Classification of equilibrium states and hydrodynamics of quantum Fermi-liquid mixtures with the vector order parameter

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Abstract

The classification of possible superfluid phases of the Fermi-liquids mixture is constructed. Thermodynamics is constructed and hydrodynamic equations of superfluid Fermi-liquid mixture with vector order parameter are derived. Thermodynamic values connected with spontaneous breaking of symmetry to phase transformations and spin rotations are presented in terms of the order parameter. A new entrainment effect connected with mutual influence of magnetic and orbital degrees of freedom is predicted. Spectra of collective excitations of the considered quantum Fermi-liquid mixture are obtained. Differences of this state from the states with scalar and tensor order parameters are considered. The cases of Galilean and relativistic invariance are considered.

Keywords: Superfluidity; Fermi-liquid; Solutions; Vector order parameter; Entrainment effect

1. Introduction

Theoretical researches in the field of superfluidity and superconductivity till now are an actual problem. Within the framework of these studies special interest is connected with a dynamical problem and the interaction of quantum macroscopical condensates

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and elementary excitations in non-relativistic and relativistic quantum systems. Interest in a superfluidity problem at given stage is caused first of all by fast progress and improvement of experimental and computational research methods and observation of dynamic matter properties on both a microscopical level, and astrophysical scales. So new kinds of superconductors [1–11] (including superconductors with heavy fermions (see, e.g. Refs. [5,8]), superconductors with d- and f-wave pairing [10,11] superconductors, and also organic superconductors [6,9]), mixtures of quantum gases in magnetic traps [12], and nuclear matter superfluidity in astrophysical objects [13–15] have been discovered.

Two chiral vacuum condensates can coexist in physical QCD vacuum [16,17]. The similar coexistence and interaction of two relativistic superfluid phases with different OP can be realized at a certain evolution stage [13,14] in astrophysical objects such as superfluidity of baryon matter component in neutron stars. So in the framework of BCS theory in papers [18–26] equilibrium properties for different kinds of baryon matter were considered and analyzed. Moreover, the assumption about the existence of superfluidity in the hyperon substance and quark–gluon plasma is being made.

Multi-zoned superconductors, mixtures of quantum Fermi-gases in magnetic traps, and superfluid nuclear matter by the nature there are Fermi-liquid solutions. Therefore, construction of their non-equilibrium dynamics represents certain interest. To derive hydrodynamics equations and obtain thermodynamical relations one can use methods, developed under the description of Bose- and Fermi-liquids solutions. Refs. [27–30] is concerned with the investigations of Bose- and Fermi-liquids solutions only. In these papers, the states described by scalar OP and equilibrium dynamics are considered.

In this paper we consider equilibrium states and non-equilibrium dynamics of superfluid states of the mixture of two Fermi-liquids characterized by vector OP. Here, we carry out a relativization of obtained equations and relations. These expressions can be used in studying superfluidity in astrophysical objects. The problem of relativization of Bose- and Fermi-liquids mixtures was solved in Refs. [28,31,32].

The Fermi-liquid approach (FLA) based on Landau's theory and used for the description of superfluid liquids with singlet [33], and triplet pairings [34,35], magnetic systems with spontaneous broken symmetry [36], and superfluidity of electron–positron plasma [37] is an effective instrument of construction of quantum liquids theory (The detailed information about the FLA is reflected in reviews [38,39] too). In the framework of this approach in Refs. [40–45] the possible phase transitions with scalar and tensor OP were investigated, the spectra of quasi-particles and temperature of phase transitions in superfluid nuclear matter were found.

In this work we investigate the equilibrium state and formulate the symmetry properties of the Fermi-liquid mixture with vector OP. Additional thermodynamical parameters are introduced on the base of vector OP, the second law of thermodynamics is obtained and the flux densities of additive motion integrals in terms of density of thermodynamical potential in local-equilibrium state are found. The classification of possible equilibrium states of a system with vector OP is carried out. The derivation of the equations of ideal hydrodynamics without using concrete dynamic symmetry of the energy functional is given and the spectra of proper excitations are obtained. The comparison of the obtained hydrodynamics equations with well-known equations of hydrodynamics in superfluid mixtures described by scalar and tensor OPs is performed. It is shown that the entrainment effect which connected with the influence of superfluid motion of one component on motion of another component is appeared in the considered mixtures. Moreover, the new entrainment effect happens as the mutual influence of spin degrees of freedom on orbital dynamics takes place. The influence of these effects on the spectra of collective excitations in the considered system is investigated.

2. Order parameters in quantum Fermi-liquid mixtures. Classification of possible equilibrium states

In order to construct a thermodynamics and derive hydrodynamic equations of quantum systems with spontaneously broken symmetry the total set of parameters of reduced description would contain except densities of additive motion integrals (AMI) also additional thermodynamical values connected with broken symmetry and determined in terms of the OP. Therefore, the study of the influence of possible OP structure in each particular case of symmetry violation in the equilibrium state on the thermodynamics, the equations of hydrodynamics and the collective spectra excitations is necessary.

We shall consider a mixture of two Fermi-liquids. Multi-zone superconductors, mixtures of quantum gases in magnetic gaps, and superfluid nuclear matter are systems of this kind. In these systems a pairing of objects, both one kind and different kinds (for example, protons and neutrons), is possible. The new types of OP in comparison with one-component superfluid are possible in this case. Besides "*scalar*"

$$\hat{\Delta}_a(\mathbf{x}) = \frac{\mathrm{i}}{2}\, \psi_{a\mu}(\mathbf{x})\,(\sigma_2)_{\mu\mu'}\, \psi_{a\mu'}(\mathbf{x}), \quad a = 1,2\,. \tag{2.1}$$

and "*tensor*" OP

$$\hat{\Delta}_{a,\alpha k}(\mathbf{x}) = (\nabla_k\, \psi_{a\mu}(\mathbf{x}))\,(\sigma_2\sigma_\alpha)_{\mu\mu'}\, \psi_{a\mu'}(\mathbf{x}) - \psi_{a\mu}(\mathbf{x})\,(\sigma_2\sigma_\alpha)_{\mu\mu'}\, \nabla_k\, \psi_{a\mu'}(\mathbf{x})\,, \tag{2.2}$$

describing each component separately the "*vector*" OP can be introduced also as

$$\hat{\Delta}_\alpha(\mathbf{x}) = \frac{\mathrm{i}}{2}\, \psi_{a\mu}(\mathbf{x})\,(\tau_2)_{aa'}\,(\sigma_2\sigma_\alpha)_{\mu\mu'}\, \psi_{a'\mu'}(\mathbf{x}) \tag{2.3}$$

which is built from field operators of both components of mixture. Here σ_α, τ_α are spin and isospin Pauli matrices, ψ, ψ^+ are fermion field operators. The vector OP can be expressed in terms of gradients of field operators. It describes spatial anisotropic liquid crystal and superfluid states which are isotropic in spin space.

OP (2.3) describes superfluid states which are anisotropic in spin and isotropic in configurational spaces. In this paper we consider the states of superfluid Fermi-liquid mixtures, described only by OP type (2.3), and we discuss the differences in thermodynamics and hydrodynamical equations arising from cases of Fermi-liquid mixtures with scalar (2.1) and tensor (2.2) OP.

Using the representation of the AMI operators in terms of creation-annihilation Fermi-operators and their commutation relations, it is easy to see that the following relations for OP operator (2.3) are correct:

$$[\hat{S}_\alpha, \hat{\Delta}_\beta(\mathbf{x})] = \mathrm{i}\varepsilon_{\alpha\beta\gamma}\hat{\Delta}_\gamma(\mathbf{x}), \quad [\hat{N}_1, \hat{\Delta}_\alpha(\mathbf{x})] = [\hat{N}_2, \hat{\Delta}_\alpha(\mathbf{x})] = -\hat{\Delta}_\alpha(\mathbf{x}) ,$$
$$[\hat{\mathscr{P}}_k, \hat{\Delta}_\alpha(\mathbf{x})] = -\mathrm{i}\nabla_k\hat{\Delta}_\alpha(\mathbf{x}) . \tag{2.4}$$

Formulas (2.4) permit to build the classification of possible equilibrium states described by vector OP.

Let us consider from the first general principles a classification without concretization of a type of order parameter. The description of condensed matter with spontaneous symmetry violations is essentially connected with OP representation. The OPs $\hat{\Delta}_A(\mathbf{x})$ (here A are all tensor indexes) in secondary quantization representation are constructed from field creation and annihilation operators. And now we shall formulate the commutation properties of the operators $\hat{\Delta}_A(\mathbf{x})$. The condition of translational invariancy has a form

$$\mathrm{i}[\hat{\mathscr{P}}_k, \hat{\Delta}_A(\mathbf{x})] = -\nabla_k\hat{\Delta}_A(\mathbf{x}) . \tag{2.5}$$

For the generator of phase transformations group for single-component matter is a number of particles operator $\hat{N}$. The following relation is correct:

$$[\hat{N}, \hat{\Delta}_A(\mathbf{x})] = -\mathrm{g}\hat{\Delta}_A(\mathbf{x}) . \tag{2.6}$$

Included in (2.6) the constants g depend on the tensor dimensions of the OP operator. In multi-component systems the number of phase transformation generators is larger than $1(\hat{N}_a,\ a = 1,\ldots,n)$. In this case instead of (2.6) it is necessary to consider

$$[\hat{N}_a, \hat{\Delta}_A(\mathbf{x})] = -\mathrm{g}_a\hat{\Delta}_A(\mathbf{x}) . \tag{2.7}$$

For transformations, connected with group of internal symmetries with generators $\hat{S}_\alpha$, the operators $\hat{\Delta}_A(\mathbf{x})$ change on representations of this group, this leads to the relation:

$$[\hat{S}_\alpha, \hat{\Delta}_A(\mathbf{x})] = -\mathrm{g}_{\alpha,AB}\hat{\Delta}_B(\mathbf{x}) . \tag{2.8}$$

Where $\mathrm{g}_{\alpha,AB}$ are some constants. The generators of group of internal symmetries $\hat{S}_\alpha$ satisfy, as is known, the relation $[\hat{S}_\alpha, \hat{S}_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma}\hat{S}_\gamma$, where an antisymmetric tensor $\varepsilon_{\alpha\beta\gamma}$ plays the role of structure constants. From formulas (2.8), using the Jacobi identity for the operators $\hat{S}_\alpha$ and $\hat{\Delta}_A(\mathbf{x})$, we obtain the following relation:

$$[\hat{\mathrm{g}}_\alpha, \hat{\mathrm{g}}_\beta] = -\varepsilon_{\alpha\beta\gamma}\hat{\mathrm{g}}_\gamma , \tag{2.9}$$

where $(\hat{\mathrm{g}}_\alpha)_{AB} = \mathrm{g}_{\alpha,AB}$.

