CONDENSED-STATE PHYSICS

FORCED CAPILLARY WAVES IN THE POOL COVERED WITH A THIN FILM

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Forced capillary waves are investigated in the pool of finite dimensions within the framework of linear dynamics of an incompressible liquid. Analytical expression for the velocity potential of the liquid in such waves is derived. It is demonstrated that amplitude poles of the forced capillary waves determine the frequency spectrum of free capillary waves.

Many works are devoted to theoretical investigations of waves of various types in liquids (for example, see [1, 2]). Capillary waves play the important role in various technical and geophysical applications. An active interest has been expressed in a study of the effects arising in the capillary waves with allowance for viscoelastic liquid properties [3], interaction of a dielectric liquid layer with a charged layer [4], and nonlinear dynamics of capillary waves in a viscous liquid jet [5].

The present work studies the problem of forced capillary wave propagation in a tank (pool) of finite dimensions. Special attention is given to the general problem of coupling of free oscillation frequencies with amplitudes of forced oscillations.

Let us consider a liquid in the pool of finite depth h the surface of which is covered with a thin film from an elastic material insoluble in the liquid. The film thickness can be neglected in comparison with the pool depth. We consider the pool shape to be cylindrical (with an arbitrary rather than circular contour C). We consider also that the surface oscillations of the film are small and the liquid remains always in contact with the film due to the forces of surface tension.

Let us direct the Oz axis along the cylinder axis toward the pool depth and assume that the pool surface covered with the film is located on the boundary $z = 0$ in the xOy plane. We designate by M points from region $B = \{(M_2, z): M_2 \in D \subset R^2, 0 < z < h\} \subset R^3$ occupied by the liquid and by M_2 points of the surface (film) from the region D of a plane figure of any arbitrary shape with smooth boundary C.

Let us consider that the liquid is ideal and incompressible. Then the velocity potential of the liquid $\varphi(M, t)$ at point $M \in B$ and time t is determined from the Laplace equation:

$$\Delta\varphi = 0, \tag{1}$$

where Δ is the Laplace operator.

The potential must satisfy the conditions of flow along the motionless pool walls, corresponding to the equality to zero of the normal velocity components:

$$\left.\frac{\partial\varphi}{\partial n}\right|_C = 0,\ M \in C, \tag{2}$$

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$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=h} = 0\,,\ M_2 \in D. \qquad (3)$$

On the film surface, the potential satisfies the specific boundary condition.

Let this surface have the shape $z = \zeta(M_2, t)$, $M_2 \in D$. Considering that its deviations from the equilibrium position are small, we can write the Laplace formula for pressure p in the liquid and variable external pressure on the film p_{ex} in the plane $z = 0$ [1]:

$$p - p_{ex} = -\sigma\Delta_2\zeta, \qquad (4)$$

where σ is the coefficient of surface tension at the liquid-film interface, Δ_2 is the two-dimensional Laplace operator with respect to the coordinates $M_2 \in D$.

After differentiation of condition (4) with respect to t, taking advantage of the well-known relations

$$p = -\rho g\zeta + \rho\frac{\partial\varphi}{\partial t}\,,\ \frac{\partial\zeta}{\partial t} = \frac{\partial\varphi}{\partial z}\,, \qquad (5)$$

where g is the free fall acceleration, we can reduce condition (4) to the form

$$\rho\frac{\partial^2\varphi}{\partial t^2} - \rho g\frac{\partial\varphi}{\partial z} - \frac{\partial p_{ex}}{\partial t} = -\sigma\frac{\partial}{\partial z}\Delta_2\varphi\,. \qquad (6)$$

The external periodic force applied to the film produces pressure depending on time by a harmonic law:

$$p_{ex}(M_2, t) = p_{ex}(M_2)e^{-i\omega t}, \qquad (7)$$

where ω is the frequency of the external force and $p_{ex}(M_2)$ is the distribution of the external pressure over the film surface. The force is considered to be small so that it does not rupture the film covering the pool. External pressure (7) caused by forces of surface tension engenders forced capillary waves in the liquid.

