HARDY SPACE OF SOLUTIONS OF ELLIPTIC SYSTEMS ON THE PLANE

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ABSTRACT. The space indicated in the title is introduced and studied.

CONTENTS

1. First-Order Elliptic Systems

In a domain $D \subseteq \mathbb{C}$ on the plane, let us consider the first-order elliptic system

$$A_1 \frac{\partial U}{\partial x} + A_2 \frac{\partial U}{\partial y} = 0 \tag{1}$$

having constant coefficients $A_i \in \mathbb{C}^{l\times l}$. The ellipticity condition means that $\det A_i \neq 0$, $i = 1, 2$, and the characteristic equation $\det(A_1 + A_2 z) = 0$ has no real roots. It is well known that solutions $U = (U_1, \dots, U_l)$ of this system are real-analytic in the domain D.

Let the boundary $\Gamma = \partial D$ of the domain be a piecewise-Lyapunov contour, i.e., its connected components are homeomorphic to the circle and admit a representation in the form of the union of finitely many Lyapunov arcs that can pairwise intersect only along their endpoints. Recall that an arc is Lyapunov if it admits a smooth parametrization of class $C^{1,\mu}$ with a certain $0 < \mu < 1$. Choose a sequence of contours $\Gamma_n \subseteq D$, $n = 1, 2 \dots$, converging to Γ in the following sense. There exists a homeomorphic mapping $\alpha_n : \Gamma \to \Gamma_n$ with piecewise-continuous derivative α_n' such that $\alpha_n(t) - t \to 0$ as $n \to \infty$ with respect to the sup-norm and the derivatives α_n' are uniformly bounded.

We introduce the Hardy space $H^p(D)$, $1 < p < \infty$, of solutions of the elliptic system by the finiteness conditions of the norm

$$|U| = \sup_n |U|_{L^p(\Gamma_n)}. \tag{2}$$

Here, the domain D can be finite, as well as infinite, and, in the latter case, the solutions u of system (1) are assumed to be bounded at infinity. The L^p norm of a function $U(t)$ on Γ_n is understood with respect to the numerical function $|U(t)|$, where $|\cdot|$ denotes a certain fixed norm in $\mathbb{R}^l$. The definition of H^p can also be considered for $p = 1$, but we exclude this case in what follows.

For analytic functions, this definition generalizes the classical Hardy class over the unit disk and is due to V. I. Smirnov, M. A. Lavrent'ev, and M. V. Keldysh [3]. The notation $E^p(D)$ is also used for it.

Systems (1) of the particular form

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0, \tag{3}$$

where the spectrum $\sigma(J)$ of the matrix $J \in \mathbb{C}^{l\times l}$ lies in the upper half-plane, are of especial interest. System (3) is a natural generalization of the Cauchy–Riemann system and was introduced by A. Douglis [2]. The main properties of analytic functions admit a generalization [6] to solutions of this system; for this reason, they are called hyperanalytic functions or Douglis analytic functions (in short, J-analytic functions).

In the general case, the solutions of system (1) can be uniquely expressed through the solutions of (3). Indeed, there exist a block-diagonal matrix $J = \operatorname{diag}(J_1, \overline{J}_2)$, where $J_k \in \mathbb{C}^{l_k\times l_k}$ and $\sigma(J_k) \subseteq \{\operatorname{Im} z > 0\}$, and an invertible matrix $B \in \mathbb{C}^{l\times l}$ such that $A_1 B + A_2 B J = 0$. In accordance with this, we have the representation $U = B\phi$, $\phi = (\phi_1, \overline{\phi}_2)$, of a solution of system (1) through J_k-analytic functions ϕ_k. This fact allows us to perform all the consideration for system (3) in what follows.

If a J-analytic function ϕ is continuous in a closed domain $\overline{D} = D\cup\Gamma$, then the Cauchy formula [6]

$$2\phi(z) = \frac{1}{\pi i}\int\limits_{\Gamma}(t-z)_J^{-1}dt_J\phi^+(t), \quad t \in D, \tag{4}$$

holds, where we have accepted the notation $(x_1+ix_2)_J = x_1 1 + x_2 J$, $x_j \in \mathbb{R}$; the writing $(dx_1+idx_2)_J = (dx_1)1 + (dx_2)J$ for the matrix differential has an analogous meaning, and the contour $\Gamma = \partial D$ is positively oriented with respect to D. Replacing the boundary value ϕ^+ by an arbitrary density φ in it, we obtain a generalized integral of Cauchy type denoted by $(I\varphi)(z)$.

The definition of the Hardy spaces naturally extends to the weighted case. Let F be a finite subsets of points $\tau \in \Gamma$. Starting from the family $\lambda = (\lambda_\tau, \tau \in F)$ of real numbers (weighted order), we denote by $C_\lambda(\Gamma, F)$ the class of functions continuous in $\Gamma \setminus F$ and having the behavior $O(1)|t-\tau|^{\lambda_\tau}$ as $t \to \tau$. A scalar-valued function $\rho \in C_\lambda$ is said to be weighted (of order λ) if it does not vanish and $\rho^{-1} \in C_{-\lambda}$. The dependence on λ is shown by the notation $\rho = \rho_\lambda$.

In a similar way, we define weight functions in the domain D with respect to the corresponding class $C_\lambda = C_\lambda(\overline{D}, F)$. In this case, the J-analytic $l \times l$-matrix weight functions $R(z) = R_\lambda(z)$ satisfying system (3) and subject to the additional condition $R(z)J = JR(z)$, $z \in D$, are of main interest. The latter condition means that the operation $\phi \to R\phi$ of multiplication by a weight function preserves the J-analytic vector-valued functions ϕ. The weight functions $R_\lambda(z)$ indicated always exist. For example, if the domain D is finite and Γ is a simple contour, then we can set

$$R_\lambda(z) = \prod_{\tau\in F}(z-\tau)_J^{\lambda_\tau}.$$

Here, in notation (4), the factor

$$(z-\tau)_J^{\alpha} = \exp[\alpha \ln(z-\tau)_J]$$

is understood in the sense of a function of matrices starting from a branch of the logarithm $\ln(z-\tau)$ continuous in D.