At transformations connected with group of spatial rotations the OP operator will be transformed under the law

$$[\hat{\mathscr{L}}_k, \hat{\Delta}_A(\mathbf{x})] = -\mathrm{g}_{k,AB}\hat{\Delta}_B(\mathbf{x}) - \varepsilon_{kij}x_j\nabla_i\hat{\Delta}_A(\mathbf{x}) . \tag{2.10}$$

Thus for $(\mathrm{g}_k)_{AB} = \mathrm{g}_{k,AB}$, one takes the relation

$$[\hat{\mathrm{g}}_i, \hat{\mathrm{g}}_j] = -\varepsilon_{ijk}\hat{\mathrm{g}}_k . \tag{2.11}$$

From the phenomenological theory it is known that for the adequate description of thermodynamics and non-equilibrium processes in condensed matter with the perturbed symmetry, generally speaking, it is necessary to introduce into the theory, new thermodynamic parameters irrelevant with respect to the conservation laws, and conditioned by the physical nature of a thermodynamic phase. In a case of normal condensed matter the thermodynamic parameters are determined only in density of AMIs. Let us formulate the symmetry properties of an equilibrium state of the degenerate condensed matter we also shall introduce on this basis additional thermodynamic parameters. In the beginning let us consider translational-invariant subgroups of an unperturbed symmetry H of the full symmetry group G. The translational invariance means that the equilibrium statistical operator leads to a relation of symmetry

$$[\hat{f}, \hat{\mathscr{P}}_k] = 0 \ . \tag{2.12}$$

The analysis of translational-invariant subgroups of an unperturbed symmetry of equilibrium state pursuant to Ref. [46] is feasible, outgoing from a relation

$$[\hat{f}, \, \hat{T}] = 0 \ , \tag{2.13}$$

where the operator of an unperturbed symmetry $\hat{T}$ in case of a single-component liquid represents a linear combination of the motion integrals:

$$\hat{T} \equiv a_i \hat{\mathscr{L}}'_i + b_\alpha \hat{S}'_\alpha + c \hat{N} \equiv \hat{T}(\xi) \tag{2.14}$$

with some real parameters $(a_i, b_\alpha, c \equiv \xi)$. For a multi-component solution the operator $\hat{T}$ has the form

$$\hat{T} \equiv a_i \hat{\mathscr{L}}'_i + b_\alpha \hat{S}'_\alpha + c_a \hat{N}_a \equiv \hat{T}(\xi), \quad (\xi \equiv a_i, b_\alpha, c_a) \ . \tag{2.15}$$

The unitary transformations $U(\xi) = \exp(\mathrm{i}\,\hat{T}(\xi))$ will lead to continuous subgroups of an unperturbed symmetry $U(\xi)U(\xi') = U(\xi''(\xi, \xi'))$ equilibrium states. From the equations

$$\mathrm{i}\,\mathrm{Sp}[\hat{f}, \, \hat{T}]\hat{\Delta}_A(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{\mathscr{P}}_k]\hat{\Delta}_A(\mathbf{x}) = 0 \ ,$$

allowing relations (2.5)–(2.7), (2.10) and definition (2.14) or (2.15) we obtain equations in the single-component case

$$\begin{aligned} &a_i \left(\mathrm{g}_{iAB} \Delta_B + \varepsilon_{ikl} Y_k \frac{\partial \Delta_A}{\partial Y_l} \right) + b_\alpha \left(\mathrm{g}_{\alpha AB} \Delta_B + \varepsilon_{\alpha\beta\gamma} Y_\beta \frac{\partial \Delta_A}{\partial Y_\gamma} \right) + \mathrm{i} c \mathrm{g} \Delta_A = 0, \\ &\nabla_k \Delta_A = 0 \end{aligned} \tag{2.16}$$

and in the multi-component case

$$\begin{aligned} &a_i \left(\mathrm{g}_{iAB} \Delta_B + \varepsilon_{ikl} Y_k \frac{\partial \Delta_A}{\partial Y_l} \right) + b_\alpha \left(\mathrm{g}_{\alpha AB} \Delta_B + \varepsilon_{\alpha\beta\gamma} Y_\beta \frac{\partial \Delta_A}{\partial Y_\gamma} \right) + \mathrm{i} c_a \mathrm{g}_a \Delta_A = 0 \ , \\ &\nabla_k \Delta_A = 0 \ . \end{aligned} \tag{2.17}$$

They establish definite connections on the parameters ξ. For a simplicity let us consider a case, when $Y_k = Y_\alpha = 0$. Thus for single-component Fermi liquids

$$T_{AB} \Delta_B = 0, \quad T_{AB} \equiv a_i \mathrm{g}_{iAB} + b_\alpha \mathrm{g}_{\alpha AB} + \mathrm{i} c \mathrm{g} \delta_{AB} \ . \tag{2.18}$$

For multi-component mixture it is necessary to select T_{AB} by the way

$$T_{AB} \equiv a_i \mathrm{g}_{iAB} + b_\alpha \mathrm{g}_{\alpha AB} + \mathrm{i} c_a \mathrm{g}_a \delta_{AB} \ . \tag{2.19}$$

The existence condition of the non-trivial solution $\Delta_A \neq 0$ of set of linear equations (2.18) and (2.19) results in equation

$$\det|T_{AB}| = 0 \ , \tag{2.20}$$

superimposing limitations on acceptable values of parameters of the generator unperturbed symmetry ξ.

The classification of states described by scalar OP does not represent a special interest as one state exists in this case only. The classification of possible states described by tensor OP was considered in many papers (see, for example Refs. [47–50]). Now we shall consider equilibrium states not having the translational-invariance property (2.12). In condensed matter with the spontaneously perturbed symmetry, different physical ways of violation of translational invariance can be realized. A spatial symmetry for such equilibrium states is set by a relation

$$[\hat{f}, \hat{P}_k] = 0, \quad \hat{P}_k \equiv \hat{\mathscr{P}}_k - p_k \hat{N} - q_{k\alpha} \hat{S}_\alpha - t_{kj} \hat{\mathscr{L}}_j \tag{2.21}$$

for the single-component case and

$$\hat{P}_k \equiv \hat{\mathscr{P}}_k - p_{ak} \hat{N}_a - q_{k\alpha} \hat{S}_\alpha - t_{kj} \hat{\mathscr{L}}_j \tag{2.22}$$

for the multi-component. Here $p_k(p_{ak}), q_{k\alpha}, t_{kj}$ are some real parameters. The generator of a perturbed symmetry as the condition the operator of a momentum is

$$\hat{T} \equiv a_i \hat{\mathscr{L}}'_i + b_\alpha \hat{S}'_\alpha + c \hat{N} + h_i \hat{\mathscr{P}}_i \equiv \hat{T}(\xi), \quad (\xi \equiv a_i, b_\alpha, c, h_i) \tag{2.23}$$

for a single-component liquid and

$$\hat{T} \equiv a_i \hat{\mathscr{L}}'_i + b_\alpha \hat{S}'_\alpha + c_a \hat{N}_a + h_i \hat{\mathscr{P}}_i \equiv \hat{T}(\xi), \quad (\xi \equiv a_i, b_\alpha, c_a, h_i) \tag{2.24}$$

for the multi-component liquid.

In the presence of connections between parameters ξ and construction of relation Δ_A from coordinate it is necessary to take advantage of relation

$$\mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{T}]\hat{\Delta}_A(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{P}_k]\hat{\Delta}_A(\mathbf{x}) = 0 \ . \tag{2.25}$$

In the solution of a problem on classification for some kinds of condensed matters relation (2.25) is not enough. It is necessary to supplement it by relations like

$$\mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{\Sigma}_\alpha]\hat{\Delta}_A(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{L}_k]\hat{\Delta}_A(\mathbf{x}) = 0 \ . \tag{2.26}$$

In which we have introduced

$$\hat{\Sigma}_\alpha \equiv \hat{S}_\alpha - u_\alpha \hat{N} - v_{k\alpha} \hat{\mathscr{P}}_k - w_{j\alpha} \hat{\mathscr{L}}_j, \qquad \hat{L}_i \equiv \hat{\mathscr{L}}_i - y_i \hat{N} - z_{ik} \hat{\mathscr{P}}_k - k_{i\alpha} \hat{S}_\alpha \tag{2.27}$$

in the single-component case and

$$\hat{\Sigma}_\alpha \equiv \hat{S}_\alpha - u_{ak} \hat{N}_a - v_{k\alpha} \hat{\mathscr{P}}_k - w_{j\alpha} \hat{\mathscr{L}}_j, \qquad \hat{L}_i \equiv \hat{\mathscr{L}}_i - y_{ai} \hat{N}_a - z_{ik} \hat{\mathscr{P}}_k - k_{i\alpha} \hat{S}_\alpha \tag{2.28}$$

in the multi-component case. Here $u_{ak}, v_{k\alpha}, w_{j\alpha}, y_{ai}, z_{ik}, k_{i\alpha}$ are also material parameters. For an unambiguous definition of all parameters it is not enough to consider only relations (2.25) and (2.26). They should be supplemented still by conditions on parameters of an unperturbed and spatial symmetry, which are a consequence of the Jacobi identity for triples of operators $\hat{f}, \hat{\mathscr{A}}, \hat{\mathscr{B}}$ ($\hat{\mathscr{A}}, \hat{\mathscr{B}} \equiv \{ \hat{T}, \hat{P}_k, \hat{L}_k \}$):

$$\mathrm{i}\,\mathrm{Sp}[\hat{f}, [\hat{\mathscr{A}}, \hat{\mathscr{B}}]]\hat{\Delta}_A(\mathbf{x}) = 0 \; . \tag{2.29}$$

The method described above for the classification of translational-invariant states is applicable to the solution of the problem of superfluid Fermi-liquids depicted by a vector OP (2.3). For the convenience of consideration we shall introduce new variables in the operator of an unperturbed symmetry (2.15): $c_\pm = \frac{1}{2}(c_1 \pm c_2)$, $\hat{N}_\pm = (\hat{N}_1 \pm \hat{N}_2)$. Thus

$$\hat{T} \equiv a_i \hat{\mathscr{L}}'_i + b_\alpha \hat{S}'_\alpha + c_+ \hat{N}_+ + c_- \hat{N}_- \; . \tag{2.30}$$

Substituting in the formulas $\mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{T}]\hat{\Delta}_\alpha(\mathbf{x}) = 0$, $\mathrm{i}\,\mathrm{Sp}[\hat{f}, \hat{P}_k]\hat{\Delta}_\alpha(\mathbf{x}) = 0$, relation (2.30) and commutation relation for the vector OP operator (2.3), we obtain the equations for OP

$$(b_\alpha \varepsilon_{\alpha\beta\gamma} + 2\mathrm{i}c_+ \delta_{\beta\gamma})\Delta_\gamma(\mathbf{x}) = 0, \quad \nabla_i \Delta_\alpha(\mathbf{x}) = 0 \; .$$

From here it is visible, that $\Delta_\alpha(\mathbf{x}) = \Delta_\alpha \equiv \mathrm{const}$ and, therefore, it is possible to write

$$(b_\alpha \varepsilon_{\alpha\beta\gamma} + 2\mathrm{i}c_+ \delta_{\beta\gamma})\Delta_\gamma = 0 \; . \tag{2.31}$$

Eq. (2.31) has non-trivial solutions for OP only under the condition that the matrix determinant is zero

$$\det|b_\alpha \varepsilon_{\alpha\beta\gamma} + 2\mathrm{i}c_+ \delta_{\beta\gamma}| = 2\mathrm{i}c_+(b^2 - 4c_+^2) = 0 \; .$$

From this equation we shall find the solutions for a constant c_+:

$$c_+ = 0, \pm 1/2(b \neq 0); \qquad c_+ = b = 0 \; .$$

To these values c_+ correspond the following kinds of symmetry conditions:

$$[\hat{f}, \hat{\mathscr{L}}_i] = [\hat{f}, \hat{N}_-] = [\hat{f}, d_\alpha \hat{S}_\alpha] = 0, \qquad [\hat{f}, \hat{\mathscr{L}}_i] = [\hat{f}, \hat{N}_-] = 0 \; ,$$
$$[\hat{f}, \hat{\mathscr{L}}_i] = [\hat{f}, \hat{N}_-] = [\hat{f}, d_\alpha \hat{S}_\alpha \pm 1/2\,\hat{N}_+] = 0 \; . \tag{2.32}$$

We can construct the equilibrium OP as a linear combination of unitary vector triad $\mathrm{d}_\alpha^{(s)} \equiv d_\alpha, e_\alpha, f_\alpha$: $\Delta_\alpha = \sum_s A_s \mathrm{d}_\alpha^{(s)}$. Here the A_s are its amplitudes. Using properties of symmetry (2.32) and the definition of the equilibrium OP, let us find amplitudes for each case. After some calculations we obtain the form of OP corresponding to each case of a symmetry:

$$\Delta_\alpha = A_1 d_\alpha, \qquad \Delta_\alpha = A_2(e_\alpha \mp \mathrm{i} f_\alpha) \; ,$$
$$\Delta_\alpha = A_1 d_\alpha + A_2 e_\alpha + A_3 f_\alpha \; . \tag{2.33}$$