By virtue of the fact that the external pressure varies by harmonic law (7), we can seek the velocity potential of the liquid in the forced capillary wave in the form

$$\varphi(M, t) = \Phi(M)e^{-i\omega t}. \qquad (8)$$

With allowance for Eqs. (7) and (8) at $z = 0$, condition (6) assumes the form

$$\rho\omega^2\Phi + \rho g\frac{\partial\Phi}{\partial z} - \sigma\frac{\partial}{\partial z}\Delta_2\Phi = i\omega p_{ex}(M_2)\,. \qquad (9)$$

Thus, collecting Eqs. (1)–(3) and (9), we obtain that the examined model can mathematically be formulated as the following boundary problem for the velocity potential of the liquid in the forced capillary wave Φ:

$$\Delta\Phi(M) = 0\,,\ M \in B, \qquad (10)$$

$$\left.\frac{\partial\Phi}{\partial n}\right|_{C} = 0\,,\ M_2 \in C,\ 0 < z < h, \qquad (11)$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=h} = 0\,,\ M_2 \in D, \qquad (12)$$

$$\left.\left(\rho\omega^2\Phi+\rho g\frac{\partial\Phi}{\partial z}-\sigma\frac{\partial}{\partial z}\Delta_2\Phi\right)\right|_{z=0}=i\omega p_{\text{ex}}(M_2)\,,M_2\in D. \tag{13}$$

A solution of boundary problem (10)–(13) is searched as an expansion of the potential in a generalized Fourier series of complete system of functions $\{\psi_n\}$ [6]:

$$\Phi(M)=\sum_n u_n(z)\psi_n(M_2)\,, \tag{14}$$

where $u_n(z)$ are the Fourier coefficients.

For the complete system of functions $\{\psi_n\}$, we choose the eigenfunctions of the following Storm–Liouville problem (SLP):

$$\Delta_2\psi_n(M_2)+\lambda_n\psi_n(M_2)=0\,,\ M_2\in D\,, \tag{15}$$

$$\left.\frac{\partial\psi_n}{\partial n}\right|_C=0\,,\ M_2\in C, \tag{16}$$

where λ_n are the eigenvalues of SLP (15)–(16). The eigenfunctions of SLP (15)–(16) are called the eigenfunctions of a membrane [6]; in our context, we call them the eigenfunctions of the film.

Since problem (15)–(16) is two-dimensional, the subscript n designates the double subscript, and Fourier series (14) is also considered doubled. By its construction, the solution in the form of series (14) satisfies boundary condition (11) on the pool contour.

Formal substitution of series (14) into equation (10) with boundary conditions (12)–(13) yields the following boundary problem for finding the Fourier coefficients $u_n(z)$:

$$u''(z)-\lambda_n u(z)=0\,,\ 0<z<h, \tag{17}$$

$$u_n'(h)=0\,, \tag{18}$$

$$\rho\omega^2 u_n(0)+(\rho g+\sigma\lambda_n)u_n'(0)=i\omega\tilde{p}_n\,, \tag{19}$$

where primes denote derivatives with respect to z, and the Fourier coefficients characterizing the force distribution over the film are determined by the expression

$$\tilde{p}_n=\frac{1}{\|\psi_n\|^2}\iint_D p_{ex}(M_2)\psi_n(M_2)dS_M\,, \tag{20}$$

where the norm of the eigenfunctions is

$$\|\psi_n\|^2=\iint_D|\psi_n(M_2)|^2\,dS_M \tag{21}$$

and dS_M is the element of the film surface.

Thus, the Fourier coefficients $u_n(z)$ are found by solving boundary problem (17)–(19) for the second-order ordinary differential equation.

A solution of boundary problem (17)–(19) for $\tilde{p}_n\neq 0$ does exist, is unique, and can be written as follows:

$$u_n(z) = A_n(\omega)\cosh\sqrt{\lambda_n}(h-z), \quad (22)$$

where the amplitude of oscillations is determined by the expression

$$A_n(\omega) = \frac{i\omega\tilde{p}_n}{\Delta_n(\omega)}, \quad (23)$$

$$\Delta_n(\omega) = \rho\omega^2\cosh\sqrt{\lambda_n}h - \sqrt{\lambda_n}(\rho g + \sigma\lambda_n)\sinh\sqrt{\lambda_n}h. \quad (24)$$

Having substituted Eq. (22) into Eq. (14), we obtain the solution of boundary problem (10)–(13):

$$\Phi(M) = \sum_n A_n(\omega)\psi_n(M_2)\cosh\sqrt{\lambda_n}(h-z), \quad (25)$$

determining the velocity potential of the liquid in the forced capillary wave by formula (8).

