LOCALIZATION OF WAVES NEAR THE INTERFACE BETWEEN NONLINEAR MEDIA WITH SPATIAL DISPERSION

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The nonlinear stationary excitations localized near the interface between media with spatial dispersion are investigated. In the study of the given problem, of prime importance is a consideration of interaction of excitations with the interface between the media. In this case, the arising localized excitations are nonlinear interface waves. It is demonstrated that nonlinear interface waves of several types exist in the examined system. The conditions of their existence are formulated.

1. In connection with the intensive development of physics of nonlinear phenomena, interest in problems of localization of excitations of various nature in nonlinear inhomogeneous media with spatial dispersion has increased recently [1–3]. The nonlinear excitations localized near the interface between the media are widely covered both theoretically and experimentally. Problems of electromagnetic wave localization at the interface between nonlinear media without dispersion were discussed, for example, in [4, 5]. The field profile of these excitations is asymmetric (in contrast with the field of freely propagating solitons). Such excitations were loosely called nonlinear surface waves, because they were localized on a defect in the bulk of a solid rather than on its surface.

In [4, 5], the interaction between the wave and the interface was considered passive in character in the sense that there was no parameter describing the intensity of wave interaction with the interface, and the boundary conditions were reduced to the continuity condition for the sought-after field and its derivative at the interface. In most cases, the interaction of linear [6-9] and nonlinear excitations [10, 11] with defects (and in particular, with the interface between the media) must be taken into account even in the simplest short-range force approximation. Hereinafter, we consider that the interface between the media interacts strongly with the wave and has the parameter characterizing this effect.

The dispersion of the medium and the interaction of the field with a defect were simultaneously taken into account, for example, in [12] based on the generalized Korteweg–de Vries equation with the short-range potential describing a point defect.

In modern physics of nonlinear phenomena it is well known [13, 14] that stationary nonlinear waves can be formed only when there are two competing factors, namely, the nonlinearity of the medium and the linear wave dispersion. In this regard, it is of interest to analyze conditions of existence of nonlinear excitations near the interface between nonlinear media with spatial dispersion considering the wave interaction with the interface. The present study is devoted to a solution of exactly this problem. The waves examined in the present study are called interface waves, because their localization is studied with allowance for their interaction with the interface. In the present study, conditions of existence of nonlinear interface waves of different types are formulated for media with and without spatial dispersion. A physical nature of the examined excitations is not specified not to limit generality of the results obtained.

2. We now consider the case in which a wave propagates along a flat interface between two media with the same spatial dispersions and different nonlinearities. The wave dispersion is characterized by the parameter β, and the nonlinearities on both sides of the interface differ by ε. It is important that the wave interacts with the interface between the media, which is modeled by a preset effective potential.

Let us generalize the model described in [5] where the problem was reduced to the one-dimensional nonlinear Schrödinger equation in which the parameter t played the role of the coordinate in the plane of the interface between the media. By analogy, we consider a wave propagating in the *xOt* plane. We also assume that the dispersion of the medium in the direction perpendicular to the interface (along the *Ox* axis) is much greater than that in the interface plane (along the *Ot* axis). As a result, we obtain the generalized nonlinear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t}+\alpha\frac{\partial^2\psi}{\partial x^2}-\beta\frac{\partial^4\psi}{\partial x^4}+\gamma|\psi|^2\psi=U\left(x,|\psi|^2\right)\psi\,, \tag{1}$$

where $\alpha > 0$, β is the dispersion parameter the sign of which is determined by the dispersive properties of the medium (we consider separately the cases of $\beta > 0$ and $\beta < 0$), and $\gamma > 0$ characterizes the nonlinearity of the medium on the left of the interface. The effective potential has the form

$$U\left(x,|\psi|^2\right)=\begin{cases}U_0\partial(x), & x<0,\\ U_0\partial(x)-\Omega-\varepsilon|\psi|^2, & x>0.\end{cases} \tag{2}$$

Here the parameter U_0 characterizes the intensity of wave interaction with the interface in the plane perpendicular to the *Ox* axis and passing through the point $x = 0$, $\delta(x)$ is the Dirac delta-function, Ω is the potential difference between the media, and ε is the difference between the nonlinearities of the media.