Now we shall consider the condition of translational-invariance. For our convenience let us rewrite the operators $\hat{P}_i$ and $\hat{T}$ in terms of the operators $\hat{N}_\pm$ assuming in the operator of a generalized momentum $q_{i\alpha} = t_{kj} = 0$. Then we have

$$\hat{P}_k \equiv \hat{\mathscr{P}}_k - p_{+,k}\hat{N}_+ - p_{-,k}\hat{N}_- ,$$
$$\hat{T} \equiv a_i \hat{\mathscr{L}}_i + b_\alpha \hat{S}_\alpha + c_+ \hat{N}_+ + c_- \hat{N}_- + h_i \hat{\mathscr{P}}_i \; . \tag{2.34}$$

Let us introduce the operator of the generalized orbital moment

$$\hat{L}_k \equiv \hat{\mathscr{L}}'_k - r_{+,k}\hat{N}_+ - r_{-,k}\hat{N}_- , \tag{2.35}$$

Here the operator $\hat{\mathscr{L}}'_k$ contains a differential part, and vector $r_{\pm,k}$ is a function of superfluid momenta. Using the relations

$$\mathrm{i}\,\mathrm{Sp}[\hat{f},\hat{T}]\hat{\Delta}_\alpha(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\hat{f},\hat{P}_k]\hat{\Delta}_\alpha(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\hat{f},\hat{L}_k]\hat{\Delta}_\alpha(\mathbf{x}) = 0 ,$$

we obtain the following equations:

$$(b_\alpha\varepsilon_{\alpha\beta\gamma} + 2\mathrm{i}\underline{c}_+\delta_{\beta\gamma})\Delta_\gamma(\mathbf{x}) = 0, \qquad \nabla_i\Delta_\alpha(\mathbf{x}) + 2p_{+,i}\Delta_\alpha(\mathbf{x}) = 0 ,$$
$$2r_{+,i}\Delta_\alpha(\mathbf{x}) = 0 . \tag{2.36}$$

Here, $\underline{c}_+ = c_+ + \mathbf{hp}_+$. From the second equation (2.36) we find the spatial form of the OP:

$$\Delta_\alpha(\mathbf{x}) = \underline{\Delta}_\alpha(0)\exp(2\mathrm{i}(\mathbf{p}_+\mathbf{x} + \theta_0)) . \tag{2.37}$$

From Eq. (2.36) after inserting (2.37) we obtain the relation between parameters $\underline{c}_+$ and b:

$$2\mathrm{i}\underline{c}_+(b^2 - 4\underline{c}_+^2) = 0 ,$$

whence $\underline{c}_+ = 0, \pm b/2$ $(b \neq 0)$ and $\underline{c}_+ = b = 0$. And at last from the third equation we obtain $r_{+,i} = 0$. In the following we assume that $\underline{\Delta}_\alpha(0) \neq 0$. Using relation (2.29), we find additional conditions for vectors **h** and **a**:

$$[\mathbf{ap}_+] = 0, \qquad [\mathbf{hp}_+] = 0 . \tag{2.38}$$

It is necessary to find a value $r_{-,i}$. We shall find this vector from a closed algebra condition for the operators $\hat{L}_i$. For which we shall take advantage of the relation

$$[\hat{L}_i, \hat{L}_k] = \mathrm{i}\varepsilon_{ikl}\hat{L}_l .$$

From which we obtain the equation

$$\tfrac{1}{2}\{[\mathbf{p}_a\partial r_{-,i}/\partial\mathbf{p}_a]_j - [\mathbf{p}_a\partial r_{-,j}/\partial\mathbf{p}_a]_i\} = \varepsilon_{ijk}r_{-,k} . \tag{2.39}$$

Except for the trivial solution of an Eq. (2.39) there is also the solution

$$r_{-,i} = |[\mathbf{p}_+\mathbf{p}_-]|^{-1}[\mathbf{p}_+\mathbf{p}_-]_i . \tag{2.40}$$

The case $r_{-,i} = 0$ corresponds to the state in which both vector and scalar OP are distinct from zero simultaneously. More interesting case in which only vector OP is different from zero takes place at a parameter $r_{-,i}$ described by formula (3.7). The properties of a symmetry of such state can be written in the form

$$[\hat{f},\hat{\mathscr{P}}_k - q_k n_\alpha\hat{S}'_\alpha - p_{ak}\hat{N}_a] = [\hat{f},\underline{d}_\alpha\hat{S}'_\alpha - m_s\hat{N}_+] = [\hat{f},\hat{\mathscr{L}}'_k + r_{-,k}\hat{N}_-] = 0 . \tag{2.41}$$

The form of the equilibrium value vector OP will be retrieved below.

3. Description of equilibrium state

According to general FLA the system energy can be represented as a functional of a statistical operator $E = E(\hat{f}) \equiv \int d^3x\, e(\mathbf{x}, \hat{f})$, where $e(\mathbf{x}, \hat{f})$ is the density of the system energy functional and $\hat{f}$ is the statistical operator. The energy operator $\hat{\varepsilon}(\hat{f})$ and quasi-particle energy density operator $\hat{\varepsilon}(\mathbf{x}, \hat{f})$ are determined by the equality:

$$\hat{\varepsilon}(\hat{f}) = \frac{\delta E}{\delta \hat{f}} = \int d^3x\ \hat{\varepsilon}(\mathbf{x}, \hat{f})\ .$$

The differential conservation laws for the densities of AMI ζ_r according to Ref. [33] can be expressed in the form:

$$\dot{\zeta}_r(\mathbf{x}, \hat{f}) = -\nabla_k \zeta_{rk}(\mathbf{x}, \hat{f})\ , \tag{3.1}$$

where $\zeta_r(\mathbf{x}, \hat{f}) = \mathrm{Sp}\, \hat{f}\, \hat{\zeta}_r(\mathbf{x})$ are densities of AMI $(r = 0, i, \alpha, a = 1, 2)$, $\hat{\zeta}_r(\mathbf{x}, \hat{f}) = \{\hat{\varepsilon}(\mathbf{x}, \hat{f}), \hat{\pi}_i(\mathbf{x}), \hat{s}_\alpha(\mathbf{x}), \hat{n}_a(\mathbf{x})\}$, and $\zeta_{rk}(\mathbf{x}, \hat{f}) = \mathrm{Sp}\, \hat{f}\, \hat{\zeta}_{rk}(\mathbf{x})$ are the fluxes, appropriate to the densities. The operators $\hat{\zeta}_{rk}(\mathbf{x})$ according to Ref. [33] can be presented in the terms of the operators $\hat{\zeta}_r(\mathbf{x})$:

$$\hat{w}_k(\mathbf{x}, \hat{f}) = \frac{\mathrm{i}}{2} \int d^3x'\, x'_k \int_0^1 d\rho [\,\hat{\varepsilon}(\mathbf{x} - (1-\rho)\mathbf{x}', \hat{f}),\, \hat{\varepsilon}_0(\mathbf{x} + \rho\mathbf{x}', \hat{f})]\ ,$$

$$\hat{\zeta}_{r'k}(\mathbf{x}, \hat{f}) = \mathrm{i} \int d^3x'\, x'_k \int_0^1 d\rho [\,\hat{\varepsilon}(\mathbf{x} - (1-\rho)\mathbf{x}', \hat{f}),\ \hat{\zeta}_{r'}(\mathbf{x} + \rho\mathbf{x}')]\ ,$$

$$\mathrm{Sp}\, \hat{f}\, \hat{t}_{kj}(\mathbf{x}, \hat{f}) = -e(\mathbf{x}, \hat{f})\delta_{kj} + \mathrm{i} \int d^3x'\, x'_k$$

$$\times \int_0^1 d\rho\, \mathrm{Sp}\, \hat{f} [\,\hat{\varepsilon}(\mathbf{x} - (1-\rho)\mathbf{x}', \hat{f}),\, \hat{\pi}_j(\mathbf{x} + \rho\mathbf{x}')]\ , \tag{3.2}$$

where $\hat{\zeta}_{r'k} = (\hat{i}_{ak}, \hat{j}_{\alpha k})$, $(r' = \alpha, a)$.

The equilibrium statistical operator of the normal state of a mixture of two Fermi-liquids $\hat{f}$ can be found from the requirement of maximal entropy at fixed values of AMIs γ_r. From the extremum condition for the thermodynamical potential Ω we obtain the self-consistent equation for equilibrium operator

$$\hat{f}_0 = [\exp(Y_r\, \hat{\gamma}_r(\hat{f})) + 1]^{-1}\ . \tag{3.3}$$

Here $Y_r = (Y_0, Y_\alpha, Y_k, Y_a)$ are thermodynamical forces, appropriate to the values $\gamma_r = (\varepsilon(\hat{f}), S_\alpha, \mathscr{P}_k, N_a)$. The equilibrium state of a normal mixture of Fermi-liquids is characterized by thermodynamical parameters Y_r only. The state, described by statistical operator (3.3), has symmetry properties

$$[\hat{f}_0, \hat{\mathscr{P}}_i] = [\hat{f}_0, \hat{S}'_\alpha] = [\hat{f}_0, \hat{\varepsilon}(\hat{f}_0)] = [\hat{f}_0, \hat{N}_a] = [\hat{f}_0, \hat{L}'_i] = 0\ . \tag{3.4}$$

here $\hat{S}'_\alpha = \hat{S}_\alpha - \mathrm{i}\varepsilon_{\alpha\beta\gamma} Y_\beta \partial/\partial Y_\gamma$, $\hat{L}'_i = \hat{L}_i - \mathrm{i}\varepsilon_{ikl} Y_k \partial/\partial Y_l$ are the generalized spin and orbital moment operators, and $\hat{L}_i = \int d^3x\, \varepsilon_{ijk} x_j\, \hat{\pi}_k(\mathbf{x})$ is the orbital angular moment.

For the description of superfluid states according to quasi-average concept [51], we determine the equilibrium average by the equality

$$a(\mathbf{x},\hat{f}) = \mathrm{Sp}\,\hat{f}\,\hat{a}(\mathbf{x}) = \lim_{\nu\to 0}\lim_{V\to\infty}\mathrm{Sp}\,\hat{f}_\nu\,\hat{a}(\mathbf{x})\,,$$

$$\hat{f}_\nu = [\exp(Y_r\hat{\gamma}_r + \nu Y_0\hat{G}) + 1]^{-1}\,, \tag{3.5}$$

where $\hat{a}(\mathbf{x})$ is an arbitrary quasi-local operator. The source $\hat{G}$ breaks the relations of symmetry (3.5) and is a linear functional of OP operators:

$$\hat{G} = \int \mathrm{d}^3x\{g_\alpha(\mathbf{x},t)\hat{\Delta}_\alpha(\mathbf{x}) + g_a(\mathbf{x},t)\hat{\Delta}_a(\mathbf{x}) + g_{a,\alpha k}(\mathbf{x},t)\hat{\Delta}_{a,\alpha k}(\mathbf{x}) + H.c.\}\,.$$

On repeating indexes the summation is meant. The particular form of the functions $g_a(\mathbf{x},t)$, $g_\alpha(\mathbf{x},t)$ and $g_{a,\alpha k}(\mathbf{x},t)$ is determined by properties of equilibrium-state symmetry.