Now let introduce the weighted Banach space $L^p_\lambda(\Gamma, F)$ over the boundary $\Gamma = \partial D$ such that $L^p_\lambda = L^p$ for $\lambda = -1/p$ and the weighted transformation $\varphi \to \rho_\nu\varphi$ is an isomorphism of L^p onto $L^p_{\nu-1/p}$. Note that the class

$$C_{\lambda+0}(\Gamma, F) = \bigcup_{\varepsilon>0} C_{\lambda+0}$$

is contained and dense in $L^p_\lambda(\Gamma, F)$.

In a similar way, we define the space $L^p_\lambda(D,F)$ over the domain D with respect to the condition $L^p_\lambda = L^p$ for $\lambda = -2/p$. In exactly the same way, we define the family of weighted Hardy spaces $H^p_\lambda(D,F)$ with respect to the condition $H^p_\lambda = H^p$ for $\lambda = -1/p$ and the weighted transformation $\phi \to R_\nu \phi$. Obviously, for $\lambda = 0$, the space L^p_0 over Γ and over D is the L^p-space of measure $d\mu = \rho_{-1}(t)ds$, $t \in \Gamma$ and, respectively, of measure $d\mu = \rho_{-2}(z)dxdy$, $z = x + iy \in D$.

The following theorem unites the main properties of the Hardy spaces.

Theorem 1. (a) *Let $\phi \in H^p_\lambda(D)$ be a solution of system* (1). *Then almost everywhere, there exist angular limit values $\phi^+(t)$, which defines the function $\phi^+_\lambda \in L^p_\lambda(\Gamma)$. The space $H^p_\lambda(D)$ is Banach, the norm $|\phi| = |\phi^+|_{L^p_\lambda}$ is equivalent to* (2), *and the dense continuous embeddings*

$$C_{\lambda+0}(\overline{D},F) \subseteq H^p_\lambda(D,F) \subseteq L^p_\lambda(D,F)$$

hold.

(b) *If $\phi \in H^p_\lambda(D)$, then the restriction of ϕ to each subdomain $D_0 \subseteq D$ bounded by a piecewise-Lyapunov contour belongs to $H^p_\lambda(D_0)$. Conversely, let subdomains $D_1,\dots,D_m$ of this type be pairwise disjoint, and let $\overline{D}_1 \cup \dots \cup \overline{D}_m = \overline{D}$. Then if a solution ϕ of system* (1) *in the domain D belongs to $H^p_\lambda(D_j)$, $j = 1,\dots,m$, it follows that $\phi \in H^p_\lambda(D)$; moreover, the norm in $H^p_\lambda(D)$ is equivalent to $|\phi| = \sum\limits_j |\phi|_{H^p_\lambda(D_j)}$.*

(c) *The Cauchy-type integral $I\varphi$ defines a bounded linear operator*

$$L^p_\lambda(\Gamma,F) \to H^p_\lambda(D,F), \quad -1 < \lambda < 1,$$

and for almost all $t_0 \in \Gamma$, the Sokhotskii–Plemejl formula

$$(I\varphi)^+(t_0) = \varphi(t_0) + \frac{1}{\pi i}\int\limits_\Gamma (t - t_0)^{-1}_J dt_J \varphi(t),$$

holds, where the integral is singular and is understood in the Cauchy principal value sense.

The classes $C_{\lambda+0}$ and $L^p_\lambda(D)$ in assertion (b) of the theorem are considered with respect to J-analytic functions. In particular, it follows from (a) and (b) that the space H^p_λ is independent of the sequence Γ_n in (2) and can be defined as the completion of the class $C_{\lambda+0}$ with respect to the norm $|\phi| = |\phi^+|_{L^p_\lambda}$. For the usual analytic functions, this scheme was realized in [7]. Under more general assumptions on the boundary of the domain D and weights, these spaces were studied in [16] for analytic functions. Also, Theorem 1(b) allows us to naturally extend the definition of $H^p_\lambda(D)$ to domains D with arbitrary piecewise-Lyapunov boundary. Assertion (c) of the theorem is used as a base for proving (a) and (b) and is proved completely analogously to [12].

An analog of the N. I. Muskhelishvili theorem on the representation of ϕ by the integral $I\varphi$ with a real vector-valued function φ proved in [6] in the Hölder classes is also preserved for the Hardy classes.

Recall [5] that a bounded operator $N : X \to Y$ acting between Banach spaces X and Y is said to be Fredholm if its kernel $\ker N$ is finite-dimensional, the image $\operatorname{Im} N$ is closed, and there exists a finite-dimensional subspace $Y_0 \subseteq Y$ such that $Y = Y_0 \oplus \operatorname{Im} N$. This subspace is conveniently called the co-image of the operator N and is denoted by $\operatorname{koIm} N = Y \ominus \operatorname{Im} N$, although it is defined by N non-uniquely. The difference $\operatorname{ind} N$ between the dimensions of the kernel and co-image is called the index of operator N.

Theorem 2. (a) *Let the domain D be bounded by a Lyapunov contour Γ. Then the operator*

$$I : L^p_\lambda(\Gamma,F) \to H^p_\lambda(D,F), \quad -1 < \lambda < 0,$$

is Fredholm, and its index is equal to $l(s-2)$, *where* s *is the number of connected components of the contour. Moreover, its kernel satisfies*

$$\ker I \subseteq C^{+0}(\Gamma) = \bigcup_{\mu>0} C^{\mu}(\Gamma),$$

and there exists a co-image $\operatorname{koIm} I \subseteq C^{+0}(\overline{D})$.

