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CHARACTERISTICALLY CLOSED DOMAINS FOR FIRST ORDER STRICTLY HYPERBOLIC SYSTEMS IN THE PLANE

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We consider a first order strictly hyperbolic system of n equations with constant coefficients in a bounded domain. It is assumed that the domain is strictly convex relative to characteristics, so that the projection along each characteristic is an involution having two fixed singular points. The natural statement of boundary value problems for such systems requires that singular points go to singular points under such transformations. We present a necessary and sufficient condition for the existence of such domains, called characteristically closed. Bibliography: 4 titles.

Dedicated to the memory of V. V. Zhikov

In a bounded domain D in the plane, we consider the first order hyperbolic system

$$\frac{\partial u}{\partial x_2} - A\frac{\partial u}{\partial x_1} = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1}$$

for a vector $u = (u_1, u_2, \dots, u_n)$, where the real constant $n \times n$-matrix $A \in \mathbb{R}^{n\times n}$ has distinct real eigenvalues $\nu_1, \nu_2, \dots, \nu_n$. Characteristics of this system are lines parallel to the lines

$$l_k: \ x_1 + \nu_k x_2 = 0, \quad k = 1, 2, \dots, n. \tag{2}$$

Apparently, the systematic study of boundary value problems for hyperbolic systems with many (including multiple) characteristics comes back to [1] and is further developed in [2]. As a rule, such problems were studied in infinite sector type domains with curved boundaries. In the case of finite domains, there is no systematic theory of boundary value problems for hyperbolic type equations with many characteristics. The particular case of three characteristics was considered in [3]. However, even in this case, a correct statement of boundary valued problems requires certain conditions on the domain D where the solution is looked for. The situation becomes much more complicated if $n > 3$, where n is the number of characteristics. In this paper, we describe such domains in the case $n > 3$.

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The first natural assumption on the domain D is that the interior points of any characteristic segment with endpoints in $\overline{D}$ are contained in D. Such domains are called *characteristically convex*. In particular, the boundary Γ of D contains no characteristic segments. By the same reason, for each $k = 1, 2, \ldots, n$ there are two supporting characteristics l_k^0 and l_k^1 that are parallel to l_k and pass through the boundary points τ_k^0 and τ_k^1, called *singular*, between which the domain D is located. The set of singular points is denoted by F. Since several supporting characteristics can pass through the same singular point, elements of F are located between 2 and $2n$.

We show that a characteristically convex domain is bounded by a simple Jordan contour Γ divided by the points τ_k^0 and τ_k^1 into two arcs Γ_k^0 and Γ_k^1 with common endpoints τ_k^0 and τ_k^1.

Indeed, let us fix k and consider the Cartesian coordinates ξ, η with the η-axis directed along l_k. Then the domain D can be described by the inequalities $f(\xi) < \eta < g(\xi)$, $a < \xi < b$, where the functions f and g take the same values at the points $\xi = a$ and $\xi = b$ (so that the points $(a, f(a))$ and $(b, f(b))$ are singular). It suffices to show that f and g are continuous. Assume the contrary. Let there exist a sequence $c_j \in [a, b]$ converging to c such that it has the limit $d = \lim f(c_j)$ different from $f(c)$. But, in this case, the sequence of boundary points $(c_j, f(c_j))$ converges to a point $(c, d) \in \Gamma$. By the characteristic convexity of the domain, the interior of the segment with endpoints (c, d) and $(c, f(c))$ is contained in D, which is impossible.

By the characteristic convexity of the domain, the projection along l_k is a homeomorphism, denoted by δ_k, from the contour Γ to itself. It is obvious that δ_k is involutive, i.e., $\delta_k[\delta_k(t)] = t$, $t \in \Gamma$, fixed points are the singular points τ_k^0, τ_k^1, and Γ_k^0 goes to Γ_k^1 under the transformation δ_k.

In the case of three characteristics, the statement of a boundary value problem proposed in [3] suggests that the contour Γ is divided into two curves $\Gamma^{\pm}$ such that each involution δ_k is a homeomorphism from Γ^+ onto Γ^-. Then the boundary value problem consists of two linear combinations of components of the solution u on Γ^+ and one combination on Γ^-. However, not every characteristically convex domain admits such a division. In the case $n > 3$, as we will see below, it is necessary to impose some conditions on the eigenvalues ν_k of the matrix A of the hyperbolic system (1). As above, provided that such a division exists, the correct statement of a boundary value problem for the hyperbolic system (1) consists of $m \leqslant n$ linear combinations of components of the solution u on Γ^+ and $n - m$ such combinations on Γ^-. The case $n = 0$ or $n = m$ is not excluded and corresponds to the Cauchy problem.