3. For a nonlinear medium without dispersion and defects (when $\beta = 0$, $\varepsilon = 0$, $\Omega = 0$, $U_0 = 0$, and the remaining parameters are nonzero), Eq. (1) has an exact solution in the form of a freely propagating soliton [13, 14]:

$$\psi(x,t)=k\sqrt{\frac{2\alpha}{\gamma}}\frac{e^{i\alpha k^2 t}}{\cosh k(x-x_0)}\,, \tag{3}$$

localized in the direction perpendicular to the direction of its motion with a maximum lying on the straight line $x = x_0$, where the wave number k is the free parameter.

By direct substitution, we can verify that in a nonlinear medium without defects but with dispersion (when $\varepsilon = 0$, $\Omega = 0$, $U_0 = 0$, and the remaining parameters are nonzero), Eq. (1) has exact solutions of two types:

1) even with respect to the variable x

$$\psi(x,t)=A\frac{e^{i\omega t}}{\cosh^2 k(x-x_0)}\,, \tag{4}$$

where

$$\omega=4k^2(\alpha-4\beta k^2)=\frac{4\alpha^2}{25\beta}\,, \tag{5}$$

$$A=2k^2\sqrt{\frac{30\beta}{\gamma}}=\frac{\alpha}{10}\sqrt{\frac{30}{\gamma\beta}}\,, \tag{6}$$

$$k=\pm\frac{1}{10}\sqrt{\frac{5\alpha}{\beta}}\,, \tag{7}$$

which exists for $\beta < 0$, and

2) odd with respect to the variable x

$$\psi(x,t) = A\frac{\sinh k(x-x_0)}{\cosh^2 k(x-x_0)}e^{i\omega t}, \tag{8}$$

where

$$\omega = k^2(\alpha + |\beta| k^2) = \frac{11\alpha^2}{100|\beta|}, \tag{9}$$

$$A = 2k^2\sqrt{\frac{30|\beta|}{\gamma}} = \frac{\alpha}{5}\sqrt{\frac{30}{\gamma|\beta|}}, \tag{10}$$

$$k = \pm\sqrt{\frac{\alpha}{10|\beta|}}, \tag{11}$$

which exists for $\beta < 0$.

4. We now consider the case in which the wave interacts with the interface between the nonlinear media without dispersion. From Eq. (1) for $\beta = 0$ and the remaining parameters being nonzero, we obtain the well-known boundary conditions

$$\psi(+0,t) = \psi(-0,t), \alpha\left\{\psi'(+0,t) - \psi'(-0,t)\right\} = U_0\psi(0,t), \tag{12}$$

corresponding to the continuity condition of the function $\psi(x, t)$ at the interface and the discontinuity condition for the first derivative. Following [5] with allowance for Eq. (3), we seek a solution of Eq. (1) in the form

$$\psi(x,t) = \begin{cases} k\sqrt{\dfrac{2\alpha}{\gamma}}\dfrac{e^{i\omega_1 t}}{\cosh k(x-X_1)}, & x<0, \\ k\sqrt{\dfrac{2\alpha}{\gamma+\varepsilon}}\dfrac{e^{i\omega_2 t}}{\cosh k(x-X_2)}, & x>0, \end{cases} \tag{13}$$

with dispersion laws, according to Eq. (3), $\omega_1 = \alpha k_1^2$ and $\omega_2 = \alpha k_2^2$. Since the frequency remains unchanged in the interface plane, from the equality $\omega_1 = \omega_2$ we obtain a relationship between wave numbers $k_2^2 = k^2 - \Omega/\alpha$, where the subscript "1" for the wave number to the left of the interface has been omitted to simplify the subsequent presentation of the material.

Substituting Eq. (13) into Eq. (12), we find the parameters characterizing the position of a maximum of the nonlinear interface wave:

$$X_1 = \frac{1}{k}\operatorname{arctanh}\frac{1}{k}\left\{\frac{U_0\gamma}{\alpha\varepsilon} \pm \sqrt{k^2 - k_c^2}\right\}, \quad X_2 = \frac{1}{\sqrt{k^2-\Omega/\alpha}}\operatorname{arctanh}\frac{1}{\sqrt{k^2-\Omega/\alpha}}\left\{\frac{U_0}{\alpha}\left(1+\frac{\gamma}{\varepsilon}\right) \pm \sqrt{k^2-k_c^2}\right\}. \tag{14}$$

Thus, there are two nonlinear interface waves with the asymmetry determined by the maximum at $x = X_1$ or $x = X_2$; however, only one of them is stable. In order that such waves exist, conditions $k_2 > \Omega/\alpha$ and $\varepsilon > -\gamma$ must be satisfied together with one of the following conditions:

1) if the nonlinearity to the right of the interface is stronger than to the left of it (for $\varepsilon > 0$), the condition $k_c^2 < 0$ must be satisfied for $\varepsilon > \varepsilon_c$; then the values specified by Eq. (14) are real;

2) if the nonlinearity to the right of the interface is weaker than to the left of it (for $\varepsilon < 0$), $k^2 > 0$ for $-\gamma < \varepsilon < \varepsilon_c$, and then the values specified by Eq. (14) are real only when $k^2 > k_c^2$.