We consider one of the possible equilibrium states of Fermi-liquids mixture, having symmetry properties [30]:

$$[\hat{f}^s, \hat{\mathscr{P}}_k - p_{ak}\hat{N}_a] = [\hat{f}^s, \hat{\varepsilon}(\hat{f}) + p_{a0}\hat{N}_a] = [\hat{f}^s, \hat{S}'_\alpha] = [\hat{f}^s, \hat{L}''_k] = 0\,, \tag{3.6}$$

where $\hat{L}''_i = \hat{L}'_i - \mathrm{i}\varepsilon_{ikl}\,p_{ak}\partial/\partial p_{al}$, $p_{a0} = Y_0^{-1}(Y_a + Y_k p_{ak})$, and p_{ak} are superfluid momenta in equilibrium states. Due to the commutation relations,

$$[\hat{s}_\alpha(\mathbf{x}), \hat{\Delta}_\beta(\mathbf{x}')] = \mathrm{i}\varepsilon_{\alpha\beta\gamma}\hat{\Delta}_\gamma(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\,,$$

$$[\hat{n}_1(\mathbf{x}), \hat{\Delta}_\alpha(\mathbf{x}')] = [\hat{n}_2(\mathbf{x}), \hat{\Delta}_\alpha(\mathbf{x}')] = -\hat{\Delta}_\alpha(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\,,$$

$$[\hat{\pi}_k(\mathbf{x}), \hat{\Delta}_\alpha(\mathbf{x}')] = -\mathrm{i}\hat{\Delta}_\alpha(\mathbf{x})\nabla_k\delta(\mathbf{x}-\mathbf{x}')\,. \tag{3.7}$$

$$\mathrm{i}[\hat{s}_\alpha(\mathbf{x}), \hat{\Delta}_{a,\beta i}(\mathbf{x}')] = \varepsilon_{\alpha\beta\gamma}\hat{\Delta}_{a,\gamma i}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') + \delta_{\alpha\beta}\nabla_i(\hat{\Delta}_a(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}'))\,,$$

$$[\hat{n}_a(\mathbf{x}), \hat{\Delta}_{b,\beta i}(\mathbf{x}')] = -2\delta_{ab}\hat{\Delta}_{b,\beta i}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\,,$$

$$\mathrm{i}[\hat{\pi}_k(\mathbf{x}), \hat{\Delta}_{a,\alpha i}(\mathbf{x}')] = \hat{\Delta}_{a,\alpha i}(\mathbf{x})\nabla_k\delta(\mathbf{x}-\mathbf{x}') + \nabla_i(\hat{\Delta}_{a,\alpha k}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}'))\,, \tag{3.8}$$

and symmetry properties of state (3.6), it is easy to see, that the average values of vector and tensor OP in this state approach to zero (OP (2.2), (2.3) are equal to zero due to the third relation of symmetry properties (3.6)), and the equilibrium value of scalar OP can be presented in the form:

$$\Delta_a^{(s)}(\mathbf{x}) = \Delta_a(\mathrm{inv})\mathrm{e}^{\mathrm{i}(\mathbf{p}_a\mathbf{x}+\theta_a)}, \qquad \Delta_\alpha^{(s)}(\mathbf{x}) = \Delta_{a,\alpha k}^{(s)}(\mathbf{x}) = 0, \quad a = 1,2\,. \tag{3.9}$$

Here the index "s" corresponds to equilibrium state (3.6). We call such states *singlet* states, because they are characterized by non-zero singlet OP in this state. Such states were investigated in papers [30,52].

The other possible equilibrium state of Fermi-liquids mixture is the state, having properties of symmetry [34]:

$$\begin{aligned}[\hat{f}^t, \hat{\mathscr{P}}_k - q_k n_\alpha\hat{S}'_\alpha - p_{ak}\hat{N}_a] &= [\hat{f}^t, \hat{\varepsilon}(\hat{f}) + \omega_{\alpha 0}\hat{S}'_\alpha + p_{a0}\hat{N}_a] \\ &= [\hat{f}^t, \hat{L}''_k + a'_{k\alpha}\hat{S}'_\alpha] = 0\,,\end{aligned} \tag{3.10}$$

where $\hat{L}''_i = \hat{L}'_i - \mathrm{i}\varepsilon_{ikl}(p_{ak}\partial/\partial p_{al} + q_k\partial/\partial q_l)$, $\omega_{\alpha0} = Y_0^{-1}(Y_\alpha + Y_k q_k n_\alpha)$, $p_{a0} = Y_0^{-1}(Y_a + Y_k p_{ak})$. Here $n_\alpha = Y_\alpha/|Y|$ is defined from compatibility ratios of spatial homogeneity and stationarity (first and second relations from properties (3.10)). For such state due to relations (3.7),

$$[\hat{s}_\alpha(\mathbf{x}), \hat{\Delta}_a(\mathbf{x}')] = 0, \quad [\hat{n}_a(\mathbf{x}), \hat{\Delta}_b(\mathbf{x}')] = -2\delta_{ab}\hat{\Delta}_b(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$[\hat{\pi}_k(\mathbf{x}), \hat{\Delta}_a(\mathbf{x}')] = -\mathrm{i}\hat{\Delta}_a(\mathbf{x})\nabla_k\delta(\mathbf{x}-\mathbf{x}') \quad (a,b=1,2) \tag{3.11}$$

and the properties of symmetry (3.10) we find, that scalar and vector OP approach to zero (OP (2.1), (2.3) approach to zero due to the third relation of the symmetry relations (3.10)), and tensor OP in this state can be presented in the form:

$$\Delta^{(t)}_{a,\alpha k}(\mathbf{x}) = \Delta_a(\mathrm{inv})a'_{k\beta}a_{\beta\alpha}\mathrm{e}^{\mathrm{i}(\mathbf{p}_a\mathbf{x}+\theta_a)}, \qquad \Delta^{(t)}_a(\mathbf{x}) = \Delta^{(t)}_\alpha(\mathbf{x}) = 0, \quad a=1,2\,. \tag{3.12}$$

The index "t" corresponds to the equilibrium state (3.12). On analogy this state can be called a *triplet*. This state describes the class of possible equilibrium states like B-phase superfluid He-3, where the equilibrium values of spin and the vector of a magnetic spiral in a mixture of two quantum Fermi-liquids differ from zero.

The equilibrium state of Fermi-liquids mixture with vector OP possess the following symmetry properties:

$$[\hat{f}^v, \hat{\mathscr{P}}_k - q_k n_\alpha \hat{S}'_\alpha - p_{ak}\hat{N}_a] = [\hat{f}^v, \hat{\varepsilon}(\hat{f}) + \omega_{\alpha0}\hat{S}'_\alpha + p_{a0}\hat{N}_a] = 0\,,$$
$$[\hat{f}^v, \underline{d}_\alpha\hat{S}'_\alpha - m_s(\hat{N}_1 + \hat{N}_2)] = [\hat{f}^v, \hat{L}''_k + r_{-,k}(\hat{N}_1 - \hat{N}_2)] = 0\,. \tag{3.13}$$

Here, $\hat{L}''_i = \hat{L}'_i - \mathrm{i}\varepsilon_{ikl}(p_{ak}\partial/\partial p_{al} + q_k\partial/\partial q_l)$, $\omega_{\alpha0} = Y_0^{-1}(Y_\alpha + Y_k q_k n_\alpha)$, and $n_\alpha = Y_\alpha/|Y|$ are defined in the same way as in case (3.8), q_i is the vector of magnetic spiral, $\underline{d}_\alpha$ is the vector of magnetic anisotropy, $m_s = \frac{1}{2}$ and $r_{-,i} = [\mathbf{p}_2 \times \mathbf{p}_1]_i/|[\mathbf{p}_2 \times \mathbf{p}_1]|$.

Therefore, using expressions (3.8), (3.11) and symmetry properties (3.13), we see, that scalar and tensor OP are vanished (OP (2.1), (2.2) are equal to zero due to the fourth symmetry properties (3.13)), and vector OP in this state is not equal to zero and has a form:

$$\Delta^{(v)}_\alpha(\mathbf{x}) = (\Delta_{\mathrm{I}}e_\alpha(\mathbf{x}) + \mathrm{i}\Delta_{\mathrm{II}}f_\alpha(\mathbf{x}))\mathrm{e}^{\mathrm{i}((\mathbf{p}_1+\mathbf{p}_2)\mathbf{x}+\theta')}\,,$$
$$\Delta^{(v)}_a(\mathbf{x}) = 0, \qquad \Delta^{(v)}_{a,\alpha k}(\mathbf{x}) = 0, \quad a=1,2\,. \tag{3.14}$$

In the expression for $\Delta^{(v)}_\alpha$ values $\Delta_{\mathrm{I}}, \Delta_{\mathrm{II}}$ are functionals of all invariants of the system. The dependence of $\Delta^{(v)}_\alpha$ from vector $r_{-,i}$ contains $\Delta_{\mathrm{I}}, \Delta_{\mathrm{II}}$ only. Here index "v" corresponds the equilibrium state (3.13). This state, we shall call the *vector*.

Taking into account (3.7), (3.8), (3.11), (3.13), we find the functions g in this state:

$$g_a(\mathbf{x},t) = \exp(-\mathrm{i}\varphi_a(\mathbf{x},t)), \quad a=1,2\,,$$
$$g_{a,\alpha k}(\mathbf{x},t) = a'_{k\beta}a_{\beta\alpha}(\theta(\mathbf{x},t))\exp(-2\mathrm{i}\varphi_a(\mathbf{x}))\,,$$
$$g_\alpha(\mathbf{x},t) = (e_\alpha(\mathbf{x},t) \pm \mathrm{i}f_\alpha(\mathbf{x},t))\exp(-\mathrm{i}(\varphi_1(\mathbf{x},t) + \varphi_2(\mathbf{x},t)))\,,$$

$$e_\alpha(\mathbf{x},t)=\underline{e}_\beta a_{\beta\alpha}(\mathbf{x},t), \qquad f_\alpha(\mathbf{x},t)=\underline{f}_\beta a_{\beta\alpha}(\mathbf{x},t)\,,$$

$$\theta_\alpha(\mathbf{x},t)=n_\alpha(\mathbf{q}\mathbf{x}-q_0t)+\theta_\alpha^0, \qquad \varphi_a(\mathbf{x},t)=\mathbf{p}_a\mathbf{x}-p_{a0}t+\varphi_{a0}\,. \tag{3.15}$$

Equilibrium state (3.13) is characterized by the following additional thermodynamical parameters: superfluid phases and momenta, magnetic anisotropy vector, vector of magnetic spiral, and orthogonal matrix of uniform spin rotation.

4. Local-equilibrium states

The statistical operator describing local-equilibrium degenerate state, we define by the equality:

$$\begin{aligned}\hat f\{Y(\mathbf{x}),\vartheta(\mathbf{x})\}=\Big[\exp\int \mathrm{d}^3x(Y_r(\mathbf{x})\,\hat\zeta_r(\mathbf{x},\hat f)\\ +vY_0(\mathbf{x})U^+_{\varphi1}U^+_{\varphi2}U^+_\theta(g_a(\mathbf{x})\hat\Delta_a(\mathbf{x})+g_{a,\alpha k}(\mathbf{x})\hat\Delta_{a,\alpha k}(\mathbf{x})\\ +g_\alpha(\mathbf{x})\hat\Delta_\alpha(\mathbf{x})+H.c.)U_\theta U_{\varphi2}U_{\varphi1}+1\Big]^{-1}\,,\end{aligned} \tag{4.1}$$

where $\vartheta(\mathbf{x})\equiv \mathrm{d}(\mathbf{x}),q(\mathbf{x}),z(\mathbf{x}),\varphi_a(\mathbf{x})$, and $g_a(\mathbf{x}),g_\alpha(\mathbf{x}),g_{a,\alpha k}(\mathbf{x})$ are some c-number functions. In this state the thermodynamical parameters $Y(\mathbf{x})$, the superfluid phases $\varphi_a(\mathbf{x})$, $a=1,2$ and the orthogonal matrix $a(\mathbf{x})$ are arbitrary functions of coordinates. In total equilibrium the thermodynamical forces $Y(\mathbf{x})=Y$ do not depend upon coordinates, and the structure of superfluid phases $\varphi_a(\mathbf{x})$ and orthogonal matrices $a_{\alpha\beta}(\mathbf{x})$ are determined by formulas (3.15).

