)^{1/c}\} < \frac{1}{2}}} \sum_{x,y} 1.$$

Then the following equality holds:

$$G_1(n) = \frac{1}{2}G(n)(1 + O(Q^{-\eta})),$$

where

$$G(n) = c_0 \operatorname{Li}\left(\frac{n}{Q}\right)\pi(Q^{1/a})\ln n\left(1 + O\left(\frac{1}{\sqrt{\ln n}}\right)\right),$$

$\eta > 0$ is an absolute constant, and

$$c_0 = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{\varphi(d)d}.$$

Proof. We split the proof of Theorem 2 into several steps.

1. We obtain an asymptotic formula for $G(n)$. First, we restrict the range of the variable x. We obtain

$$G(n) = 2G'(n) - G''(n), \tag{21}$$

where

$$G'(n) = \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy=1}} \sum_{x\le\sqrt{n}} 1, \quad G''(n) = \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy=1}} \sum_{\substack{x\le\sqrt{n}\\ y\le\sqrt{n}}} 1.$$

We estimate the sum $G''(n)$ representing it as the sum of two terms $G''(n) = G_1''(n) + G_2''(n)$, where

$$G_1''(n) = \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy=1}} \sum_{\substack{x\le\sqrt{n}Q^{-1}\\ y\le\sqrt{n}}} 1, \qquad G_2''(n) = \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy=1}} \sum_{\substack{\sqrt{n}Q^{-1}<x\le\sqrt{n}\\ y\le\sqrt{n}}} 1.$$

Note that

$$G_1''(n) \le \sum_{m\le nQ^{-1}+1} \tau(m-1)t_1(m), \qquad \text{where} \quad t_1(m) = \sum_{\substack{p_1p_2^a=m\\ p_1\in A_1\\ p_2\in A_2}} 1.$$

Obviously, $t_1(m) = 1$ if $m = p^{a+1}$ or $m = p_1p_2^a$ and $t_1(m) = 0$ in all other cases. Taking this remark into account, we obtain from Lemma 3 the estimate

$$G_1''(n) \le \sum_{m\le nQ^{-1}+1} \tau(m-1) \ll \frac{n\ln n}{Q}.$$

Next. we estimate $G_2''(n)$. Since in the case where $p_2 \mid x$ the equation $p_1p_2^a - xy = 1$ has no solutions, we conclude that

$$G_2''(n) = \sum_{\sqrt{n}Q^{-1}<x\le\sqrt{n}} \sum_{\substack{p_2\in A_2\\ \gcd(p_2,x)=1}} \sum_{\substack{p_1\in A_1\\ p_1\equiv p_2^* \pmod{x}}} 1 = \sum_{\sqrt{n}PQ^{-1}<x\le\sqrt{n}} \sum_{\substack{p_2\in A_2\\ \gcd(p_2,x)=1}} \pi(nQ^{-1}, p_2^*, x),$$

where p_2^* is a solution of the congruence $p_2^a t \equiv 1 \pmod{x}$. Applying Lemmas 1 and 2, we obtain

$$G_2''(n) \ll \frac{n}{Q\ln n} \sum_{p_2\in A_2} \sum_{\sqrt{n}Q^{-1}<x\le\sqrt{n}} \frac{1}{\varphi(x)} \ll \frac{n}{Q}\pi(Q^{1/a})\frac{\ln Q}{\ln n}.$$

From the estimates for $G_1''(n)$ and $G_2''(n)$ we get

$$G''(n) \ll \frac{n}{Q}\pi(Q^{1/a})\frac{\ln Q}{\ln n}. \tag{22}$$

To estimate the sum $G'(n)$, we represent it, similarly to $G''(n)$, as the sum of two terms:

$$G'(n) = G_1'(n) + G_2'(n) = \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy=1}} \sum_{x\le\sqrt{n}Q^{-1}} 1 + \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy=1}} \sum_{\sqrt{n}Q^{-1}<x\le\sqrt{n}} 1.$$