The following theorem completely describes characteristically convex domains that admit such a division of the boundary. By definition, a domain D is *characteristically closed* if all n transformations δ_k are invariant on the set F of singular points.

A Jordan arc Γ_0 is said to be *noncharacteristic* if each characteristic can intersect Γ_0 at most at one point. It is obvious that for such an arc the segment L_0 joining the arc endpoints is not characteristic. It is clear that for any point $y \in \Gamma_0$ the characteristic parallel to l_k intersects L_0 exactly at one point $p_k(y)$ and the projection $p_k : \Gamma_0 \to L_0$ is a homeomorphism.

Theorem 1. *Let a domain D be characteristically closed. Then the number of singular points is even and is equal to 2 or $2n$. The arcs on which the contour Γ is divided by these points are noncharacteristic. These arcs are denoted by $\Gamma^{\pm}$ in the first case and by $\Gamma_j^{\pm}$, $1 \leqslant j \leqslant n$, where the arcs of the same sign are pairwise disjoint, in the second case. Then each δ_k sends an arc to an arc of the opposite sign, in particular, the curve $\Gamma^+ = \Gamma_1^+ \cup \ldots \cup \Gamma_m^+$ goes to a similar curve Γ^-. If the number of singular points is equal to $2n$, then the polygon P with vertices at singular points is convex and characteristically closed. The converse assertion is also true*: *If a polygon P of the above type is given and noncharacteristic arcs $\Gamma_j^{\pm}$ have common endpoints with* *its sides $L_j^{\pm}$, then these arcs form the contour Γ bounding a characteristically closed domain D with singular points at the vertices of the polygon P.*

Proof. We fix k and consider the arcs Γ_k^0 and Γ_k^1 with common endpoints at the singular points τ_k^0 and τ_k^1. Since these points are fixed under the involution δ_k, the set $F \setminus \{\tau_k^0, \tau_k^1\}$ is invariant under the involution δ_k and, consequently, is divided into pairs of singular points which are transformed to each other. In particular, the set F consists of $2m$ elements. Denote by $e_k^1, \dots, e_k^{m-2}$ the segments of characteristics passing through these pairs of points along the arc Γ_k^0 from τ_k^0 to τ_k^1.

It is obvious that two arcs in the family $(\Gamma_j^{\pm})$ with a common endpoint τ^0 have opposite signs. Moving along the arc Γ_k^0 from τ^0 to τ^1, we see that a similar property holds for other pairs of arcs of this family that are transformed by this involution to each other.

Assume that $1 < m < n$. Then for some k another supporting line l_s^0, $s \neq k$, passes through the point τ_k^0, so that $\tau_k^0 = \tau_s^0$ and $\tau_k^1 = \tau_s^1$. Then there is at least one singular point between l_k^0 and e_s^1, which contradicts the fact that there are $2m$ points on e_s^i, $1 \leqslant i \leqslant 2m-2$.

Let the number of singular points be equal to $2n$, and let the segments $L_j^{\pm}$ join the endpoints of $\Gamma_j^{\pm}$. We first show that the polygon P with sides $L_j^{\pm}$ is characteristically convex and, consequently, is convex in the usual sense. It suffices to verify that any characteristic can intersect the boundary at most at two points. Without loss of generality we can assume that the characteristic l does not pass through singular points and, for the sake of definiteness, is parallel to the line l_1. Assume the contrary. Let l intersect three sides L_1, L_2, L_3 of the polygon P at points τ_1, τ_2, τ_3 respectively. Assume that arcs Γ_1, Γ_2, Γ_3 of the family $\Gamma_j^{\pm}$ are associated with L_1, L_2, L_3. Then the points $p_1^{-1}(\tau_j)$ lie on the contour Γ, which contradicts the characteristic convexity of D. It is obvious that the small diagonals of P are parallel to the characteristics.