We determine a local-equilibrium thermodynamical potential by the equality

$$\Omega(\hat f(\underline{Y},\vartheta),Y)=-\mathscr{S}(\hat f)+\int \mathrm{d}^3x\,\underline{Y}_r(\mathbf{x})\,\mathrm{Sp}\,\hat f\,\hat{\underline{\zeta}}_r(\mathbf{x})\,. \tag{4.2}$$

Here, the $\underline{Y}_r(\mathbf{x})$ are Lagrange multipliers, corresponding to the values $\underline{\zeta}_r(\mathbf{x})$ connected with the values $\zeta_r(\mathbf{x})=\mathrm{Sp}\,\hat f\,\hat\zeta_r(\mathbf{x})$. The local-equilibrium thermodynamical potential Ω is a functional of thermodynamical parameters $\underline{Y}_r=\{Y_0,\underline{Y}_\alpha=a_{\alpha\beta}Y_\beta,Y_i,Y_a\}$, transversal Cartan's form $\underline{z}_{\alpha k}(\mathbf{x}')$, the vector of magnetic spiral q_i, the vector of magnetic anisotropy d_α, and superfluid momenta $\mathbf{p}_a=\nabla\theta_a$:

$$\Omega=\Omega(\underline{Y}(\mathbf{x}'),\vartheta'(\mathbf{x}'))=\int \mathrm{d}^3x\,\omega(\mathbf{x},\underline{Y}(\mathbf{x}'),\vartheta(\mathbf{x}'))\,, \tag{4.3}$$

where ω is the density of thermodynamical potential. The value ϑ' is ϑ, where φ_a is replaced with $\mathbf{p}_a$. Varying (4.3) on thermodynamical parameters, we obtain

$$\delta\omega=\frac{\partial\omega}{\partial\underline{Y}_r}\,\delta\underline{Y}_r+\frac{\partial\omega}{\partial\underline{z}_{\alpha k}}\,\delta\underline{z}_{\alpha k}+\frac{\partial\omega}{\partial q_k}\,\delta q_k+\frac{\partial\omega}{\partial p_{ak}}\,\delta p_{ak}\,. \tag{4.4}$$

We receive the expressions for flux densities of particle numbers of each type. We consider potential (4.2), which is connected with the group of arbitrary phase transformations. We note the variation in the form:

$$\delta_{\varphi a}\Omega = -\mathrm{i}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\delta\varphi_a(\mathbf{x}')\,\mathrm{Sp}\,\hat{f}[\,\hat{n}_a(\mathbf{x}'), Y_r(\mathbf{x})\,\hat{\zeta}_r(\mathbf{x})], \quad a = 1,2\ .$$

Using now the commutation relations for AMIs, we obtain

$$\delta_{\varphi a}\Omega = \int \mathrm{d}^3x \left(Y_0(\mathbf{x}) i_{ak}(\mathbf{x}) + Y_k(\mathbf{x}) \frac{\partial\omega}{\partial Y_a} \right) \nabla_k \delta\varphi_a(\mathbf{x})\ .$$

On the other hand, we can write

$$\delta_{\varphi a}\Omega = \int \mathrm{d}^3x\,\delta_{\varphi a}\omega(\mathbf{x}) = \int \mathrm{d}^3x\,\frac{\partial\omega}{\partial p_{ak}}\,\nabla_k\delta\varphi_a(\mathbf{x})\ .$$

Comparing two last expressions, we find the flux density of particle numbers of each kind

$$i_{ak} = \frac{1}{Y_0}\,\frac{\partial\omega}{\partial p_{ak}} - \frac{Y_k}{Y_0}\,\frac{\partial\omega}{\partial Y_a}\ . \tag{4.5}$$

Now we find the expression for density of spin flux. The variation of the potential (5.2), connected with spin rotations, can be expressed as

$$\delta_\theta\Omega = -\mathrm{i}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\delta R_\alpha(\mathbf{x}')\,\mathrm{Sp}\,\hat{f}[\,\hat{s}_\alpha(\mathbf{x}'), Y_r(\mathbf{x})\,\hat{\zeta}_r(\mathbf{x})], \qquad \delta R_\alpha = \tfrac{1}{2}\,\varepsilon_{\alpha\beta\gamma}a_{\rho\gamma}\delta a_{\rho\beta}\ .$$

We take the advantage again of the commutation relations for AMIs, and we obtain

$$\begin{aligned}
\delta_\theta\Omega &= \int \mathrm{d}^3x \left\{ -\left(Y_0(\mathbf{x})\,j_{\alpha k}(\mathbf{x}) + Y_k(\mathbf{x})\frac{\partial\omega}{\partial Y_\alpha} \right) \nabla_k\delta R_\alpha - \varepsilon_{\alpha\beta\gamma}Y_\beta\,\frac{\partial\omega}{\partial Y_\gamma}\delta R_\alpha \right\} \\
&= -\int \mathrm{d}^3x \left\{ \left(Y_0(\mathbf{x})\underline{j}_{\|k}(\mathbf{x}) + Y_k(\mathbf{x})\frac{\partial\omega}{\partial \underline{Y}_\|} \right) \delta q_k \right. \\
&\quad \left. + \left(Y_0(\mathbf{x})\underline{j}^\perp_{\alpha k}(\mathbf{x}) + Y_k(\mathbf{x})\frac{\partial\omega}{\partial \underline{Y}^\perp_\alpha} \right) \delta \underline{z}_{\alpha k} \right\}\ .
\end{aligned}$$

On the other hand, we have

$$\delta_\theta\Omega = \int \mathrm{d}^3x\,\delta_\theta\omega(\mathbf{x}) = \int \mathrm{d}^3x \left\{ \frac{\partial\omega}{\partial q_k}\,\delta q_k(\mathbf{x}) + \frac{\partial\omega}{\partial \underline{z}_k}\,\delta\underline{z}_k(\mathbf{x}) \right\}$$

comparing the last expressions, we find expressions for longitudinal and transversal components of spin fluxes: $\underline{j}_{\|k} = d_\alpha \underline{j}_{\alpha k}$, $\underline{j}^\perp_{\alpha k} = \delta^\perp_{\alpha\beta}(\mathbf{d})\underline{j}_{\beta k}$. The total spin flux is:

$$\underline{j}_{\alpha k} = a_{\alpha\beta}j_{\beta k} = \frac{1}{Y_0}\,\frac{\partial\omega}{\partial q_k}\,d_\alpha + \frac{1}{Y_0}\,\frac{\partial\omega}{\partial \underline{z}_{\alpha k}} - \frac{Y_k}{Y_0}\,\frac{\partial\omega}{\partial \underline{Y}_\alpha}\ . \tag{4.6}$$

Substituting (4.5), (4.6) in expression (4.4) we obtain second thermodynamical law for the local-equilibrium case:

$$\begin{aligned}
\delta\omega &= \underline{\zeta}_r\delta\underline{Y}_r + (Y_0\underline{j}_{\|k} + Y_k\underline{s}_\|)\delta q_k + (Y_0\underline{j}^\perp_{\alpha k} + Y_k\underline{s}^\perp_\alpha)\delta\underline{z}_{\alpha k} + (Y_0 i_{ak} + Y_k n_a)\delta p_{ak} \\
&\equiv \underline{\zeta}_r\delta\underline{Y}_r + (Y_0\underline{j}_{\alpha k} + Y_k\underline{s}_\alpha)(\underline{d}_\alpha\delta q_k + \delta\underline{z}_{\alpha k}) + (Y_0 i_{ak} + Y_k n_a)\delta p_{ak}\ , \\
\underline{\zeta}_r &= \mathrm{Sp}\,\hat{\underline{f}}\,\hat{\zeta}_r = \{\varepsilon, \pi_i, \underline{s}_\alpha = a_{\alpha\beta}s_\beta, n_a\}\ .
\end{aligned} \tag{4.7}$$

And now we will derive the expression for the momentum flux. For this purpose we consider the modified potential, connected with the group of arbitrary deformations $\eta_i(\mathbf{x})$

$$\Omega(f(Y,a,\varphi_a,\eta),Y) \equiv -\mathscr{S}(\hat{f}) + \int \mathrm{d}^3x\, Y_r(\mathbf{x})\, \mathrm{Sp}\, U_\eta \hat{f} U_\eta^+\, \hat{\zeta}_r(\mathbf{x})\,. \tag{4.8}$$

This variation is expressed as

$$\delta_\eta \Omega = -\mathrm{i}\int \mathrm{d}^3x\, \mathrm{d}^3x'\, \delta\eta_i(\mathbf{x}')\, \mathrm{Sp}\, \hat{f}[\,\hat{\pi}_i(\mathbf{x}'), Y_r(\mathbf{x})\,\hat{\zeta}_r(\mathbf{x})]\,.$$

Using the commutation relations for AMIs, we obtain

$$\delta_\eta \Omega = \int \mathrm{d}^3x\, \delta\eta_i(\mathbf{x}) \left\{ -\nabla_k \left(Y_0 t_{ik} + \frac{\partial \omega Y_k}{\partial Y_i} \right)(\mathbf{x}) \right.$$
$$\left. + \frac{\partial\omega}{\partial p_{ak}} \nabla_i p_{ak}(\mathbf{x}) + \frac{\partial\omega}{\partial \underline{\omega}_{ak}} \nabla_i \underline{\omega}_{ak}(\mathbf{x}) \right\}\,.$$

On the other hand, according to the obvious type of superfluid momenta $p_{ak} \equiv p_{ak}(U_\eta \hat{f} U_\eta^+)$, the vector of magnetic spiral $q_k \equiv q_k(U_\eta \hat{f} U_\eta^+)$ and transversal Cartan's form $\underline{z}_{\alpha k} \equiv \underline{z}_{\alpha k}(U_\eta \hat{f} U_\eta^+)$, we find their transformation properties under the infinitesimal transformations of deformation: $\delta p_{ak} = -\nabla_k \delta\eta_i\, p_{ai} - \nabla_i\, p_{ak}\delta\eta_i, \delta q_k = -\nabla_k \delta\eta_i q_i - \nabla_i q_k \delta\eta_i, \delta\underline{z}_{\alpha k} = -\nabla_k \delta\eta_i \underline{z}_{\alpha i} - \nabla_i \underline{z}_{\alpha k}\delta\eta_i$. From this we obtain the variation $\delta_\eta\Omega$

$$\delta_\eta\Omega = \int \mathrm{d}^3x\, \delta_\eta\omega(\mathbf{x}) = \int \mathrm{d}^3x \left\{ \frac{\partial\omega}{\partial p_{ak}}\, \delta_\eta p_{ak}(\mathbf{x}) + \frac{\partial\omega}{\partial q_k}\, \delta_\eta q_k(\mathbf{x}) + \frac{\partial\omega}{\partial \underline{z}_{\alpha k}}\, \delta_\eta \underline{z}_{\alpha k}(\mathbf{x}) \right\}$$
$$= \int \mathrm{d}^3x \left\{ \frac{\partial\omega}{\partial p_{ak}} (p_{ai}\nabla_k\delta\eta_i + \nabla_i p_{ak}\delta\eta_i) + \frac{\partial\omega}{\partial q_k} (q_i\nabla_k\delta\eta_i + \nabla_i q_k\delta\eta_i) \right.$$
$$\left. + \frac{\partial\omega}{\partial \underline{z}_{\alpha k}} (\underline{z}_{\alpha i}\nabla_k\delta\eta_i + \nabla_i \underline{z}_{\alpha k}\delta\eta_i) \right\}\,.$$

Comparing these two last expressions, we find the momentum flux

$$t_{ik} = -\frac{\partial}{\partial Y_i}\frac{\omega Y_k}{Y_0} + \frac{p_{ai}}{Y_0}\frac{\partial\omega}{\partial p_{ak}} + \frac{q_i}{Y_0}\frac{\partial\omega}{\partial q_k} + \frac{\underline{z}_{\alpha i}}{Y_0}\frac{\partial\omega}{\partial \underline{z}_{\alpha k}}\,. \tag{4.9}$$