(b) *Let the matrix* J *be triangular, let the domain* D *be bounded by a piecewise-Lyapunov contour* Γ *without cusp points, and let the set* F *contain all corner points of the contour. Then the operator*

$$I : L^p_\lambda(\Gamma, F) \to H^p_\lambda(D, F), \quad -1/2 < \lambda < 0,$$

is Fredholm, and its index is equal to $l(s-2)$. *Moreover, the elements of the kernel are constant on connected components of* Γ, *and it is possible to choose the co-image whose elements are constant in* D.

More precisely, in assertion (b), the kernel and the co-image of the operator I are described as follows. If the domain D is infinite, then $\ker I$ consists of all locally constant functions, and we can set $\operatorname{koIm} I = \mathbb{C}^l$. If the domain D is finite, then $\ker I$ consists of all locally constant functions equal to zero on the exterior component of the contour Γ and $\operatorname{koIm} I = \{i\xi \mid \xi \in \mathbb{R}^l\}$. Here, by the exterior component we understood the connected component Γ inside which the domain D is contained.

Another approach based on the arguments of homogeneity with respect to dilatations is also possible for the definition of weighted spaces. Let F consist of one point, which we assume to be the origin for convenience. Let us cover Γ by smooth curves $\Gamma_1, \Gamma_2, \dots$ so that

$$\tilde{\Gamma}_n = 2^n \Gamma_n \subseteq \{1/2 \le |t| \le 2\}$$

for sufficiently large n. Then with a function $\varphi \in L^p_\lambda(\Gamma, 0)$, we can associate the sequence of functions $\tilde{\varphi}_n(t) = \varphi(2^{-n}t) \in L^p(\tilde{\Gamma}_n)$, and the space $L^p_\lambda(\Gamma, 0)$ can be described by using the equivalent norm

$$|\varphi| = \Big(\sum_n |\xi_n|^p\Big)^{1/p}, \qquad \xi_n = 2^{-n\lambda} |\tilde{\varphi}_n|_{L^p(\tilde{\Gamma}_n)}. \tag{5}$$

An analogous fact holds for the weighted Hardy space.

Theorem 3. *Let* D *be represented in the form of the union of domains* $D_1, D_2, \dots$ *for which*

$$\tilde{D}_n = 2^n D_n \subseteq \{1/2 \le |z| \le 2\}$$

for sufficiently large n, *and let the boundaries* $\partial \tilde{D}_n$ *converge to a certain contour* $\tilde{\Gamma}$ *in the same sense as in* (2). *Then for* $\phi \in H^p_\lambda(D, 0)$, *the functions satisfies* $\tilde{\phi}_n(z) = \phi(2^{-n}z) \in H^p\tilde{D}_n)$, *and the space* $H^p_\lambda(D, 0)$ *can be described by using the equivalent norm*

$$|\phi| = \Big(\sum_n |\xi_n|^p\Big)^{1/p}, \qquad \xi_n = 2^{-n\lambda} |\tilde{\phi}_n|_{H^p(\tilde{D}_n)}. \tag{6}$$

We see from this theorem that the introduced families of spaces monotonically decrease in each of the parameters p and λ_τ in the sense of Banach space embeddings. Note that the contours ∂D_n in the theorem can be chosen to be smooth. Therefore, the weighted Hardy spaces can be introduced starting from the definition of the spaces H^p in domains with smooth boundary. By using the translation, the previous definition is extended to the case $F = \{\tau\}$ with an arbitrary point $\tau \neq 0$. In the general case, the domain D can be represented in the form of the union of domains D_τ, $\tau \in F$, where D_τ is bounded by a contour Γ_τ smooth outside τ. Then the space $H^p_\lambda(D, F)$ can be defined by the condition

$\phi \in H^p_{\lambda_\tau}(D_\tau, \tau)$ for all $\tau \in F$. According to Theorem 1(b), this definition is independent of the choice D_τ.

Based on Theorem 2, we can extend the results of [12] from the weighted Hölder spaces to the Hardy spaces. In particular, this concerns the Riemann–Hilbert problem with a piecewise-continuous matrix coefficient G. We restrict ourselves to the simplest case $G = 1$ of the Schwarz problem

$$\operatorname{Re}\phi^+ = f \tag{7}$$

with a real right-hand side $f \in L^p_\lambda(\Gamma, F)$. The Fredholm property and the index of the problem is understood relative to the $\mathbb{R}$-linear operator $H^p_\lambda \to L^p_\lambda$ of its boundary condition.

The criterion for the Fredholm property is formulated in terms of the endpoint symbol, the family $x_\tau(\zeta)$, $\tau \in F$, of entire functions of the complex variable ζ defined as follows. With each pair of distinct unit vectors $a = a_1 + ia_2$ and $b = b_1 + ib_2$, we associate the analytic function ω of two variables ζ and u, $\operatorname{Im} u \neq 0$, by the formula

$$\omega(a, b; u, \zeta) = \left(-\frac{a_1 + ua_2}{b_1 + ub_2}\right)^\zeta, \qquad \left|\arg\left(-\frac{a_1 + ua_2}{b_1 + ub_2}\right)\right| < \pi.$$

Since the spectra of the matrices J and $\overline{J}$ lie in the upper and lower half-planes, respectively, where ω is analytic as a function of u, we can introduce the value of this functions at the above matrices. As a result, we can define the entire function of the variable ζ by the formula

$$h(a, b; \zeta) = \det[e^{\pi i\zeta}\omega(a, b; J, \zeta) - e^{-\pi i\zeta}\omega(a, b; \overline{J}, \zeta)]. \tag{8}$$

It is easy to verify that in each band $\lambda_1 < \operatorname{Re}\zeta < \lambda_2$ of finite width, the function h has finitely many zeros. Therefore, the projection of the zero set of this function on the real axis is a discrete subset of $\mathbb{R}$ denoted by $\Delta(a, b)$.