If the nonlinearities on both sides of the interface are identical, only one mode exists:

$$X_1 = -\frac{1}{k}\operatorname{arctanh}\frac{U_0^2 + \alpha\Omega}{2\alpha U_0 k}, \quad X_2 = \frac{1}{\sqrt{k^2 - \Omega/\alpha}}\operatorname{arctanh}\frac{U_0^2 - \alpha\Omega}{2\alpha U_0\sqrt{k^2 - \Omega/\alpha}}.$$

It should be noted that the equality $U_0 = 0$ means only that the wave does not interact with the interface between the media, but the interface itself remains unchanged. In this case, from Eq. (14) we derive the formulas presented in [5]:

$$X_1 = \pm\frac{1}{k}\operatorname{arctanh}\sqrt{1 - \frac{k_c^2}{k^2}}, \quad X_2 = \pm\frac{1}{\sqrt{k^2 - \Omega/\alpha}}\operatorname{arctanh}\sqrt{\frac{k^2 - k_c^2}{k^2 - \Omega/\alpha}},$$

that determine two waves called nonlinear surface waves in [5]. In the derivation of the above formulas, the boundary conditions of the continuity of the function $\psi(x, t)$ and its first derivative were used that follow from Eq. (12) at $U_0 = 0$. For these nonlinear waves localized at the passive interface between the media, the condition of their existence is $k^2 > k_c^2$, where $k_c^2 = -\gamma\Omega/\varepsilon\alpha$. This condition is satisfied if the nonlinearity parameters ε and γ are opposite in sign. This means that wave amplitude (13) must exceed the critical value $A_c = \sqrt{-2\Omega/\varepsilon}$, which is the case for ε and Ω opposite in sign. In [5] it was also demonstrated that the wave with a maximum at $x = X_1$ is stable when $\varepsilon < 0$ ($\Omega < 0$ in [5]).

It can be seen that the consideration of wave interaction with the interface between the media causes the existence of nonlinear interface waves for a wider range of the parameters of the medium.

5. Now we study the existence of nonlinear interface waves in media with spatial dispersion. Given that all the parameters of Eq. (1) are nonzero, we naturally obtain the following system of the boundary conditions [8, 9, 12]:

$$\psi(+0) = \psi(-0), \quad \frac{\partial\psi(+0)}{\partial x} = \frac{\partial\psi(-0)}{\partial x}, \quad \frac{\partial^2\psi(+0)}{\partial x^2} = \frac{\partial^2\psi(-0)}{\partial x^2}, \quad \frac{\partial^3\psi(+0)}{\partial x^3} - \frac{\partial^3\psi(-0)}{\partial x^3} + \frac{U_0}{\beta}\psi(0) = 0. \tag{15}$$

Let us restrict ourselves to an examination of localization of waves described by Eq. (4) for $\beta > 0$, because the problem of localization of waves described by Eq. (8) for $\beta < 0$ gives no qualitatively new results, but the formulas become more cumbersome.

In the examined case, we seek a solution of Eq. (1) in the form

$$\psi(x,t) = \begin{cases} 2k_1^2\sqrt{\dfrac{30\beta}{\gamma}}\dfrac{e^{i\omega_1 t}}{\cosh^2 k_1(x - X_1)}, & x < 0, \\ 2k_2^2\sqrt{\dfrac{30\beta}{\gamma + \varepsilon}}\dfrac{e^{i\omega_2 t}}{\cosh^2 k_2(x - X_2)}, & x > 0, \end{cases} \tag{16}$$

where according to Eq. (5), dispersion laws are $\omega_1 = 4k_1^2(\alpha - 4\beta k_1^2)$ and $\omega_2 = \Omega + 4k_2^2(\alpha - 4\beta k_2^2)$. Since the frequency is retained in the interface plane, from the equality $\omega_1 = \omega_2$ it follows that

$$k_2 = \sqrt{k_m^2 \pm \sqrt{\left(k_m^2 - k_1^2\right)^2 + \Omega/16\beta}}, \tag{17}$$

where $k_m^2 = \alpha/8\beta > 0$. Two signs under the radical sign in Eq. (17) indicate the existence of several interface waves under specific conditions. Below we demonstrate that the condition $k_1k_2 < 0$ must be satisfied; therefore, for definiteness we set $k_1 < 0$ and $k_2 > 0$. For real wave number (17), we obtain that