The last assertion of the theorem can be proved by using the projections $p_k : \Gamma_k^{\pm} \to L_k^{\pm}$. $\square$

A question arises whether there exists a characteristically closed $2n$-gon P. Segments joining vertices of a convex polygon are called *sections*. Two sections are *equivalent* if two arcs in the division of the part of the polygon boundary lying between these sections contain the same number of vertices. It is obvious that each section is equivalent to either a side or a small diagonal. Moreover, in an n-gon, any diagonal is equivalent to a side if n is odd and to either a pair of sides or a pair of small diagonals if n is even.

We consider a characteristically closed $2n$-gon P. The small diagonals of P are characteristics since for a supporting line l at a vertex τ the nearest parallel characteristic passes through the small diagonal. Thus, those and only those diagonals that are equivalent to small diagonals are characteristics. The vertex set F can be divided into two subsets F_1 and F_2 of n elements in such a way that between any two neighboring points of one set there is a point of the other set. Then the diagonals with endpoints in the same set and only they are characteristics. Consequently if an n-gon Q_j is generated by vertices $\tau \in F_j$, then all its sides and diagonals are characteristics. Such a polygon Q is called *characteristic*. It is obvious that any two equivalent sections of such a polygon are parallel. It is clear that for $n = 3$ any triangle with characteristic sides is characteristic, whereas for $n = 4$ we have a parallelogram with characteristic sides and diagonals.

The following assertion yields a simple description of characteristic polygons.

Theorem 2. *A characteristic n-gon is affine equivalent to a regular one, i.e., is the image of a regular n-gon under an affine transformation of the plane.*

Proof. It suffices to consider the case $n > 4$. We first show that all the vertices of a characteristic n-gon Q lie on some ellipse. Since each convex pentagon can be uniquely inscribed into an ellipse, it suffices to prove that any six sequential vertices lie on some ellipse. It is convenient to denote by $cbaa'b'c'$ these vertices so that the points $a'b'c'$ are successively located, starting with the first of them, while moving along ∂Q in the positive direction, whereas the points abc move in the opposite direction. Then the segments bb' and cc' are parallel to the side aa'. Applying a suitable affine transformation, if necessary, we can assume that the trapezoid $abb'a'$ is isosceles.

It is obvious that the section $a'b$ is parallel to $b'c$, whereas the section ab' is parallel to $b'c$. Consequently, $bmb'n$ is a parallelogram, where m is the intersection point of ab' and ba', whereas n is the intersection point of cb' and bc'. But the trapezoid $abb'a'$ is isosceles and, consequently, $bm = mb'$, i.e., this parallelogram is a rhombus. But then $nb = nb'$ and the trapezoid $bb'cc'$ is isosceles, so that the line mn is the symmetry axis of the hexagon $cbaa'b'c'$. Therefore, the intersection points d_1 and d_2 of the opposite sides and, respectively, bc, $a'b'$ and ab, $b'c'$ are also symmetric. Hence the line d_1d_2 is parallel to the sides aa' and cc' constituting the third pair.

Thus, the points d_1 and d_2 and the ideal point corresponding to the last pair of sides belong to the same line. By the Pascal theorem [4], it is possible to inscribe an ellipse into the hexagon under consideration, which is required.

Thus, the polygon Q can be inscribed into an ellipse which can be transformed to a circle by an affine transformation, as we will assume below. It suffices to verify that all n arcs have the same length. Since $n > 4$, we can consider only two disjoint arcs with endpoints a, b and a', b'. The enumeration is taken in such a way that the points b and b' lie on the same side of the section aa'. But, in this case, the section aa' is parallel to bb' and, consequently, the arcs under consideration have the same length, which completes the proof. $\square$

The question arises under what conditions on the parameters ν_k defining the characteristics (2) there exist characteristic n-gons.

Lemma 1. *Let the characteristics* (2) *are enumerated in the order of counter-clockwise rotations. Then for $n > 3$ a characteristic n-gon exists if and only if*

$$\nu_k \neq \frac{\nu_1 \sin\theta \cos(k-1)\theta + (q_2\nu_2 - \nu_1 \sin\theta)\sin(k-1)\theta}{\sin\theta\cos(k-1)\theta - (\cos\theta - q_2)\sin(k-1)\theta}, \quad k = 4, \dots, n, \tag{3}$$

where

$$\theta = \frac{\pi}{n}, \quad q_2 = \frac{\nu_1 - \nu_3}{2(\nu_2 - \nu_3)\cos\theta}.$$