If $a = -b$, then the function satisfies $\omega = 1$, and hence, with accuracy up to a constant factor, the function $h(a, -a; \zeta)$ coincides with $\sin^l \pi\zeta$. Thus,

$$\Delta(a, -a) = \mathbb{Z}. \tag{9}$$

In the scalar case $l = 1$ where $J = \nu \in \mathbb{C}$, definition (8) passes to

$$h(a, b; \zeta) = 2i|q|^\zeta \sin\theta\zeta, \qquad q = \frac{b_1 + \nu b_2}{a_1 + \nu a_2},$$

where we have set $\theta = \arg q$, $0 < \theta < 2\pi$. Obviously, in this case,

$$\Delta(a, b) = \{(\pi/\theta)k,\ k \in \mathbb{Z}\}. \tag{10}$$

We now turn to problem (7). In a small neighborhood of a corner point $\tau \in \Gamma$, the set $S_\tau = D \cap \{|z - \tau| < r\}$ is a curvilinear sector whose lateral sides are denoted by $\Gamma_{\tau\pm0}$ (it is assume that a turn counterclockwise inside the sector is performed from $\Gamma_{\tau+0}$ to $\Gamma_{\tau-0}$). Let $q_{\tau\pm0}$ be unit vectors tangent to the lateral sides of $\Gamma_{\tau\pm0}$ at the point τ. Let these side be not tangent to one another, i.e., let τ be not a cusp point of the contour Γ. Then $q_{\tau-0} \neq q_{\tau+0}$, and we can consider the function $x_\tau(\zeta) = h(q_{\tau+0}, q_{\tau-0}; \zeta)$, called the endpoint symbol of the problem at the point τ. Denote by Δ_τ the set $\Delta(q_{\tau+0}, q_{\tau-0})$ corresponding to it for short. Let an integer-valued function $\chi_\tau(t)$ be constant on intervals of the complement to Δ, and for $t \in \Delta_\tau$, let the jump $\chi_\tau(t-0) - \chi_\tau(t+0)$ coincide with the number of zeros of the function $x_\tau(\zeta)$ on the line $\operatorname{Re}\zeta = t$ with account for their multiplicity. Therefore, this function monotonically decreases and is defined with accuracy up to an additive constant, which can be fixed by the condition $\chi(-0) = 0$.

Theorem 4. *Let the domain D be bounded by a piecewise-Lyapunov contour without cusp points, and let the set F contain all corner points of the contour. Then the Fredholm property of problem (7) in the class $H^p_\lambda(D,F)$ is equivalent to the condition*

$$\lambda_\tau \notin \Delta_\tau, \quad \tau \in F, \tag{11}$$

and its index $\varkappa$ is given by the formula

$$\varkappa = l(2-s) + \sum_\tau \chi_\tau(\lambda_\tau),$$

where s is the number of connected components of Γ.

Note that in the scalar case where $l = 1$ and $J = \nu$, the set Δ_τ is defined by relation (10), where the quantity $\theta = \theta_\tau$ is geometrically the angle of the sector to which the curvilinear sector S_τ passes under the affine transformation $x + iy \to x + \nu y$. In the particular case $\nu = i$ of analytic functions, the quantity θ_τ coincides with the angle of the sector S_τ itself. In the scalar case considered, condition (11) reduces to $\theta_\tau\lambda_\tau/\pi \notin \mathbb{Z}$, so that $\chi_\tau(t) = -[\theta_\tau t/\pi] + 1$, where the square brackets denote the integral part of a number. If the contour Γ is smooth, then $q_{\tau+0} = -q_{\tau-0}$ at points $\tau \in F$, and, according to (9), we have the same situation as in the scalar case.

For now, the boundary Γ of the domain D was assumed to be finite. Now let the curve Γ be unbounded, i.e., $\infty \in \Gamma$, and let it be a piecewise-Lyapunov contour on the Riemann sphere $\mathbb{C} \cup \infty$. The convergence of contours $\Gamma_n \subseteq D$ to Γ is understood in the same sense as above but with the additional requirement $\infty \in \Gamma_n$ for all n (in particular, the function $\alpha_n(t)$ is unbounded, i.e., $\alpha_n(\infty) = \infty$). Under this conditions, the Hardy space $H^p(D)$ is as before defined by condition (2). In particular, in the case of the half-plane $\{z = x + iy, y > 0\}$, as Γ_n, we can take the lines $y = \varepsilon_n$, where $\varepsilon_n \to 0$. Applied to analytic functions, this yields the classical Hardy space [4].

We now turn to the definition of weighted spaces in the case considered. We include the infinitely distant point ∞ to F and define the class $C_\lambda(\Gamma, F)$ as before but with the additional requirement $\varphi(t) = O(1)|t|^{-\lambda_\infty}$ as $t \to \infty$. Starting from this class, we define weight scalar-valued functions $\rho(t) = \rho_\lambda(t)$ and J-analytic $l \times l$-matrix functions $R(z) = R_\lambda(z)$ analogous to the above. Using these functions, we define the weighted spaces as before with the only difference that $L^p_\lambda(\Gamma, F) = L^p(\Gamma)$ and $H^p_\lambda(\Gamma, F) = H^p(\Gamma)$ for the weight order λ assuming the value $-1/p$ at finite points $\tau \in F$ and the value $1/p$ at the point $\tau = \infty$, and, analogously, $L^p_\lambda(D, F) = L^p(D)$ for the weight order λ assuming the value $-2/p$ at finite points $\tau \in F$ and the value $2/p$ at the point $\tau = \infty$. Note that in the case of the half-plane, the weighted Hardy space for J-analytic functions was introduced and studied in [8].