The sum $G_2'(n)$ can be estimated exactly the same as $G_2''(n)$; therefore,

$$G_2'(n) \ll \frac{n}{PQ}\pi(Q^{1/a})\frac{\ln Q}{\ln n}.$$

We obtain an asymptotic formula for the sum $G_1'(n)$:

$$G_1'(n) = \sum_{x\le\sqrt{n}Q^{-1}} \sum_{\substack{p_2\in A_2\\ \gcd(p_2,\ x)=1}} \pi\left(\frac{n}{Q}, p_2^*, x\right)$$

$$= \mathrm{Li}\left(\frac{n}{Q}\right) \sum_{p_2\in A_2} \sum_{\substack{x\le\sqrt{n}Q^{-1}\\ \gcd(p_2,\ x)=1}} \frac{1}{\varphi(x)} + O\left(\sum_{p_2\in A_2} \sum_{\substack{x\le\sqrt{n}Q^{-1}\\ \gcd(p_2,\ x)=1}} \left|\pi\left(\frac{n}{Q}, p_2^*, x\right) - \frac{\mathrm{Li}\left(\frac{n}{Q}\right)}{\varphi(x)}\right|\right),$$

where p_2^* is a solution of the congruence $p_2^a t \equiv 1 \pmod{x}$. Therefore, due to the Bombieri–Vinogradov theorem (see Lemma 4),

$$G_1'(n) = \mathrm{Li}\left(\frac{n}{Q}\right) \sum_{p_2\in A_2} \sum_{\substack{x\le\sqrt{n}Q^{-1}\\ \gcd(p_2,\ x)=1}} \frac{1}{\varphi(x)} + O\left(\frac{n\pi(Q^{1/a})}{Q\ln n}\right).$$

For estimating the interior sum in the first term of the right-hand side of the equality obtained, we argue similarly to the step 3 of the proof of Theorem 1. We obtain

$$\sum_{\substack{x\le\sqrt{n}Q^{-1}\\ \gcd(p_2,x)=1}} \frac{1}{\varphi(x)} = c_0 \ln\left(\frac{\sqrt{n}}{Q}\right) + O\left(\frac{\ln n}{p_2}\right),$$

where

$$c_0 = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{\varphi(d)d}.$$

Summing both sides of the equality obtained by $p_2 \in A_2$, we obtain

$$\sum_{p_2\in A_2} \sum_{\substack{x\le\sqrt{n}Q^{-1}\\ \gcd(p_2,x)=1}} \frac{1}{\varphi(x)} = c_0 \ln\left(\frac{\sqrt{n}}{Q}\right) \pi(Q^{1/a}) + O\left(\ln n \ln\ln Q\right).$$

Therefore, we arrive at the asymptotic formula

$$G_1'(n) = c_0 \,\mathrm{Li}\left(\frac{n}{Q}\right) \pi(Q^{1/a}) \ln\left(\frac{\sqrt{n}}{Q}\right) \left(1 + O\left(\frac{\ln Q}{\ln n}\right)\right).$$

Therefore, taking into account the estimate for $G_2''(n)$, we obtain

$$G'(n) = \frac{c_0}{2}\,\mathrm{Li}\left(\frac{n}{Q}\right) \pi(Q^{1/a}) \ln n \ \left(1 + O\left(\frac{\ln Q}{\ln n}\right)\right). \tag{23}$$

Finally, substituting (23) and (22) into (21), we get the asymptotic formula

$$G(n) = c_0 \,\mathrm{Li}\left(\frac{n}{Q}\right) \pi(Q^{1/a}) \ln n \ \left(1 + O\left(\frac{1}{\sqrt{\ln n}}\right)\right). \tag{24}$$

2. Consider the sum

$$G_1(n) = \sum_{p_1\in A_1} \sum_{\substack{p_2\in A_2\\ p_1p_2^a - xy = 1\\ \{\frac{1}{2}(p_1p_2^a)^{1/c}\}<\frac{1}{2}}} \sum_{x,y} 1.$$

Further, we assume that $n \geq n_0 > 0$, where n_0 is a sufficiently large number. Similarly to the proof of Theorem 2, we use the auxiliary periodic function

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ 0 & \text{if } 1/2 \leq x < 1. \end{cases}$$

Obviously,

$$G_1(n) = \sum_{p_1 \in A_1} \sum_{\substack{p_2 \in A_2 \\ p_1 p_2^a - xy = 1}} \chi\left(\frac{1}{2}(p_1 p_2^a)^{1/c}\right).$$

As in the proof of Theorem 1, we apply Lemma 5 and choose the parameters r, Δ, α, and β in two ways.