Proof. We introduce the directed vectors $e_k = (-1, \nu_k) \in \mathbb{R}^2$ of the lines l_k and set

$$e_k^0 = (\cos(k-1)\theta, \sin(k-1)\theta), \quad 1 \leqslant k \leqslant n,$$

By Theorem 2, the assertion of the lemma is equivalent to the existence of a nonsingular matrix $a \in \mathbb{R}^{2\times 2}$ such that the vectors ae_k^0 are proportional to e_k for all $k = 1, \dots, n$ (with a nonzero proportionality coefficient). For such a matrix we take

$$a = \begin{pmatrix} -\sin\theta & \cos\theta - q_2 \\ \nu_1 \sin\theta & q_2\nu_2 - \nu_1\cos\theta \end{pmatrix}$$

and, in addition to q_2, introduce the nonzero coefficients $q_1 = 1$ and $q_3 = (\nu_1 - \nu_2)/(\nu_2 - \nu_3)$. A direct verification shows that for these parameters we have $ae_k^0 = (\sin\theta)q_k e_k$, $k = 1, 2, 3$. It remains to note that similar identities for $k = 4, \dots, n$ with some nonzero q_k hold if and only if the condition (3) is satisfied. $\square$

We illustrate the lemma by an example of four characteristics, $n = 4$. In this case, $\theta = \pi/4$, $\sqrt{2}q_2 = (\nu_1 - \nu_3)/(\nu_2 - \nu_3)$, and the condition (3) becomes

$$\frac{\nu_1\nu_2 + \nu_2\nu_3 - 2\nu_1\nu_3}{2\nu_2 - \nu_1 - \nu_3} = \nu_4. \tag{4}$$

We consider orthogonal lines l_1 and l_3 such that $\nu_1\nu_3 = -1$. Then the characteristic 4-gon with sides parallel to l_1 and l_3 is a rectangle whose diagonals are parallel to l_2 and l_4. In particular, they form equal angles with l_1. Let l_k form the angle θ_k with the x_1-axis so that $\nu_k = \tan\theta_k$. Then $\theta_2 - \theta_1$ and $\theta_1 - \theta_4$ coincide up to a summand multiple to π. Therefore, their tangents are equal, which leads to the equality

$$\frac{\nu_2 - \nu_1}{1 + \nu_1\nu_2} = \frac{\nu_1 - \nu_4}{1 + \nu_1\nu_4} \quad \text{or} \quad \nu_4 = \frac{2\nu_1 + \nu_1^2\nu_2 - \nu_2}{2\nu_1\nu_2 + 1 - \nu_1^2}.$$

This equality coincides with (4) with $\nu_3 = -1/\nu_1$.

As was already noted, small diagonals of a characteristically closed $2n$-gon P form two characteristic n-gons Q_1 and Q_2. We note that for odd n their sides run over all the characteristics. For even n the sides of Q_1 run over the group of $n/2$ characteristics, whereas the diagonals of Q_1 run over the group of the remaining characteristics. Relative to Q_2 these groups exchange. In such a case, we say that these polygons are conjugate. For example, in the case of a characteristic rectangle, the conjugate of a tetragon is a rhombus. For $n = 3$ the conjugate of a triangle is a triangle of the same type.

If a similar conjugacy property is valid for two regular n-gons Q_1^0 and Q_2^0 inscribed into the same circle, then one of these polygons is obtained from the other by rotation by the angle π/n. Moreover, in the case of odd n, they are also centrally symmetric. In the general case, Q_2^0 is obtained from Q_1^0 by successively applying the rotation by the angle π/n about the center (or the central symmetry in the case of odd n) and then homothety from this center with positive coefficient and translation.

Thus, by Theorem 2, the characteristic n-gons Q_1 and Q_2 are affine conjugate, i.e., they are the images of two mutually conjugate regular polygons Q_1^0 and Q_2^0 under some affine transformation. Owing to this fact, we can explicitly construct the characteristic polygon conjugate to a given one. Since the central symmetry operation is preserved under an affine transformation, for odd n the affine-conjugate n-gon Q_2 is obtained from Q_1 by successively applying homothety with negative coefficient and translation. This property is also generalized to characteristic triangles in the case $n = 3$.

Thus, one can construct a characteristically closed $2n$-gon$P$ as follows. We choose two conjugate characteristic n-gons Q_1 and Q_2 such that the vertices of Q_1 lie outside Q_2 and conversely. Then their common vertices form a $2n$-gon P with the required properties.

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