It should be noted that potential (25) can have another form, namely,

$$\Phi(M) = \iint_D p_{\text{ex}}(N_2)G(M_2, N_2, z)dS_N, \quad (26)$$

where dS_N means that integration is carried out over points N_2 of the film surface, and the Green's function has been introduced for boundary problem (10)–(13):

$$G(M_2, N_2, z) = \sum_n \frac{\psi_n(M_2)\psi_n(N_2)}{\|\psi_n\|^2}\frac{i\omega\cosh\sqrt{\lambda_n}(h-z)}{\Delta_n(\omega)}. \quad (27)$$

Poles of oscillation amplitudes (23), that is, zeros of function (24) specify the discrete spectrum of frequencies

$$\omega_n^2 = \sqrt{\lambda_n}\left(g + \lambda_n\frac{\sigma}{\rho}\right)\tanh\sqrt{\lambda_n}h. \quad (28)$$

Spectrum (28) comprises frequencies of free (or natural) capillary waves (that is, waves without external force, when $p_{\text{ex}} \equiv 0$) in the pool of finite dimensions specified by Eq. (22) in which the amplitude A is an arbitrary constant.

For short capillary waves for which $\lambda_n h^2 >> 1$, the discrete spectrum

$$\omega_n^2 = \sqrt{\lambda_n}\left(g + \lambda_n\frac{\sigma}{\rho}\right) \quad (29)$$

is obtained from Eq. (28), and for long capillary waves for which $\lambda_n h^2 << 1$, the discrete spectrum assumes the form

$$\omega_n^2 = h\lambda_n\left(g + \lambda_n\frac{\sigma}{\rho}\right). \quad (30)$$

For an infinite pool of finite depth $\lambda_n \to k^2$, and the well-known dispersion law of capillary waves is derived from Eq. (28) [1, 2].

Let us consider now pools having concrete shapes. First we consider a rectangular pool with sides a and b; then $D = \{(x, y)\colon 0 < x < a,\ 0 < y < b\}$. For this case, the eigenfunctions and eigenvalues of SLP (15)–(16) are well known

[6]. Then the frequency spectrum of free capillary waves in the rectangular pool assumes the form (further the double subscript is written in an explicit form)

$$\omega_{nm}^2 = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\left\{g + \pi^2\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]\frac{\sigma}{\rho}\right\}\tanh\left(\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,\pi h\right), \tag{31}$$

$n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots$.

For a pool shaped as a circular cylinder with radius R, the surface $D = \{(r, \theta)\colon 0 < r < R,\ 0 < \theta < 2\pi\} \subset R^2$. Using the well-known eigenfunctions and eigenvalues of SLP (15)–(16) in this circle D [6], from Eq. (28) we obtain the spectrum of frequencies of free capillary waves in the circular pool:

$$\omega_{nm}^2 = \frac{\mu_m^{(n)}}{R}\left\{g + \left(\frac{\mu_m^{(n)}}{R}\right)^2\frac{\sigma}{\rho}\right\}\tanh\left(\frac{\mu_m^{(n)}}{R}h\right), \tag{32}$$

$n = 0, 1, 2, \ldots, m = 1, 2, \ldots$, where $\mu_m^{(n)}$ are zeros of the mth-order of the derivative of the nth order Bessel functions, that is, roots of the equation $J_n'(\mu_m^{(n)}) = 0$, where J_n are the nth order Bessel functions.

In conclusion it should be noted that if the frequency of the external force coincides with one of the frequencies of discrete spectrum (28), the amplitude of oscillations (23) will be infinite. This means that a resonance occurs. In actual situation, the energy is always dissipated. Consideration of any dissipation process, for example, viscosity of the liquid in the model would prevent infinite amplitudes of forced oscillations at certain frequencies. Nevertheless, as is well known from the theory of oscillations [7], the maximum amplitude in the model that takes into account dissipation processes is reached at the frequency close to the resonant frequency at which the amplitude of oscillations in the model disregarding dissipation processes tends to infinity. Therefore, expression (28) can be used to estimate approximately the resonant frequencies at which the amplitude of true oscillations is maximum.

In this work, the analytical expression has been derived for the potential of the liquid in the forced capillary wave in the pool of arbitrary shape with finite dimensions. Of special significance is the fact that the amplitude poles of the forced capillary wave are observed at frequencies of free capillary waves. This is analogous to the fact that amplitude poles of an elastic wave and quasiparticle scattering on crystal defects determine the spectra of frequencies of their natural oscillations [8–11].

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