In the case considered, we first set

$$r = [\ln n], \quad \Delta = \frac{1}{\ln^2 n}, \quad \alpha = \Delta, \quad \beta = \frac{1}{2} - \Delta.$$

We denote by $\chi_1(x)$ the function whose existence follows from Lemma 5. Next, for the same r and Δ, we set $\alpha = -\Delta$ and $\beta = 1/2 + \Delta$ and denote by $\chi_2(x)$ the corresponding function. Then Lemma 5 implies that $\chi_1(x) \leq \chi(x) \leq \chi_2(x)$ and, therefore,

$$G_{11}(n) \leq G_1(n) \leq G_{12}(n), \tag{25}$$

where

$$G_{1i}(n) = \sum_{p_1 \in A_1} \sum_{p_2 \in A_2} \sum_{\substack{x \leq \sqrt{n} \\ p_1 p_2^a - xy = 1}} \chi_i\left(\frac{1}{2}(p_1 p_2^a)^{1/c}\right), \quad i = 1, 2.$$

If we obtain asymptotic formulas for $G_{11}(n)$ and $G_{12}(n)$ with the same leading terms and remainders, then the inequality (25) means that the same formula is also valid for $G_1(n)$.

3. To deduce an asymptotic formula for $G_{11}(n)$, we expand the function $\chi_1\left(\frac{1}{2}(p_1 p_2^a)^{1/c}\right)$ into a Fourier series:

$$G_{11}(n) = \left(\frac{1}{2} - 2\Delta\right) G(n) + R_1(n) + R_2(n), \tag{26}$$

where

$$R_1(n) = \sum_{0 < |m| \leq \Delta^{-1} \ln n} |g_m||V_m(n)|, \quad R_2(n) = \sum_{|m| > \Delta^{-1} \ln n} |g_m||V_m(n)|,$$

$$V_m(n) = \sum_{p_1 \in A_1} \sum_{\substack{p_2 \in A_2 \\ p_1 p_2^a - xy = 1}} t_2(p_1 p_2^a - 1) e^{\pi i m (p_1 p_2^a)^{1/c}}, \quad t_2(k) = \sum_{\substack{xy = k \\ x \leq \sqrt{n} Q^{-1}}} 1,$$

and g_m is the mth Fourier coefficient of the function χ_1.

Now we estimate $R_2(n)$. Lemma 5 implies

$$|g_m| \leq \frac{1}{\pi |m|} \left(\frac{r}{\pi \Delta |m|}\right)^r.$$

Moreover, $|V_m(n)| \leq G_1(n) \leq n$; therefore,

$$R_2(n) = O\left(n \sum_{|m| > \Delta^{-1} \ln n} \left(\frac{r}{\Delta}\right)^r m^{-r-1}\right) = O(1). \tag{27}$$

Estimate $R_1(n)$:

$$|R_1(n)| \le \sum_{0<|m|<\Delta^{-1}\ln n} \frac{1}{\pi|m|}|V_m(n)|, \quad |V_m(n)| \le \sum_{n_1 \le \frac{n}{Q}} \left| \sum_{p_2 \in A_2} t_2(n_1 p_2^a - 1) e^{\pi i m (n_1 p_2^a)^{1/c}} \right|,$$

where n_1 runs over the set of natural numbers. Applying the Cauchy inequality, we obtain

$$|V_m(n)|^2 \le \frac{n}{Q} \sum_{n_1 \le n/Q} \left| \sum_{p_2 \in A_2} t_2(n_1 p_2^a - 1) e^{\pi i m (n_1 p_2^a)^{1/c}} \right|^2$$
$$= \frac{n}{Q} \sum_{p_2 \in A_2} \sum_{p_2' \in A_2} \sum_{n_1 \le n/Q} t_2\big(n_1 p_2^a - 1\big) t_2\big(n_1 (p_2')^a - 1\big) e^{\pi i m (p_2^{a/c} - (p_2')^{a/c}) n_1^{1/c}}$$
$$= \frac{n}{Q}\big(V_0(n) + V_1(n)\big), \quad (28)$$

where the sum $V_0(n)$ corresponds to term in which $p_2 = p_2'$, whereas $p_2 \ne p_2'$ in the sum $V_1(n)$.