Theorems 1 and 2 are also preserved in the case considered with the only difference that the conditions $-1 < \lambda < 0$ and $-1/2 < \lambda < 0$ for $\tau = \infty$ are replaced by $0 < \lambda_\infty < 1$ and $0 < \lambda_\infty < 1/2$, respectively. Also, an analog of Theorem 3 holds. Let the set F consist of the single point $\tau = \infty$, i.e., the unbounded curve Γ is smooth. Let us cover Γ by smooth curves Γ_n, $n \leq 1$, so that

$$\tilde{\Gamma}_n = 2^{-n}\Gamma_n \subseteq \{1/2 \leq |z| \leq 2\}$$

for sufficiently large n. Let the domains D_n have an analogous meaning with respect to D. Then the space $L^p_\lambda(\Gamma, \infty)$ can be given by norm (5). The same also concerns the space $H^p_\lambda(D, \infty)$ in Theorem 3 with respect to (6). Therefore, as in the case of finite contours, the family of spaces considered monotonically decreases in each of the parameters p, λ_τ, $\tau \in F$.

Theorem 4 also extends to the case considered; we need to only define the enumeration of arcs $\Gamma_{\tau\pm0}$ with the endpoint $\tau = \infty$ and the corresponding unit tangent vectors $q_{\tau\pm0}$ to these arcs at this endpoint. Recall that in the case $\tau \neq \infty$, the enumeration of lateral sides $\Gamma_{\tau\pm 0}$ of the curvilinear sector

$$S_\tau = D \cap \{|z-\tau| < \varepsilon\},$$

where $\varepsilon > 0$ is sufficiently small, with vertex τ corresponds to the following rule. Going around the boundary ∂S_τ through the point τ from $\Gamma_{\tau-0}$ to $\Gamma_{\tau+0}$, the sector S_τ remains to the left. In this case the arc $\Gamma_{\tau\pm 0}$ can be given by the parametric equation $z-\tau = rq_{\tau\pm 0}\exp[ih(r)]$, $0 \le r \le \varepsilon$, where the real-valued function $h(r)$ is continuous and is equal to zero at the point $r=0$. In these terms, we can also analogously proceed in the case $\tau = \infty$. Precisely, for a sufficiently small $\varepsilon > 0$, the curvilinear "sector"

$$S_\infty = D \cap \{|z| > 1/\varepsilon\}$$

has the "arcs" $\Gamma_{\tau\pm 0}$ with the endpoint $\tau = \infty$ as its lateral sides, which admit the parametric equation $z = rq_{\tau\pm 0}\exp[ih(r)]$ on the semi-axis $r \ge 1/\varepsilon$, where the real-valued function $h(r)$ is continuous and tends to zero as $r \to \infty$. The enumeration of these arcs is chosen in the same way as indicated above for finite vertices τ.

2. Second-Order Elliptic Systems

In a domain $D \subseteq \mathbb{C}$ on the plane, let us consider the following second-order elliptic system:

$$A_{11}\frac{\partial^2 u}{\partial x^2} + \big(A_{12}+A_{21}\big)\frac{\partial^2 u}{\partial x \partial y} + A_{22}\frac{\partial^2 u}{\partial y^2} = 0 \tag{12}$$

with real coefficients $A_{ij} \in \mathbb{R}^{l\times l}$. The ellipticity condition is that $\det A_{ii} \neq 0$ and the characteristic equation

$$\det P(z) = 0, \quad P(z) = A_{11} + (A_{12}+A_{21}) + A_{22}z^2,$$

has no real roots.

The class of elliptic systems (12) in domains with smooth boundary $\Gamma = \partial D$ for which the Dirichlet problem $u|_\Gamma = f$ is Fredholm was studied by A. V. Bitsadze [1]. These systems were called weakly coupled by him, and they can be described [9] by the condition

$$\det\left[\int_{\mathbb{R}} P^{-1}(t)dt\right] \neq 0.$$

The equivalent requirement is the existence of a matrix $J \in \mathbb{C}^{l\times l}$, $\sigma(J) \subseteq \{\operatorname{Im} z > 0\}$, such that $\det(\operatorname{Im} J) \neq 0$ and the following matrix equations is satisfied:

$$A_{11} + (A_{12}+A_{21})J + A_{22}J^2 = 0. \tag{13}$$

In this notation, in each simply connected subdomain $D_0 \subseteq D$ any solution u of system (12) is represented in the form [9]

$$u = \operatorname{Re}\phi \tag{14}$$

with a J-analytic function ϕ, and, moreover, with accuracy up to a constant summand $\eta \in \mathbb{C}^l$, $\operatorname{Re}\eta = 0$, the function ϕ is defined uniquely. The fact that formula (14) indeed yields the solutions of system (12) follows directly from (2) and the equation

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0, \tag{15}$$

which the J-analytic vector-valued function satisfies by definition. In the whole domain, the function $\phi(z)$ in representation (14) is multivalued in general and admits a branching when going around connected components of the boundary ∂D. If this boundary lies in the finite part of the plane, i.e., the domain D contains a neighborhood of the infinitely distant point ∞, and the solution u satisfies the condition

$$|\operatorname{grad} u|(z) = O(1)|z|^{-2} \quad \text{as} \quad z \to \infty, \tag{16}$$

then in a neighborhood of ∞, this solution is represented by formula (14) with a univalent bounded function ϕ. In what follows, we assume that this condition holds for the domain considered, and according to this, the simple connectedness of a domain is determined by the connectedness of its boundary.