The expression for the density of energy flux we obtain (following study in Ref. [34]), using the relation

$$\underline{Y}_r(Y_0 \underline{\zeta}_{rk} + Y_k \underline{\zeta}_r) = 0 \tag{4.10}$$

and the expression for the other fluxes (4.5), (4.6), (4.9). This flux density has the form:

$$w_k = -\frac{\partial}{\partial Y_0}\frac{\omega Y_k}{Y_0} - \frac{p_{a0}}{Y_0}\frac{\partial\omega}{\partial p_{ak}} + \frac{q_0}{Y_0}\frac{\partial\omega}{\partial q_k} + \frac{\underline{z}_{\alpha 0}}{Y_0}\frac{\partial\omega}{\partial \underline{z}_{\alpha k}}\,. \tag{4.11}$$

All local-equilibrium flux densities in the considered state could be presented in a compact form:

$$\underline{\zeta}_{rk} = -\frac{\partial}{\partial \underline{Y}_r}\frac{\omega Y_k}{Y_0} + \frac{\partial p_{a0}}{\partial \underline{Y}_r}\frac{\partial\omega}{\partial p_{ak}} + \frac{\partial q_0}{\partial \underline{Y}_r}\frac{\partial\omega}{\partial q_k} + \frac{\partial \underline{z}_{\alpha 0}}{\partial \underline{Y}_r}\frac{\partial\omega}{\partial \underline{z}_{\alpha k}}\,. \tag{4.12}$$

Here $q_0 = Y_0^{-1}(d_\alpha \underline{Y}_\alpha + Y_k q_k)$, $\underline{z}_{\alpha 0} = Y_0^{-1}(\underline{Y}_\alpha - d_\beta d_\alpha \underline{Y}_\beta + Y_k \underline{z}_{\alpha k})$. As it will be shown below these expressions allow to ascertain the influence of entrainment effects on thermodynamics and the spectra of proper excitations.

The density of thermodynamical potential ω for a mixture of two Fermi-liquids depends on the following invariants which are square-law on thermodynamic parameters: $Y_0, Y_k^2, Y_\beta^2, Y_a, Y_k p_{ak}, Y_k q_k, Y_\beta d_\beta, p_a^2, p_a^2, p_{1k} p_{2k}, p_{ak} q_k, q^2, z^2$.

To trace the interrelation between expressions obtained in this paper and the expressions for fluxes and fluxes in the "two-liquid" Landau–Tisza formulation we introduce the physical quantities

$$\rho_n = -2Y_0 \frac{\partial \omega}{\partial \mathbf{Y}^2}, \qquad \rho_{sa} = 2\frac{m_a^{*2}}{Y_0}\frac{\partial \omega}{\partial \mathbf{p}_a^2}, \quad \rho_{s12} = \frac{m_1^* m_2^*}{Y_0}\frac{\partial \omega}{\partial \mathbf{p}_1 \mathbf{p}_2}, \quad \nu = \frac{\partial \omega}{\partial \mathbf{Yq}},$$

$$\rho - \rho_n - \rho_{s2} - \rho_{s12} = m_1^* \frac{\partial \omega}{\partial \mathbf{Y}\mathbf{p}_1}, \qquad \rho - \rho_n - \rho_{s1} - \rho_{s12} = m_2^* \frac{\partial \omega}{\partial \mathbf{Y}\mathbf{p}_2},$$

$$\kappa_1 = \frac{1}{Y_0}\frac{\partial \omega}{\partial \mathbf{p_1 q}},$$

$$\tau_\parallel = \frac{2}{Y_0}\frac{\partial \omega}{\partial \mathbf{q}^2}, \qquad \tau_\perp = \frac{2}{Y_0}\frac{\partial \omega}{\partial \mathbf{z}^2}, \qquad s = \frac{\partial \omega}{\partial \mathbf{dY}},$$

$$\chi = -2Y_0 \frac{\partial \omega}{\partial Y_\alpha^2}, \qquad \kappa_2 = \frac{1}{Y_0}\frac{\partial \omega}{\partial \mathbf{p_2 q}}, \tag{4.13}$$

where the potential ω depends only on linear and quadratic on vector thermodynamical parameters invariants. In expressions (4.13) ρ_n is the normal density of mixture, ρ_{s1}, ρ_{s2} are superfluid densities of components, and m_1^*, m_2^* are effective masses of the components, and ρ_{s12} is "entrained" superfluid density. Quantities $\tau_\parallel, \tau_\perp$ represent longitudinal and transversal constants of stiffness relative to the vector of magnetic anisotropy. The value s is the parameter connected with longitudinal spin and χ is magnetic susceptibility. The physical parameters κ_1 and κ_2 describe the effect of spin–orbital entrainment, as will be shown below. All introduced flux densities can be expressed as:

$$i_{1i} = \frac{1}{m_1^*}[(\rho_n + \rho_{s2} + \rho_{s12} - \rho + m_1^* n_1)v_{ni} + \rho_{s1} v_{s1i} + \rho_{s12} v_{s2i}] + \kappa_1 q_i,$$

$$i_{2i} = \frac{1}{m_2^*}[(\rho_n + \rho_{s1} + \rho_{s12} - \rho + m_2^* n_2)v_{ni} + \rho_{s12} v_{s1i} + \rho_{s2} v_{s2i}] + \kappa_2 q_i,$$

$$j_{\alpha i}^\perp = s_\alpha^\perp v_{ni} + z_{\alpha i}\tau_\perp, \quad j_{\parallel i} = (s_\parallel - \nu)v_{ni} + q_i \tau_\parallel + m_1^* \kappa_1 v_{s1i} + m_2^* \kappa_2 v_{s2i},$$

$$t_{ik} = p\delta_{ik} + \rho_n v_{ni} v_{nk} + \rho_{s1} v_{s1i} v_{s1k} + \rho_{s2} v_{s2i} v_{s2k} + \rho_{s12}(v_{s1i} v_{s2k} + v_{s2i} v_{s1k})$$
$$+ m_1^* \kappa_1 (v_{s1i} q_k + v_{s1k} q_i) + m_2^* \kappa_2 (v_{s2i} q_k + v_{s2k} q_i) + \tau_\parallel q_i q_k + \tau_\perp \underline{z}_{\alpha i} \underline{z}_{\alpha k},$$

$$w_k = (\varepsilon - p)v_{nk}$$
$$- \left\{ \frac{1}{m_1^*}[-(\rho - \rho_n - \rho_{s2} - \rho_{s12})v_{nk} + \rho_{s1} v_{s1k} + \rho_{s12} v_{s2k}] + \kappa_1 q_k \right\} p_{10}$$

$$-\left\{\frac{1}{m_2^*}[-(\rho-\rho_n-\rho_{s1}-\rho_{s12})v_{nk}+\rho_{s12}v_{s1k}+\rho_{s2}v_{s2k}]+\kappa_2 q_k\right\}p_{20}$$

$$-\{-\nu v_{nk}+m_1^*\kappa_1 v_{s1k}+m_2^*\kappa_2 v_{s2k}+\tau_\parallel q_k\}q_0-\tau_\perp z_{\alpha k}z_{\alpha 0},$$

$$s_\parallel = s+\chi\underline{h}_\parallel, \qquad \underline{h}_\parallel = d_\alpha \underline{h}_\alpha, \qquad s_\alpha^\perp = \chi\underline{h}_\alpha^\perp, \qquad \underline{h}_\alpha^\perp = \delta_{\alpha\beta}^\perp(\mathbf{d})\underline{h}_\beta\,. \tag{4.14}$$

Fluxes (4.14) are written out with accuracy up to quadratic on thermodynamical parameters terms. We consider the vectors v_{sak}, q_k in equilibrium state to be small, so the terms with higher orders can be neglected. In formulas (4.14) $j_{\parallel i}=d_\alpha j_{\alpha i}$ is the longitudinal part of spin flux density and $j_{\alpha i}^\perp=\delta_{\alpha\beta}^\perp j_{\beta i}$ is the transversal one. The quantity ρ_{s12} is responsible for the effect connected with the entrainment superfluid movement of one component of the mixture by the other component, earlier found in papers [52,53]. As is visible from (4.14) the terms in fluxes of particle numbers proportional to v_{s1i}, v_{s2i} lead to the entrainment effect of each mixture component. Besides the above-mentioned effect, another physical entrainment effect, connected with the influence of magnetic degrees of freedom on superfluid motion exists. The terms proportional to the vector of magnetic spiral $\mathbf{q}$ in density of fluxes of particle numbers and terms proportional v_{sai} in the density of longitudinal spin flux describe this entrainment effect. In the expression for momentum flux terms proportional to $v_{s1i}v_{s2i}, v_{sai}v_{sai}$ are responsible for the entrainment effect of one superfluid component by the other, and the terms proportional $v_{sai}q_i$ at the mutual influence of orbital and spin degrees of freedom.

In obtaining formulas (4.14) the availability of Galilean invariancy of the considered physical system was not assumed.

In the case of Galilean invariancy the structure of thermodynamical potential as the function from thermodynamic parameters Y_r, p_k, q_k is simplified $\omega(Y,\mathbf{p}_a,\mathbf{q}) = \omega(Y',\mathbf{p}'_a,\mathbf{q}')$. Here the primed parameters are determined by the equality: $Y'_0=Y_0, Y'_\alpha=Y_\alpha, Y'_k=Y_k+u_iY_0, Y'_a=Y_a+m_au_iY_i+\frac{1}{2}m_au^2Y_0, p'_{ak}=p_{ak}-m_au_k$ $(a=1,2)$, $q'_k=q_k$, where m_a, are masses of particles and u_i is the parameter of Galilean transformation. In a case, where $u_i=-Y_i/Y_0\equiv v_{ni}$ the value Y'_k approaches to zero. If we take into account in the thermodynamical potential the dependence on scalar invariants accurate to quadratic, the following relations will be correct, connected with quantities (see (4.13)):

$$m_1^*=m_1,\ m_2^*=m_2, \qquad \rho=\rho_n+\rho_{s1}+\rho_{s2}+2\rho_{s12}, \qquad \nu=\kappa_1m_1+\kappa_2m_2\,,$$

where m_1, m_2 are masses of the particles of the component in mixture.

We consider now the thermodynamical expressions in a case when the system is relativistic invariant. It is convenient to perform the transition to relativistic tensor values. We introduce AMI densities and the corresponding flux densities in the local-thermodynamic equilibrium state in terms of the Gibbs potential $\omega'\equiv\omega/Y_0$, being itself the relativistic scalar:

$$t^{\mu\nu}=-\frac{\partial(Y^\nu\omega')}{\partial Y_\mu}+p_a^\mu i_a^\nu+\underline{\omega}_\alpha^\mu \underline{j}_\alpha^\nu, \quad i_a^\nu=\frac{\partial\omega'}{\partial p_{a\nu}}, \quad \underline{j}_\alpha^\nu=\frac{\partial\omega'}{\partial\underline{\omega}_{\alpha\nu}}\,. \tag{4.15}$$

Here $t^{\mu\nu}\equiv(t^{00}=\varepsilon, t^{0k}=w_k, t^{k0}=\pi_k, t^{kl}=t_{kl})$ is the energy-momentum 4-tensor, $i_a^\nu\equiv(n_a, i_{ak})$ is the charge 4-current, $\underline{j}_\alpha^\nu\equiv(\underline{s}_\alpha, \underline{j}_{\alpha k})$ is the isospin 4-current, $p_a^\nu\equiv(p_{a0}, p_{ak})$

are the superfluid 4-momenta, $\underline{\omega}_{\alpha}^{\nu} \equiv (\underline{\omega}_{0k}, \ \underline{\omega}_{\alpha k})$—Cartan's 4-form, and $Y_{\mu} = (Y_0, Y_i)$. Gibbs potential ω' is a functional of such invariants: $Y^2, p_a^2, \underline{\omega}^2, \ Yp_a, \ p_1 p_2, \ (Y\underline{\omega})^2, (p_a\underline{\omega})^2, \ (p_1\underline{\omega})(p_2\underline{\omega}), \ (Y\underline{\omega})(p_a\underline{\omega})$.