Estimate $V_0(n)$. Note that the multiplicativity of the function $\tau(n)$ and the formula for calculating its values follows that $\tau(ab) \le \tau(a)\tau(b)$. This remark and Lemma 6 imply

$$V_0(n) \le \sum_{p_2 \in A_2} \sum_{n_1 \le \frac{n}{Q}} \tau^2(n_1 p_2^a - 1) \le (a+1)^2 Q^{1/a} \sum_{n_1 \le n Q^{-1}} \tau^2(n_1) \ll n Q^{1/a - 1} \ln^3 n. \quad (29)$$

Estimate $V_1(n)$. Note that

$$V_1(n) \le \sum_{p_2 \in A_2} \sum_{p_2' \in A_2} f(n), \quad (30)$$

where

$$f(n) = \sum_{x_1 \le \sqrt{n} Q^{-1}} \sum_{x_2 \le \sqrt{n} Q^{-1}} s(m) \quad (31)$$

and

$$s(m) = \sum_{\substack{n_1 \le n Q^{-1} \\ n_1 p_2^a \equiv 1 \pmod{x_1} \\ n_1 (p_2')^a \equiv 1 \pmod{x_2}}} e^{2\pi i \frac{m}{2} (p_2^{a/c} - (p_2')^{a/c}) n_1^{1/c}}.$$

Without loss of generality, we assume that $\gcd(p_2, x_1) = 1$ and $\gcd(p_2', x_2) = 1$, sine in the opposite case the sum is equal to zero.

Let $p_2 q_2 \equiv 1 \pmod{x_1}$ and $p_2' q_2' \equiv 1 \pmod{x_2}$. We solve the system of congruences

$$\begin{cases} x \equiv q_2^a & \pmod{x_1} \\ x \equiv (q_2')^a & \pmod{x_2}. \end{cases}$$

It is solvable if and only if $\gcd(x_1, x_2) | (q_2^a - (q_2')^a)$ and the solution has the form $x = z_2 + mD$, where $D = [x_1, x_2]$. Thus,

$$s(m) = \sum_{\xi + l \le n/(DQ)} e^{2\pi i \varkappa (\xi + l)^{1/c}},$$

where

$$\varkappa = \frac{m}{2}\left(p_2^{a/c} - (p_2')^{a/c}\right), \quad \xi = \frac{z_2}{D}.$$

Obviously, $0 < \xi < 1$.

4. To obtain an asymptotic formula for G_{11}, we must estimate $s(m)$. We represent this sum in the form $s(m) = s_1(m) + s_2(m)$, where

$$s_1(m) = \sum_{\frac{n}{DQ^{3/2}} < \xi + l \le \frac{n}{DQ}} e^{2\pi i \varkappa (\xi + l)^{1/c}}, \qquad s_2(m) = \sum_{\xi + l \le \frac{n}{DQ^{3/2}}} e^{2\pi i \varkappa (\xi + l)^{1/c}}.$$

Obviously,

$$|s_2(m)| \le \sum_{\xi + l \le \frac{n}{DQ^{3/2}}} 1 \le \frac{n}{DQ^{3/2}}.$$

Therefore,

$$s(m) = s_1(m) + O\left(\frac{n}{DQ^{3/2}}\right). \tag{32}$$

We split the sum $s_1(m)$ into $O(\ln n)$ sums of the form

$$\bar{s_1}(M) = \sum_{M < \xi + l \le M_1} e^{2\pi i \varkappa (\xi + l)^{1/c}},$$

where

$$\frac{n}{DQ^{3/2}} \le M < M_1 \le 2M, \quad M_1 \le \frac{n}{DQ}.$$

Since $x_1 \le \sqrt{n} Q^{-1}$ and $x_2 \le \sqrt{n} Q^{-1}$, we have $D = [x_1, x_2] \le x_1 x_2 \le n Q^{-2}$. Therefore,

$$M \ge \frac{n}{DQ^{3/2}} \ge Q^{1/2}$$

and hence the sum $\bar{s_1}(M)$ contains “relatively many” terms.