The Hardy spaces $H^p(D)$, $1 < p < \infty$, are defined below for weakly coupled systems. First, assume that the domain D considered is bounded by a Lyapunov contour, i.e., admits a smooth parametrization of class $C^{1,\mu}$ with a certain $0 < \mu < 1$. Choose a sequence of contours $\Gamma_n \subseteq D$, $n = 1, 2 \ldots$, converging to Γ in the following sense. There exists a homeomorphic mapping $\alpha_n : \Gamma \to \Gamma_n$ of class $C^{1,\mu}$ converging to the identity mapping in the norm of this space as $n \to \infty$.

Then the space $H^p(D)$, $1 < p < \infty$, of solutions of system (12) is defined by the finiteness condition for the norm

$$|u| = \sup_n |u|_{L^p(\Gamma_n)}. \tag{17}$$

Here, the domain D can be finite as well as infinite; recall that in the latter case, condition (16) is assumed to be fulfilled.

Obviously, the linear operation (14) transforms J-analytic functions ϕ from the Hardy class $H^p(D)$ considered in [10] into an analogous class of solutions of (12). The following important theorem shows that the converse is also true.

Theorem 5. *Let $u \in H^p(D)$ be a solution of the weakly coupled system* (12) *in a domain D bounded by a Lyapunov contour Γ. Then in each simply connected subdomain $D_0 \subseteq D$ bounded by a Lyapunov contour, the function u is represented in the form* (14) *with a J-analytic function $\phi \in H^p(D_0)$.*

With account for the Banach theorem on operators with closed range, it follows from this theorem that under the additional assumption $u(z_0) = \phi(z_0) = 0$ at a fixed point $z_0 \in D_0$, the following estimate holds:

$$|\phi|_{H^p(D_0)} \le \operatorname{const} |u|_{H^p}(D_0). \tag{18}$$

We can reformulate this estimate in terms of the function $v(z)$ conjugated to the solution u of system (12). It is defined by the conditions

$$\frac{\partial v}{\partial x} = -\left(A_{21}\frac{\partial u}{\partial x} + A_{22}\frac{\partial u}{\partial y}\right) \quad \text{and} \quad \frac{\partial v}{\partial y} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial u}{\partial y}. \tag{19}$$

The correctness of this definition follows from Eq. (12). Differentiating this equation and using (15), we obtain

$$\frac{\partial u}{\partial x} = \operatorname{Re}\phi', \qquad \frac{\partial u}{\partial y} = \operatorname{Re} J\phi',$$

where ϕ' is the derivative $\partial\phi/\partial x$. Substituting these expressions in (19), for the function v, we obtain the following representation analogous to (14):

$$v(z) - v(z_0) = \operatorname{Re} C[\phi(z) - \phi(z_0)], \qquad C = -(A_{21} + A_{22}J).$$

As a result, (18) leads to the estimate

$$|v|_{H^p(D_0)} \le \operatorname{const} |u|_{H^p}(D_0), \quad u(z_0) = v(z_0) = 0, \tag{20}$$

where the norm of the function v on the left-hand side is defined analogously to (17). In the case where (12) is the Laplace equation, v is a conjugated harmonic function, and estimate (20) is the content of the classical Riesz theorem [3].

It follows from theorem 5 and the corresponding properties of the Hardy space of J-analytic functions [10] that a function $u \in H^p(D)$ has angular limit values $u^+(t)$, $t \in \Gamma$, almost everywhere, which define a function from L^p. Moreover, the continuous dense embeddings

$$C(\overline{D}) \subseteq H^p(D) \subseteq L^p(D)$$

hold.

Also, it follows from Theorem 5 that the operator $\phi \to \operatorname{Re}\phi$ is Fredholm on the spaces H^p, and, moreover, its kernel consists of constant vectors $\{i\xi,\, \xi \in \mathbb{R}^l\}$, whereas the co-image is of dimension $l(s-1)$ and can be chosen in the class $C^1(\overline{D})$. The assertion on the image of the operator $\phi \to \operatorname{Re}\phi$ presented above is established analogously to [6] by an explicit description of multivalued J-analytic function with univalent real part. In particular, the index of this operator is equal to $l(s-2)$. Combining this with the assertion of [10, Theorem 2(a)] on the Cauchy-type integral

$$(I\varphi)(z) = \frac{1}{\pi i}\int_\Gamma (t-z)_J^{-1}\, dt_J \varphi(t)$$

with a real density $\varphi \in L^p(\Gamma)$, where the notation $z_J = x + Jy \in \mathbb{C}^{l\times l}$ is accepted for $z = x + iy \in \mathbb{C}$, we arrive at the following result for the operator

$$P\varphi = \operatorname{Re} I\phi \tag{21}$$

defining the integral representation of functions $u \in H^p(D)$.

Theorem 6. *The operator $P : L^p(\Gamma) \to H^p(D)$ is Fredholm, and its index is equal to zero; moreover, its kernel satisfies*

$$\ker P \subseteq C^{+0}(\Gamma) = \bigcup_{\mu>0} C^\mu(\Gamma),$$

and there exists a co-image $\operatorname{koIm} P \subseteq C^{+0}(\overline{D})$.