Introducing entropy 4-current $\sigma^{\mu} \equiv (\sigma, \sigma_k = \sigma v_{nk})$, can write second thermodynamic law in the relativistic form:

$$\begin{aligned} \mathrm{d}\sigma^{\mu} &= Y_{\nu}\,\mathrm{d}t^{\nu\mu} + Y_a\,\mathrm{d}i_a^{\mu} + \underline{Y}_{\alpha}\,\mathrm{d}\underline{j}_{\alpha}^{\mu} - (Y^{\mu}i_a^{\nu} - Y^{\nu}i_a^{\mu})\,\mathrm{d}p_{a\nu} \\ &\quad - (Y^{\mu}\underline{j}_{\alpha}^{\nu} - Y^{\nu}\underline{j}_{\alpha}^{\mu})\,\mathrm{d}\underline{\omega}_{\alpha\nu}\,, \end{aligned} \tag{4.16}$$

where entropy 4-current σ^{μ} satisfy the adiabaticity condition:

$$\nabla_{\mu}\sigma^{\mu} = 0, \quad \sigma^{\mu} = -Y^{\mu}Y_{\nu}\frac{\partial\omega'}{\partial Y_{\nu}}\,. \tag{4.17}$$

5. Equation of hydrodynamics with vector OP

We consider the problem about the construction of ideal hydrodynamics of mixture of two superfluid Fermi-liquids described by the vector OP. In a set of parameters of abridged description, additional thermodynamical variables enter, which can be expressed through the order parameters and an arbitrary non-equilibrium statistical operator.

In a vector state the superfluid phases are entered by the equality

$$\varphi_{1,2}(\mathbf{x}, t) = \tfrac{1}{2}\,\mathrm{Im}\{\ln(\mathrm{Sp}\,\hat{f}(t)\hat{\Delta}_{\alpha}(\mathbf{x}))^2 \pm \ln(\mathrm{Sp}\,\hat{f}(t)\hat{\Delta}_{\alpha}^2(\mathbf{x}))\} \tag{5.1}$$

(index 1 corresponds the top character, and 2 the bottom one).

The vector of magnetic anisotropy, vector of magnetic spiral, and transversal Cartan's form can be expressed through the vector OP in the explicit form:

$$\begin{aligned} &d_{\alpha}(\mathbf{x}) = [\mathbf{e}(\mathbf{x})\mathbf{f}(\mathbf{x})]_{\alpha} = \{1 - (\xi_2(\mathbf{x})\xi_1(\mathbf{x}))^2\}^{-1/2}[\xi_2(\mathbf{x})\xi_1(\mathbf{x})]_{\alpha}\,, \\ &q_k(\mathbf{x}) = \tfrac{1}{2}\{1 - (\xi_2(\mathbf{x})\xi_1(\mathbf{x}))^2\}^{-1/2}(\xi_2(\mathbf{x})\nabla_k\xi_1(\mathbf{x}) - \xi_1(\mathbf{x})\nabla_k\xi_2(\mathbf{x}))\,, \\ &z_{\alpha k} = [(\nabla_k\mathbf{d}(\mathbf{x})) \times \mathbf{d}(\mathbf{x})]_{\alpha}, \qquad \xi_{a\alpha}(\mathbf{x}) = \Delta_{a\alpha}(\mathbf{x})/\Delta_a(\mathbf{x}), \quad (a = 1, 2)\,. \end{aligned} \tag{5.2}$$

The vector of magnetic spiral q_k, and transversal Cartan's form $\underline{z}_{\alpha k}$ satisfy the following relations:

$$\nabla_i q_k - \nabla_k q_i = \varepsilon_{\alpha\beta\gamma}d_{\alpha}\nabla_i d_{\beta}\nabla_k d_{\gamma}, \qquad \nabla_k\underline{z}_{\alpha i} - \nabla_i\underline{z}_{\alpha k} = -\varepsilon_{\alpha\beta\gamma}\underline{z}_{\beta i}\underline{z}_{\gamma k}\,. \tag{5.3}$$

The first of these relations is Mermin–Ho identity. The second one defines the identity for transversal Cartan's form to magnetic anisotropy direction **d**.

We construct the hydrodynamical equations of mixture of two superfluid Fermi-liquids. On hydrodynamical stage of evolution this mixture is described by a final set of parameters such as the densities of AMI ζ, superfluid momenta $\mathbf{p}_a$, the vector of magnetic spiral **q**, the vector of magnetic anisotropy **d**, and the transversal Cartan's form z. The functional hypothesis then is

$$\hat{f}(t) \underset{t \gg \tau_0}{\rightarrow} \hat{f}(\zeta(t), \vartheta(t)), \quad \vartheta(t) \equiv \{\varphi_a(t), d(t), q(t), z(t)\} ,$$

where τ_0 is the relaxation time.

We can formulate the equations for the values $\zeta(t), \vartheta(t)$ (in initial expressions we replace $\hat{f}$ to $\hat{f}(\zeta, \vartheta)$):

$$\dot{\zeta}_r(\mathbf{x}, \hat{f}(\zeta, \vartheta)) = \mathrm{i}\,\mathrm{Sp}\,\hat{f}(\zeta, \vartheta)[\,\hat{\varepsilon}(\hat{f}(\zeta, \vartheta)),\, \hat{\zeta}_r(\mathbf{x}, \hat{f}(\zeta, \vartheta))] , \tag{5.4}$$

$$\dot{d}_\alpha(\mathbf{x}, \hat{f}(\zeta, \vartheta)) = \mathrm{i}\,\mathrm{Sp}\,\hat{f}(\zeta, \vartheta)[\,\hat{\varepsilon}(\hat{f}(\zeta, \vartheta)),\, \hat{d}_\alpha(\mathbf{x}, \hat{f}(\zeta, \vartheta))] , \tag{5.5}$$

$$\dot{\varphi}_a(\mathbf{x}, \hat{f}(\zeta, \vartheta)) = \mathrm{i}\,\mathrm{Sp}\,\hat{f}(\zeta, \vartheta)[\,\hat{\varepsilon}(\hat{f}(\zeta, \vartheta)),\, \hat{\varphi}_a(\mathbf{x}, \hat{f}(\zeta, \vartheta))] . \tag{5.6}$$

The operators $\hat{d}_\alpha$, $\hat{\varphi}_a$ introduced here are determined by the variational expressions $\delta d_\alpha(\mathbf{x}, \hat{f}) = \mathrm{Sp}\,\delta \hat{f}\,\hat{d}_\alpha(\mathbf{x}, \hat{f}), \delta\varphi_a(\mathbf{x}, \hat{f}) = \mathrm{Sp}\,\delta\hat{f}\;\hat{\varphi}_a(\mathbf{x}, \hat{f})$.

We consider the equation for the vector of magnetic anisotropy d_α (5.5). Neglecting relaxation processes in (5.5), we have

$$\dot{d}_\alpha(\mathbf{x}, \hat{f}_0) = \mathrm{i}\,\mathrm{Sp}\,\hat{f}_0[\,\hat{\varepsilon}(\hat{f}_0),\, \hat{d}_\alpha(\mathbf{x}, \hat{f}_0)] ,$$

then, in view of the relations

$$\mathrm{i}\,\mathrm{Sp}\,\hat{f}[\,\hat{s}_\alpha(\mathbf{x}'),\, \hat{d}_\beta(\mathbf{x}, \hat{f})] = \varepsilon_{\alpha\beta\gamma} d_\gamma(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}'), \qquad \mathrm{i}\,\mathrm{Sp}\,\hat{f}[\,\hat{n}_a(\mathbf{x}'),\, \hat{d}_\alpha(\mathbf{x}, \hat{f})] = 0 ,$$

$$\mathrm{i}\,\mathrm{Sp}\,\hat{f}[\,\hat{\pi}_i(\mathbf{x}'),\, \hat{d}_\alpha(\mathbf{x}, \hat{f})] = \nabla_i d_\alpha(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}') ,$$

we obtain the equation for the vector of magnetic anisotropy d_α in main approximation:

$$\dot{d}_\alpha = d_\lambda \varepsilon_{\lambda\alpha\gamma}(h_\gamma + \mathbf{q}\mathbf{v}_n n_\gamma), \quad h_\gamma \equiv -Y_\gamma / Y_0,\; v_{nk} \equiv -Y_k / Y_0 .$$

Here h_γ is an internal magnetic field, v_{nk} is normal velocity.

Equations for other unity vectors, e_α, f_α, have the same form as the equation for d_α. Using the definition of magnetic spiral vector q_i transversal Cartan's form $z_{\alpha k}$, and equations for triad $d_\alpha, e_\alpha, f_\alpha$, we obtain equations of motion for $q_i, z_{\alpha k}$:

$$\dot{q}_i = -\nabla_i(d_\alpha h_\alpha + q_j v_{nj}) + \varepsilon_{\alpha\beta\gamma} d_\alpha z_{\beta i}(h_\gamma + z_{\gamma j} v_{nj}) ,$$

$$\dot{z}_{\alpha i} = -\delta^{\perp}_{\alpha\beta}(\mathbf{d})\nabla_i(h_\beta + z_{\beta j} v_{nj}) + \varepsilon_{\alpha\beta\gamma} z_{\beta i}(h_\gamma + z_{\gamma j} v_{nj}) .$$

Now we derive the equation for superfluid phases φ_a. Neglecting in (5.6) relaxation processes, we have

$$\dot{\varphi}_a(\mathbf{x}, \hat{f}_0) = \mathrm{i}\,\mathrm{Sp}\hat{f}_0[\,\hat{\varepsilon}(\hat{f}_0),\, \hat{\varphi}_a(\mathbf{x}, \hat{f}_0)] .$$

whence in view of symmetry conditions (3.21) and properties

$$\mathrm{i}\,\mathrm{Sp}\,\hat{f}[\,\hat{s}_\alpha(\mathbf{x}'),\, \hat{\varphi}_a(\mathbf{x}, \hat{f})] = 0, \quad \mathrm{i}\,\mathrm{Sp}\,\hat{f}[\,\hat{n}(\mathbf{x}'),\, \hat{\varphi}_a(\mathbf{x}, \hat{f})] = \delta(\mathbf{x} - \mathbf{x}') ,$$

$$\mathrm{i}\,\mathrm{Sp}\,\hat{f}[\,\hat{\pi}_k(\mathbf{x}'),\, \hat{\varphi}_a(\mathbf{x}, \hat{f})] = \nabla_k \varphi_a(\mathbf{x}, \hat{f})\delta(\mathbf{x} - \mathbf{x}') ,$$

we find the equations of motion for superfluid phases:

$$\dot{\varphi}_a = \mu_a + \mathbf{p}_a \mathbf{v}_n, \qquad \mu_a \equiv -Y_a/Y_0 ,$$

where the μ_a are chemical potentials.

In summary we obtain the equations of motion for AMIs. Neglecting relaxation processes in (5.4) we obtain

$$\dot{\zeta}_r(\mathbf{x}, \hat{f}_0) = \mathrm{i}\,\mathrm{Sp}\,\hat{f}_0[\,\hat{\varepsilon}(\hat{f}_0), \hat{\zeta}_r(\mathbf{x}, \hat{f}_0)] = -\nabla_i\,\mathrm{Sp}\,\hat{f}_0\,\hat{\zeta}_{ri}(\mathbf{x}, \hat{f}_0) ,$$

where the form of flux densities of AMI ζ_{rk} in terms local-equilibrium thermodynamical potential is determined by formulas (4.5), (4.6), (4.9), (4.14).