We estimate $\bar{s_1}(M)$ under the condition $\frac{n}{DQ^{3/2}} \le M \le \frac{n}{DQ}$. First, we consider the case where

$$1 \le D \le n^{0.99} \quad \Longleftrightarrow \quad M \ge \frac{n^{0.01}}{Q^{1.5}}.$$

If

$$\frac{|\varkappa| M^{1/c}}{M} \le \frac{1}{10},$$

then, approximating the sum by an integral, we obtain:

$$\bar{s_1}(M) = \int_M^{M_1} e^{2\pi i \varkappa (\xi + l)^{1/c}} dl = O(1) = O\left(M^{1 - 1/c}\right).$$

In the sequel, we assume that $|\varkappa| M^{1/c}/M > 1/10$ and use van der Corput’s method for estimating the sum $\bar{s_1}(M)$. We define a natural number k from the condition

$$\frac{1}{M^2} < \frac{|\varkappa| M^{1/c}}{M^k} \le \frac{1}{M}.$$

If $k = 2$, then

$$\frac{|\varkappa| M^{1/c}}{M^2} \asymp \frac{1}{M}.$$

Estimating $\bar{s_1}(M)$ by the second derivative, we obtain that $\bar{s_1}(M) = O(\sqrt{M})$ for $k = 2$.

Consider the case $k \ge 3$ and estimate $\bar{s_1}(M)$ by a derivative of order k (see [9, p. 66-70]). We have $\bar{s_1}(M) \ll M^{1-\delta}$, where $\delta = \delta(k) > 0$. The case $D \le n^{0,99}$ is fully explored.

Now let

$$D > n^{0,99}, \quad \frac{|\varkappa| M^{1/c}}{M} > \frac{1}{10}.$$

We estimate $\bar{s_1}(M)$ using the scheme proposed by I. M. Vinogradov for the zeta sum (see [9, p. 66-70]). Let $a = [M^{5/11}]$; then

$$|\bar{s_1}(M)| \le \frac{1}{a^2} \sum_{M<m\le M_1} |W(m)| + 2a^2,$$

where

$$W(m) = \sum_{u=1}^{a}\sum_{v=1}^{a} e^{2\pi i \varkappa(\xi+m+uv)^{1/c}}.$$

We apply the Taylor formula:

$$(\xi+m+uv)^{1/c} = \sum_{j=0}^{r} \binom{1/c}{j} (\xi+m)^{1/c-j}(uv)^j + \theta_2 \binom{1/c}{r+1} (\xi+m)^{1/c-r-1} a^{2(r+1)}, \quad |\theta_2| \le 1.$$

Thus,

$$e^{2\pi i \varkappa(\xi+m+uv)^{1/c}} = e^{2\pi i F(uv)} + 2\pi\theta_3|\varkappa|(\xi+m)^{1/c} \left(\frac{a^2}{\xi+m}\right)^{r+1}, \quad |\theta_3| \le 1,$$

where

$$F(uv) = \sum_{j=0}^{r} \binom{1/c}{j} (\xi+m)^{1/c-j}(uv)^j.$$

Introduce the following notation:

$$x_j = \binom{1/c}{j}, \quad T = |\varkappa|(\xi+m)^{1/c}, \quad \alpha_j = \frac{T \operatorname{sgn} \varkappa}{x_j(\xi+m_j)}.$$

Then

$$W(m) = W_1 + 2\pi\theta_4 T \left(\frac{a^2}{M}\right)^r a^2 M^{-\frac{1}{11}}, \quad |\theta_4| \le 1,$$

$$W_1 = \sum_{u=1}^{a}\sum_{v=1}^{a} e^{2\pi i F(uv)} = \sum_{u=1}^{a}\sum_{v=1}^{a} e^{2\pi i(\alpha_1 uv + \alpha_2 u^2v^2 + \cdots + \alpha_r u^r v^r)}.$$

We choose a natural number r from the condition

$$r - 1 < \frac{11 \ln T}{\ln M} \le r$$

and note that $|\varkappa| \le n^{0.99/c}$ since $D > 0.99$. Therefore, from the inequality $MD \le n/Q$ we conclude that $M < n^{0.01}$ and hence

$$\frac{\ln T}{\ln M} > \frac{\ln \frac{1}{2} n^{0,99/c}}{\ln n^{0,01}} > \frac{99}{c} - 1 \ge \frac{97}{2},$$

so that the choice of r specified is possible. Following Vinogradov's scheme, we obtain:

$$|W_1| \le c_2 a^2 e^{-\gamma_1\sqrt{\ln n}}, \quad c_2 > 0, \quad \gamma_1 > 0.$$

Therefore,

$$\bar{s_1}(M) \ll \frac{n}{QD} e^{-\gamma\sqrt{\ln n}}, \quad s_1(m) \ll \frac{n \ln n}{QD} e^{-\gamma\sqrt{\ln n}}.$$

Using estimates obtained above, we find from (32)

$$s(m) \ll nQ^{-1}e^{-\gamma\sqrt{\ln n}} D^{-1} \ln n + nQ^{-1.5}D^{-1} \ll nQ^{-1}e^{-\gamma\sqrt{\ln n}/2}D^{-1}, \tag{33}$$

where $D = [x,x_2]$.