In explicit form, the operator P acts by the formula

$$(P\varphi)(z) = \int_\Gamma P(t, t-z)\varphi(t) ds_t, \quad z \in D, \tag{22}$$

with the matrix kernel

$$2\pi i P(t,z) = z_J^{-1}[e(t)]_J - z_{\overline{J}}^{-1}[e(t)]_{\overline{J}},$$

where $e(t) = e_1(t) + ie_2(t)$ is a unit tangent vector at a point t of the contour Γ (so that the complex differential satisfies $dt = e(t)ds_t$). Obviously, the function $P(t,z)$ is odd and homogeneous of degree -1 in the variable z, and, moreover,

$$P[t, e(t)] = 0, \quad t \in \Gamma. \tag{23}$$

Since the contour Γ is Lyapunov, it follows that for $z = t_0 \in \Gamma$, the function $P(t, t-t_0)$ has a weak singularity at $t = t_0$. More precisely,

$$P(t, t-t_0) = |t-t_0|^{-1}k(t_0, t),$$

where the function $k(t_0, t)$ satisfies the Hölder condition, i.e., belongs to the class $C^{+0}(\Gamma \times \Gamma)$ and vanishes for $t = t_0$. Moreover, the Sokhotskii–Plemejl formula for the Cauchy-type integral $I\varphi$ in (21) implies

$$(P\varphi)^+(t_0) = \varphi(t_0) + \int_\Gamma P(t_0, t - t_0)\varphi(t)\, ds_t. \tag{24}$$

In fact, along with the boundedness of $L^p(\Gamma) \to H^p(D)$, the operator P is bounded as an operator $C(\Gamma) \to C(D)$. This fact follows from the following general assertion.

Lemma 1. *Let the contour Γ be Lyapunov, and let the function $P(t, z)$, $t \in \Gamma$, be odd, homogeneous of degree -1 in the variable $z \in \mathbb{C}$, satisfy condition* (23)*, and infinitely times differentiable in the variables $x = \operatorname{Re} z$ and $x = \operatorname{Im} z$ (so that its partial derivatives $P_1(t, z) = \partial P/\partial x$ and $pP_2(t, z) = \partial P/\partial y$ are homogeneous of degree -2 in the variable z). Let the functions $P(t, z)$, P_1, and P_2 satisfy the Hölder condition on Γ. Then operator* (22) *is bounded as an operator $C(\Gamma) \to C(D)$, and formula* (24) *holds for boundary values.*

In the case where Γ is the unit circle and the matrix J is scalar and coincides with the imaginary unit i, with accuracy up to an additive summand, the function $q(t, t - z)$ coincides with the Poisson kernel

$$P(t, t - z) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \theta_0) + r^2} + \frac{1}{2}, \quad t = e^{i\theta}, z = re^{i\theta_0}.$$

Obviously, the integral operator on the right-hand side of (24) is compact on the space $L^p(\Gamma)$, and, therefore, with account for the Riesz theorem, Theorem 6 directly implies that the operator $u \to u^+$ is Fredholm as an operator $H^p(D) \to L^p(\Gamma)$ and its index is equal to zero. In other words, the Dirichlet problem is Fredholm in the class H^p and is of index zero. More precisely, the following assertion holds.

Theorem 7. (a) *The solution space $\{u \in H^p(D), \operatorname{Re} u^+ = 0\}$ of the homogeneous Dirichlet problem is finite-dimensional and is contained in $C^+(\overline{D})$. There exists a finite-dimensional space $Y \subseteq C^+(\Gamma)$ of the same dimension such that the orthogonality conditions*

$$\int_\Gamma f(t)g(t)\, ds_t = 0, \quad g \in Y,$$

are necessary and sufficient for the solvability of the inhomogeneous problem $\operatorname{Re} u^+ = f$.

(b) *Any solution $u \in H^p(D)$ of the Dirichlet problem with the right-hand side $f \in C(\Gamma)$ belongs to the class $C(\overline{D})$.*

The second assertion of the theorem is a consequence of Lemma 1 and the compactness of the integral operator on the right-hand side of (24) on the space $C(\Gamma)$. This means that the Dirichlet problem is Fredholm in the class of continuous functions. For the first time, this effect was discovered by N. E. Tovmasyan [13] for a definite class of elliptic systems.

For now, the exponent p in definition (13) of the Hardy class was assumed to be greater than 1. We can show that the operator P is also bounded as an operator from $L^1(\Gamma)$ into $H^1(D)$ (although for the Cauchy-type operator I in (21) this fact is violated). Therefore Theorem 7 also holds for the class $H^1(D)$.

As a consequence of the Fredholm property of the Dirichlet problem, we note that the relation

$$|u| = |u^+|_{L^p(\Gamma)} + |u|_{L^p(D)} \tag{25}$$

defines an equivalent norm in the space $H^p(D)$. To prove this, we only need to take into account the following assertion.

Lemma 2. *Let an operator $R : X \to Y$ be Fredholm, and in a Banach space X, let a norm $|\,|'$ admitting the estimate $|x|' \le C|x|_X$ be given. Then the relation $|x|'' = |x|' + |Rx|_Y$ defines an equivalent norm in X.*

The definition of Hardy spaces over domains with piecewise-smooth boundary, including the case of weighted spaces, differs from the analogous definition for J-analytic functions considered in [10]. The distinction of these two cases is caused by the essence of the subject and is related to the fact that there are no weight functions the multiplication by which is invariant in the class of solutions of system (12). Definite difficulties arise even for harmonic functions. In this case, these spaces were studied by V. I. Vlasov [14, 15] in detail.

Let a domain D considered on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ be bounded by a piecewise-Lyapunov contour Γ. Choose a finite set $F \subseteq \Gamma$ containing all corner points of the curve. If the curve Γ is unbounded, i.e., contains the infinitely distant point ∞ on the Riemann sphere, then we include this point in F. In the case where Γ is connected, i.e., is a simple contour and F consists of a single point τ, this curve is called a closed Lyapunov arc with common endpoint τ. In this case, for $\tau = \infty$, Γ is a smooth curve in each finite part of the plane.

Denote by $H^p_{\rm loc}(D, F)$ the class of all solutions of system (12) in the domain D that belong to $H^p(D_0)$ in each subdomain $D_0 \subseteq D$ with Lyapunov boundary ∂D_0 disjoined with F. By Theorem 5, this definition is correct. Clearly, the functions of this class admit angular limit values u^+ belonging to $L^p_{\rm loc}(\Gamma, F)$, i.e., to $L^p(\Gamma_0)$ on each smooth arc $\Gamma_0 \subseteq \Gamma$ disjoined with F.