1) two waves with wave numbers (17) exist when $-q_2 < k_1 < -q_1$ and $q_1 < q_2$, where $q_{1,2} = \sqrt{k_m^2 \pm \sqrt{k_m^4 - \Omega/8\beta}}$,

2) one wave with wave number (17) exists with the plus sign for $k_1 < -q_2$.

In these cases, conditions $\Omega \le \alpha^2/4\beta$ and $\varepsilon > \gamma$ must also be satisfied.

Substituting Eq. (16) into Eq. (15), we obtain a system of four equations. By simple manipulations, the pair of equations from this system can be reduced to the expressions $\tanh^2 k_1 X_1 = -k_2/3k_1$ and $\tanh^2 k_2 X_2 = -k_1/3k_2$, from which the condition $k_1 k_2 < 0$ follows. Further manipulations yield solutions of the system in an explicit form:

$$k_1 = \left(\frac{U_0}{\beta}\right)^{\frac{1}{3}} F(\varepsilon/\gamma) = \Phi(\varepsilon/\gamma, \alpha/\beta, \Omega/\beta),$$

$$k_2 = \left(\frac{U_0}{\beta}\right)^{\frac{1}{3}} F(\varepsilon/\gamma)\varphi(\varepsilon/\gamma) = \Phi(\varepsilon/\gamma, \alpha/\beta, \Omega/\beta)\varphi(\varepsilon/\gamma),$$

$$X_1 = \frac{1}{\Phi}\operatorname{arctanh}\sqrt{\frac{|\varphi|}{3}}, \qquad X_2 = \frac{1}{\Phi|\varphi|}\operatorname{arctanh}\sqrt{\frac{1}{3|\varphi|}}, \tag{18}$$

where we have introduced the following functions of the parameters entering into Eq. (1):

$$\varphi(\varepsilon/\gamma) = \frac{1}{6}\left\{\sqrt{1+\frac{\varepsilon}{\gamma}} - 1 - \sqrt{\left(\sqrt{1+\frac{\varepsilon}{\gamma}} - 1\right)^2 + 36\sqrt{1+\frac{\varepsilon}{\gamma}}}\right\},$$

$$F(\varepsilon/\gamma) = \frac{1}{2}\left\{\sqrt{\frac{3}{|\varphi|}}\frac{1}{(\varphi-1)(2\varphi+3)}\right\}^{\frac{1}{3}},$$

$$\Phi_{\pm}(\varepsilon/\gamma, \alpha/\beta, \Omega/\beta) = -\frac{1}{\sqrt{\varphi^2+1}}\sqrt{k_m^2 \pm \sqrt{k_m^4 + \frac{\Omega}{16\beta}\frac{\varphi^2+1}{\varphi^2-1}}}. \tag{19}$$

Signs of these functions must be chosen to meet conditions $k_1 < 0$ and $k_2 > 0$; therefore, we obtain $\varphi < 0$ and $\Phi_{\pm} < 0$. Figure 1 shows plots of functions (19), which in fact are dependences of the wave number k_1 on the nonlinearity parameters determining two interface modes by virtue of equality (18).

The function φ and hence functions $\Phi_{\pm}$ and F will be real for $\varepsilon > -\gamma$ (this region is bounded from the left by the dashed straight line *a* in Fig. 1). The function F has a discontinuity at the nonlinearity parameter $\varepsilon = 45\gamma/4$ (shown by the dashed straight line *b* in Fig. 1).