5. We continue to deduce an asymptotic formula for $G_{11}(n)$. The formulas (26) and (27) imply

$$G_{11}(n) = \left(\frac{1}{2} - 2\Delta\right) G_1(n) + O\left(\sum_{0<|m|<\Delta^{-1}\ln n} \frac{1}{\pi|m|}\big|V_m(n)\big|\right),$$

where $\Delta = 1/\ln^2 n$. From (28) and (29) we obtain

$$|V_m(n)|^2 \le \frac{n}{Q} V_1(n) + O(n^2 Q^{1/a-2} \ln^3 n).$$

Therefore, to obtain an asymptotic formula, we need an estimate for $V_1(n)$. From (30) we have

$$V_1(n) \ll \sum_{p_2\in A_2} \sum_{p_2'\in A_2} f(n),$$

where

$$f(n) = \sum_{x_1\le\sqrt{n}Q^{-1}} \sum_{x_2\le\sqrt{n}Q^{-1}} s(m).$$

Substituting the estimate (33) into this formula, we obtain

$$f(n) \ll \frac{n}{Q} \exp\left(-\frac{\gamma}{2}\sqrt{\ln n}\right) \sum_{x_1\le\sqrt{n}Q^{-1}} \sum_{x_2\le\sqrt{n}Q^{-1}} D^{-1}.$$

Estimate the sum in this inequality:

$$\begin{aligned}\sum_{x_1\le\sqrt{n}Q^{-1}} \sum_{x_2\le\sqrt{n}Q^{-1}} D^{-1} &= \sum_{x_1\le\sqrt{n}Q^{-1}} \sum_{x_2\le\sqrt{n}Q^{-1}} \frac{\gcd(x_1,x_2)}{x_1x_2}\\ &= \sum_{x_1\le\sqrt{n}Q^{-1}} \sum_{x_2\le\sqrt{n}Q^{-1}} \frac{1}{x_1x_2} \sum_{d|(x_1,x_2)} \varphi(d)\\ &\le \sum_{d\le\sqrt{n}P^{-1}} \frac{\varphi(d)}{d^2} \left(\sum_{x\le\sqrt{n}Q^{-1}d^{-1}} \frac{1}{x}\right)^2 \ll \ln^3 n.\end{aligned}$$

Therefore, $f(n) \ll n\ln^3 n/Q^2$. This and (30) imply

$$|V_1(n)| \ll n^2 Q^{1/a-3} \ln^3 n. \tag{34}$$

Using the estimates (29) and (34), we obtain from (28)

$$|V_m(n)|^2 \ll n^2 Q^{-2+\frac{2}{a}} e^{-\frac{\gamma}{4}\sqrt{\ln n}} + n^2 Q^{-2+\frac{1}{a}} \ln^3 n \ll \left(\frac{n}{Q}\right)^2 Q^{-2+2/a} e^{-\gamma\sqrt{\ln n}/4},$$

that is,

$$|V_m(n)| \ll nQ^{-1+1/a} e^{-\gamma\sqrt{\ln n}/8}.$$

Therefore,

$$|R_1(n)| \ll nQ^{-1+1/a} e^{-\gamma\sqrt{\ln n}/8} \ln^2 n. \tag{35}$$

Substituting the estimates (27) and (35) into (26), we obtain the following asymptotic formula:

$$G_{11}(n) = \frac{1}{2} G(n)\big(1 + O(Q^{-\eta})\big), \quad \eta > 0.$$

Similar arguments for $G_{12}(n)$ yield an asymptotic formula with the same leading term and remainder. Therefore, it follows from (25) that the following asymptotic formula for $G_1(n)$ holds:

$$G_1(n) = \frac{1}{2} G(n)\big(1 + O(Q^{-\eta})\big), \quad \eta > 0.$$

Theorem 2 is proved. □

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