Starting from a family $\lambda = (\lambda_\tau,\ \tau \in F)$ of real numbers (weighted order), we introduce the Hardy space

$$H^p_\lambda(D, F) \subseteq H^p_{\rm loc}(D, F)$$

by the pattern of definition from [10, Theorem 7] for Douglis analytic functions. As in [10], we first assume that Γ is connected and F consists of a single point τ, i.e., Γ is a closed Lyapunov arc with the common endpoint τ. Obviously, it suffices to restrict ourselves to two cases $\tau = 0$ and $\tau = \infty$, which we consider separately.

(1) Let $\tau = 0$. Represent D in the form of the union of domains D_n, $n \ge 1$, with Lyapunov boundary for which

$$\tilde D_n = 2^n D_n \subseteq \{1/2 \le |z| \le 2\}$$

for sufficiently large n and whose boundaries $\partial \tilde D_n$ converge to a certain contour $\tilde\Gamma$ in the same sense as in (13). Then for $u \in H^p_{\rm loc}(D, 0)$,

$$\tilde u_n(z) = u(2^{-n}z) \in H^p(\tilde D_n),$$

and the space $H^p_\lambda(D, 0)$ is defined by the finiteness condition for the norm

$$|u| = \left(\sum_n |\xi_n|^p\right)^{1/p}, \quad \xi_n = 2^{-n\lambda}|\tilde u_n|_{H^p(\tilde D_n)}. \tag{26}$$

(2) Let $\tau = \infty$. Represent D in the form of the union of domains D_n, $n \le 1$, with Lyapunov boundary for which

$$\tilde D_n = 2^{-n} D_n \subseteq \{1/2 \le |z| \le 2\}$$

for sufficiently large n and whose boundary $\partial \tilde D_n$ converge to a certain contour $\tilde\Gamma$ in the same sense as in (13). Then for $u \in H^p_{\rm loc}(D, \infty)$,

$$\tilde u_n(z) = 2^{-n\lambda} u(2^{-n}z) \in H^p(\tilde D_n),$$

and the space $H^p_\lambda(D, \infty)$ is defined by the finitness condition for norm (26).

Theorem 8. *Let* $u \in H^p_\lambda(D,\tau)$ *be a solution of the weakly coupled system* (1) *in the domain* D *bounded by a closed bounded Lyapunov arc* Γ *with common endpoint* τ. *Let* τ *be not a cusp point of the curve* Γ (*on the Riemann sphere*). *Then* u *is represented in the form* (14) *with the* J*-analytic function* $\phi \in H^p_\lambda(D,\tau)$.

For $\lambda < 1$, it is convenient to extend the space $H^p_\lambda(D,\tau)$ by adding a constant function. This extension is a Banach space denoted by $H^p_{(\lambda)}(D,\tau)$ with respect to the corresponding norm. Obviously it coincides with $H^p_\lambda(D,\tau)$ for $\lambda < 0$ and $H^p_{(\lambda)}(D,\tau) = H^p_\lambda(D,\tau) \oplus \mathbb{R}^l$ for $0 \le \lambda < 1$. In this notation, Theorem 4 leads to the following estimate analogous to (18):

$$|\phi|_{H^p_{(\lambda)}} \le \text{const}\,|u|_{H^p_{(\lambda)}}, \quad u(z_0) = \phi(z_0) = 0,$$

where the point $z_0 \in D$ is fixed.

In the general case where the set F is multipoint, we can represent the domain D in the form of the union of domains D_τ, $\tau \in F$, where D_τ is bounded by a closed Lyapunov arc Γ_τ with common endpoint τ. Then the space $H^p_\lambda(D,F)$ can be defined by the condition $u \in H^p_{\lambda_\tau}(D_\tau,\tau)$ for all $\tau \in F$ (by the corresponding norm). According to Theorem 5, this definition is independent of the choice of D_τ. In a similar way, we also define the space $H^p_{(\lambda)}(D,F)$, $\lambda < 1$.

As in the case of Theorem 5, Theorem 8 easily implies that the operator $\phi \to \operatorname{Re}\phi$ is Fredholm on the spaces $H^p_{(\lambda)}$, and, moreover, its kernel consists of constant vectors $\{i\xi,\ \xi \in \mathbb{R}^l\}$, whereas the direct complement to its image is of dimension $l(s-1)$ and can be chosen in the class of functions satisfying the Hölder condition in the closed domain $\overline{D}$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

As in the case of smooth domains, the criterion for the Fredholm property of the Schwarz problem $\operatorname{Re}\phi^+ = f$ for J-analytic functions given in [10] simultaneously serves as a criterion for the Fredholm property of the Dirichlet problem $\operatorname{Re}u^+ = f$ for solutions of system (12). In the notation of [10], it is formulated as follows.

Theorem 9. *Let a domain* D *be bounded by a piecewise-Lyapunov contour without cusp points, and let the set* F *contain all corner points of the contour. Then the Fredholm property for the Dirichlet problem for system* (12) *in the class* $H^p_{(\lambda)}(D,F)$, *where* $0 < |\lambda_\tau| < 1$, *is equivalent to the condition*

$$\lambda_\tau \notin \Delta_\tau, \quad \tau \in F, \tag{27}$$

and its index $\varkappa$ *is given by the formula* $\varkappa = \sum\limits_{\tau} \chi_\tau(\lambda_\tau)$.

An analogous result for the weighted Hölder spaces was obtained in [11] (in slightly different terms). As above, on the basis of Theorem 9 and Lemma 2, we conclude that under assumption (27), the relation

$$|u| = |u^+|_{L^p_\lambda(\Gamma)} + |u|_{L^p_\lambda(D)}$$

defines an equivalent norm in the space $H^p_\lambda(D,F)$.

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