Thus we have derived the total set of hydrodynamics equations of Fermi-liquids mixture in the states, described by the vector OP. They have the form:

$$\dot{\underline{\zeta}}_r = -\nabla_i \underline{\zeta}_{ri} + \eta_r, \quad \dot{d}_\alpha = \varepsilon_{\alpha\beta\gamma}\left(\frac{\partial\varepsilon}{\partial \underline{s}_\beta} + \underline{z}_{\beta i}\,\frac{\partial\varepsilon}{\partial\pi_i}\right) d_\gamma ,$$

$$\dot{p}_{ai} = -\nabla_i\left(\frac{\partial\varepsilon}{\partial n_a} + p_{aj}\frac{\partial\varepsilon}{\partial\pi_j}\right),$$

$$\dot{q}_i = -\nabla_i\left(d_\alpha\,\frac{\partial\varepsilon}{\partial \underline{s}_\alpha} + q_j\,\frac{\partial\varepsilon}{\partial\pi_j}\right) + \varepsilon_{\alpha\beta\gamma} d_\alpha \underline{z}_{\beta i}\left(\frac{\partial\varepsilon}{\partial \underline{s}_\gamma} + \underline{z}_{\gamma j}\,\frac{\partial\varepsilon}{\partial\pi_j}\right) ,$$

$$\dot{\underline{z}}_{\alpha i} = -\delta^{\perp}_{\alpha\beta}(\mathbf{d})\nabla_i\left(\frac{\partial\varepsilon}{\partial \underline{s}_\beta} + \underline{z}_{\beta j}\,\frac{\partial\varepsilon}{\partial\pi_j}\right) + \varepsilon_{\alpha\beta\gamma}\underline{z}_{\beta i}\left(\frac{\partial\varepsilon}{\partial \underline{s}_\gamma} + \underline{z}_{\gamma j}\,\frac{\partial\varepsilon}{\partial\pi_j}\right) ,$$

$$\eta_r = \delta_{r\alpha}\varepsilon_{\alpha\beta\gamma}(\underline{\zeta}_{\gamma k}(q_k d_\alpha + \underline{z}_{\beta k}) + \underline{\zeta}_\gamma(q_0 d_\alpha + \underline{z}_{\beta 0})) , \tag{5.7}$$

where η_r represents the source in the equation for spin. We pay attention that different from the spin density s_α the density $\underline{s}_\alpha = a_{\alpha\beta}s_\beta$ is not the density of AMI, so in equations of hydrodynamics (5.3) the source arises. The convenience of giving the hydrodynamics equations (5.3) and the main expression of thermodynamics (4.8) consist in that they are formulated in terms of thermodynamic variables, and independent from coordinates and the time in equilibrium state. The construction of the simple perturbation theory on small space inhomogeneities for all parameters of abridged description hypothesis is then possible.

The equations of ideal hydrodynamics (5.7), the second beginning of thermodynamics (4.8) and density of AMI fluxes (5.1) describe *the non-equilibrium dynamics* of superfluid Fermi-liquids mixture with vector OP.

The equations of hydrodynamics and main thermodynamic expressions for the mixture of superfluid Fermi-liquids in the state with vector OP differ from known expressions for superfluid Fermi-liquid with singlet pairing [27,30,54]. In these equations we have variables connected with the second component of mixture and with the violation of symmetry concerning to spin rotations (the vector of magnetic spiral q_i, transversal Cartan's form $z_{\alpha i}$ and the vector of magnetic anisotropy d_α). If in the obtained equations of hydrodynamics and thermodynamic relation we omit the density of second component particle number and its superfluid momenta, and replace the vector of magnetic spiral q_i, transversal Cartan's form $z_{\alpha i}$, and the vector of magnetic anisotropy d_α on Cartan's form $\underline{\omega}_{\alpha i}$, these equations reduce to similar equations for superfluid Fermi-liquid in the state with triplet OP [34,55]. The mixture considered in the present work is described by a set of parameters, including, except for the variable of Fermi-liquids mixture with singlet pairing [30,52], also variables connected with the spontaneous violation of symmetry for spin rotations.

The equations of ideal hydrodynamics of mixture of superfluid Fermi-liquid with the vector OP in the relativistic invariant form can be written as

$$\nabla_\nu t^{\mu\nu} = 0, \qquad \nabla_\nu i_a^\nu = 0, \qquad \nabla_\nu \underline{j}_\alpha^\nu = \varepsilon_{\alpha\beta\gamma} \underline{\omega}_{\beta\mu} \underline{j}_\gamma^\mu . \tag{5.8}$$

The first of the three equations contains the equations of motion for AMI densities. The two last equations contain the equations of motion for the superfluid momenta p_{ak} and the Cartan's form $\underline{\omega}_{\alpha k}$.

The values $p_a^\mu, \underline{\omega}_\alpha^\mu$ satisfy the equations

$$\nabla^\mu p^\nu - \nabla^\nu p^\mu = 0, \qquad \nabla^\mu \underline{\omega}_\alpha^\nu - \nabla^\nu \underline{\omega}_\alpha^\mu = -\varepsilon_{\alpha\beta\gamma} \underline{\omega}_\beta^\nu \underline{\omega}_\gamma^\mu . \tag{5.9}$$

These equations contain the potentiality conditions and Maurer–Cartan identity.

We consider the question about the collective excitations in the considered mixture of Fermi-liquids. Linearizing the system of Eqs. (5.3) near equilibrium state, we obtain the dispersion equation: $\sum_{m=0}^{14} A_m(k)\omega^m = 0$, where the factors $A_m(k)$ represent the combinations of derivatives from ε on all variables.

As the dispersion equation is the most complete, for revealing of features of spectra, we consider the model energy functional

$$\begin{aligned} \varepsilon = {} & \varepsilon_0(n_1, n_2, \sigma) + \frac{1}{2\rho}\pi^2 + \frac{1}{2}\alpha_1 p_1^2 + \frac{1}{2}\alpha_2 p_2^2 + \alpha_3 \mathbf{p}_1\mathbf{p}_2 \\ & + \frac{1}{2\chi_\parallel}\underline{s}_\parallel^2 + \frac{1}{2\chi_\perp}\underline{s}_\perp^2 + \frac{1}{2}\tau_\parallel q^2 + \frac{1}{2}\tau_\perp \underline{z}^2 + \kappa_1(\mathbf{p}_1\mathbf{q})^2 + \kappa_2(\mathbf{p}_2\mathbf{q})^2 , \end{aligned} \tag{5.10}$$

containing quadratic terms on the parameters of abridged description and only two terms of the fourth order adequate for interaction between subsystems. Here $s_\parallel = d_\alpha \underline{s}_\alpha$, $s_\alpha^\perp = \delta_{\alpha\beta}^\perp(\mathbf{d})\underline{s}_\beta$.

We consider two simple cases:

(a) All vectors and z in equilibrium state are equal zero. In this case we find non-trivial solutions:

$$\begin{aligned} & \omega_1^2 = w_1^2 k^2, \qquad \omega_2^2 = w_2^2 k^2, \qquad \omega_3^2 = w_3^2 k^2, \qquad w_1^2 + w_2^2 + w_3^2 = A_1 , \\ & w_1^2 w_2^2 + w_1^2 w_3^2 + w_2^2 w_3^2 = A_2, \quad w_1^2 w_2^2 w_3^2 = A_3, \quad \omega_4^2 = B_1 k^2, \quad \omega_{5,6}^2 = B_2 k^2 , \end{aligned} \tag{5.11}$$

where

$$A_1 = a_1 + b_2 + c_3, \quad A_2 = -a_1 b_2 - a_1 c_3 - b_2 c_3 + a_3 c_1 + a_2 b_1 + c_2 b_3 ,$$

$$A_3 = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2 ,$$

$$B_1 = \tau_\parallel/\chi_\parallel, \quad B_2 = \tau_\perp/\chi_\perp, \quad a_1 = (v\sigma^0 + u_1 n_1^0 + u_2 n_2^0)/\rho, \quad a_2 = \alpha_1 u_1 + \beta u_2 ,$$

$$a_3 = \beta u_1 + \alpha_2 u_2, \quad b_1 = (\xi_1 \sigma^0 + \zeta_1 n_1^0 + \lambda n_2^0)/\rho, \quad b_2 = \alpha_1 \zeta_1 + \beta \lambda ,$$
$$b_3 = \beta \zeta_1 + \alpha_2 \lambda, \quad c_1 = (\xi_2 \sigma^0 + \lambda n_1^0 + \zeta_2 n_2^0)/\rho ,$$
$$c_2 = \alpha_1 \lambda + \beta \zeta_2, \quad u_1 = \partial p / \partial n_1 ,$$
$$c_3 = \beta \lambda + \alpha_2 \zeta_2, \quad u_2 = \partial p / \partial n_2, \quad v = \partial p / \partial \sigma, \quad \xi_1 = \partial^2 \varepsilon / \partial n_1 \partial \sigma ,$$
$$\xi_2 = \partial^2 \varepsilon / \partial n_2 \partial \sigma, \quad \zeta_1 = \partial^2 \varepsilon / \partial n_1^2, \quad \zeta_2 = \partial^2 \varepsilon / \partial n_2^2, \quad \lambda = \partial^2 \varepsilon / \partial n_1 \partial n_2 . \tag{5.12}$$

They represent three branches of sound excitations and degenerated spins modes. In the approximation $w_1^2 \gg w_2^2 \gg w_3^2$, used in Ref. [52] for obtaining the spectra of sound excitations, velocities of sound modes have the form: $w_1^2 = A_1, w_2^2 = A_2/A_1, w_3^2 = A_3/A_2$. The excitations $\omega_1^2, \omega_2^2, \omega_3^2$ correspond to the first, second and third sounds. In the case of small superfluid velocities and mixture concentration formulae reduce to results of paper [53] for hydrodynamics of superfluid liquids mixture with singlet pairing.

(b) Only $\mathbf{p}_1, \mathbf{p}_2$ are not equal zero. $\mathbf{p}_1 \perp \mathbf{p}_2 \perp \mathbf{k}$ are small. The sound modes are

$$\omega_1^2 = w_1^2 k^2, \qquad \omega_2^2 = w_2^2 k^2, \qquad \omega_3^2 = w_3^2 k^2, \qquad w_1^2 + w_2^2 + w_3^2 = A_1' ,$$
$$w_1^2 w_2^2 + w_1^2 w_3^2 + w_2^2 w_3^2 = A_2', \qquad w_1^2 w_2^2 w_3^2 = A_3' , \tag{5.13}$$

where

$$A_1' = a_1 + b_2' + c_3', \quad A_2' = -a_1 b_2' - a_1 c_3' - b_2' c_3' + a_3 c_1 + a_2 b_1 + c_2' b_3' ,$$
$$A_3' = a_1 b_2' c_3' + a_2 b_3' c_1 + a_3 b_1 c_2' - a_3 b_2' c_1 - a_2 b_1 c_3' - a_1 b_3' c_2' ,$$
$$b_2' = \alpha_1 p_1^2/\rho, \quad b_3' = \beta p_1^2/\rho, \quad c_2' = \beta p_2^2/\rho, \quad c_3' = \alpha_2 p_2^2/\rho . \tag{5.14}$$

The spin modes are:

$$\omega^2 = B_1 k^2, \qquad \omega^2 = (B_2 + B')k^2, \qquad B' = (\kappa_1 p_1^2 + \kappa_2 p_2^2)/\chi_{\parallel} .$$

Then the expressions for sound modes will be modified in comparison with variant (a), and the expressions for velocities of spin waves also will be renormalized in view of the above-mentioned entrainment effect ($p_{ak} \neq 0$).

On this basis it is possible to conclude that the system considered in this work partially has properties of superfluids mixture with singlet pairing, as well as magnetic component of superfluid Fermi-liquid with triplet pairing.

Acknowledgements

The authors wish to thank S.V. Peletminsky and N.N. Bogolyubov (Jr.) for useful discussion of our work and advices. The present work is executed on a theme "The dynamic theory of systems in stochastic fields and the hydrodynamical theory of non-unitary states of superfluid Fermi-liquid" by the financial support from the Ministry of Education and Sciences of Ukraine (Grant number N0102U000445), and also Grants RFFI N030217695 (Russia), INTAS N00-00577.

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