Our analysis of functions (19) has demonstrated that for $-\gamma\xi_1 < \varepsilon < -\gamma\xi_2$, where

$$\xi_{1,2} = 6\frac{\Omega}{\beta}\frac{\frac{\Omega}{\beta}\left(3\frac{\Omega}{\beta} + 192k_m^4\right) + 3072k_m^6 + 4\left(\mp\frac{\Omega}{\beta} \pm 16k_m^4\right)256k_m^6 - \frac{\Omega^2}{\beta^2}}{\left(5\frac{\Omega}{\beta} + 64k_m^4\right)\left(1024k_m^6 + 5\frac{\Omega^2}{\beta^2} + 144\frac{\Omega}{\beta}k_m^4\right)},$$

the functions $\Phi_{\pm}$ have no real values. This means that interface waves (16) do not exist in this region (it is narrow, in particular, for the parameters of Fig. 1, it is in the range $0.089\gamma < \varepsilon < -0.046\gamma$). For $-\gamma < \varepsilon < -\gamma\xi_1$, there are two interface waves (16) with wave numbers determined by functions $\Phi_{\pm}$ according to Eq. (18). For $\varepsilon > -\gamma\xi_2$, there is only one interface wave (16) with the wave number determined by the function Φ_{+}.

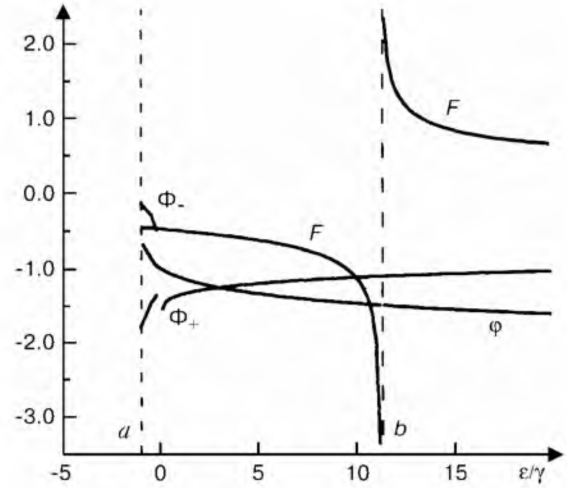

Fig. 1. Functions (19) versus the nonlinearity parameter. Dashed straight lines: a) $\varepsilon > \gamma$ and b) $\varepsilon = 45\gamma/4$. The function Φ was calculated for $\Omega/\beta = 1$ and $\alpha/\beta = 16$.

From Eq. (18) it follows that nonlinear interface waves (16) can exist only in the media with the nonlinearity parameters, the dispersion, and the intensity of wave interaction with the defect related by the expression

$$\frac{U_0}{\beta} = \left(\frac{\Phi}{F}\right)^3 . \tag{20}$$

From here it follows that the condition $U_0 < 0$ must be satisfied for $F > 0$, that is, for $\varepsilon = 45\gamma/4$, or the condition $U_0 > 0$ must be satisfied for $F < 0$, that is, for $-\gamma < \varepsilon < 45\gamma/4$. For the existence of nonlinear interface waves (16) it is also necessary to select such values of functions (19) for which the above-indicated limitations on the wave number k_1 following from Eq. (17) are met.

The results obtained are more illustrative when the potentials of the media on both sides of the interface are identical, that is, when $\Omega = 0$. From Eq. (17) we obtain two solutions: $k_2 = \sqrt{2k_m^2 - k_1^2}$ for the asymmetric wave and $k_2 = -k_1$ for the symmetric wave. From Eq. (18) it follows that symmetric nonlinear interface waves can exist in the media with identical nonlinearities on both sides of the interface, that is, when $\varepsilon = 0$. The parameters of solution (16) with allowance for Eq. (20) for the symmetric state with $k_2 = -k_1$ assume the form

$$k_1 = -k_2 = -\frac{1}{2}\left(\frac{\sqrt{3}U_0}{\beta}\right)^{\frac{1}{3}}, \; X_1 = -X_2 = -2\left(\frac{\beta}{\sqrt{3}U_0}\right)^{\frac{1}{3}} \operatorname{arctanh}\frac{1}{\sqrt{3}} .$$

6. In conclusion, it should be noted that in media with arbitrary values of the nonlinearity parameter, dispersion, and intensity of wave interaction with the defect, the asymmetric states whose structures differ significantly from Eq. (16) and, for example, are expressed through elliptic functions can be localized at the interface.

It is important that in this work, all results have been obtained in explicit forms and are exact in the sense that no approximate calculation methods, for example, of perturbation theory were used in their derivation.

In this work, we have not touched the stability of nonlinear interface waves, since this is a fundamental problem that calls for a separate consideration.

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