Nonlinear dynamics and collective excitations of spin-1 magnets

M.Yu. Kovalevsky [a], Tran Quang Vuong [b]

[a] Akhiezer Institute for Theoretical Physics, National Science Center "Kharkov Institute of Physics and Technology", Kharkov 61108, Ukraine
[b] Belgorod State University, Belgorod 308015, Russia

ABSTRACT

The Poisson brackets for macroscopic parameters are obtained and nonlinear dynamic equations of spin-1 magnets are derived. Two types of magnetic exchange Hamiltonians corresponding to two Kazimir invariants of SU(3) group are introduced. Thermodynamics of spin-1 magnets is studied and the flux densities of additive integrals of motion are found in terms of exchange energy density. The momentum of magnons is introduced and the corresponding dynamic equation is derived. The spectra of spin and quadrupole waves of magnets with various symmetry of equilibrium state with respect to time inversion are found.

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1. Introduction

In this Letter we study the dynamics of non-equilibrium states of magnets with spin $s = 1$. In contrast to the well-known Landau–Lifshitz equation [1], which specifies the evolution of spin-1/2 magnets, the description of magnets with higher spin requires the introduction of additional dynamic variables. For magnets with spin $s = 1$ it is necessary to introduce eight magnetic degrees of freedom – the densities of spin and quadrupole matrix. The investigation of such magnets was carried out in [2–4], where only the pure quantum states were studied (the local equilibrium state is characterized by four dynamic variables). The Hamiltonian approach describing nonlinear dynamics of mixed quantum states of such media is developed in [5]. The equilibrium properties of spin-1 magnets were considered and analyzed in [6–13]. These works predict a possibility of existence of nematic magnetic states, in which the equilibrium state is invariant with respect to time reversion.

The symmetries of Hamiltonian and equilibrium state define the set of dynamic variables, which is necessary for a complete macroscopic description of physical systems. In this article we study the states, which are specified by SU(3) symmetry of equilibrium state and Hamiltonian. The spin and quadrupole matrix are the integrals of motion. In other case the Hamiltonian has SU(3) symmetry and the symmetry of equilibrium state is spontaneously broken relatively to this one, thus the general number of dynamic quantities is equal to sixteen. Such states are spontaneously broken with respect to SU(3) symmetry and we do not study them in the present article. Our investigation is based on Hamiltonian formalism [14].

2. Principle of stationary action and Poisson brackets of spin-1 magnets

According to the general approach of continuum mechanics a Lagrangian of an arbitrary physical system we shall present as $L = L_k(\phi, \dot{\phi}) - H = \int d^3x\, F_a(\mathbf{x}, \phi(\mathbf{x}'))\dot{\phi}_a(\mathbf{x}') - H(\phi)$, where $L_k(\phi, \dot{\phi})$ is the kinematic part of Lagrangian and $H(\phi) = \int d^3x\, e(\mathbf{x}, \phi)$ is the Hamiltonian. The energy density $e(\mathbf{x}, \phi(\mathbf{x}'))$ and $F_a(\mathbf{x}, \phi(\mathbf{x}'))$ are the certain functionals of dynamic variables $\phi(\mathbf{x})$. From the principle of stationary action we can find the dynamic equations for $\phi(\mathbf{x})$ [14]:

$$\dot{\phi}_a(\mathbf{x}) = \int d^3x'\, J^{-1}_{ab}(\mathbf{x}, \mathbf{x}'; \phi) \frac{\delta H(\phi)}{\delta \phi_b(\mathbf{x}')} = \{\phi_a(\mathbf{x}), H(\phi)\}. \quad (1)$$

The matrix $J_{ab}(\mathbf{x}, \mathbf{x}'; \phi)$ is given by

$$J_{ab}(\mathbf{x}, \mathbf{x}'; \phi) \equiv \frac{\delta F_b(\mathbf{x}', \phi)}{\delta \phi_a(\mathbf{x})} - \frac{\delta F_a(\mathbf{x}, \phi)}{\delta \phi_b(\mathbf{x}')}. \quad (2)$$

This matrix defines the Poisson brackets for quantities $\phi_a(\mathbf{x})$:

$$\{\phi_a(\mathbf{x}), \phi_a(\mathbf{x}')\} = J^{-1}_{ab}(\mathbf{x}, \mathbf{x}'; \phi). \quad (3)$$

For arbitrary functionals $A(\phi), B(\phi)$ we introduce the Poisson brackets according to the following relation:

$$\{A, B\} \equiv \int d^3x \int d^3x' \frac{\delta A}{\delta \phi_a(\mathbf{x})} \{\phi_a(\mathbf{x}), \phi_b(\mathbf{x}')\} \frac{\delta B}{\delta \phi_b(\mathbf{x}')}. \tag{4}$$

Taking into account the definitions (2)–(4), we see that these brackets are antisymmetric with respect to permutation $a \Leftrightarrow b$, $\mathbf{x} \Leftrightarrow \mathbf{x}'$ and satisfy the Leibniz and Jacobi identities.

The finite transformations $\phi_a(\mathbf{x}) \to \phi'_a(\mathbf{x}) = \phi_a(\mathbf{x}, \phi(\mathbf{x}'))$, which leave invariant the kinematic part of Lagrangian

$$L_k(\phi, \dot\phi) = \int d^3x\, F_a(\mathbf{x}, \phi)\dot\phi_a(\mathbf{x}) = L_k(\phi', \dot\phi') = \int d^3x\, F_a(\mathbf{x}, \phi')\dot\phi'_a(\mathbf{x})$$

are canonical, if the following relation is valid:

$$F_b(\mathbf{x}', \phi) = \int d^3x\, F_a(\mathbf{x}, \phi')\delta\phi'_a(\mathbf{x})/\delta\phi_b(\mathbf{x}').$$

In the case of infinitesimal transformations $\phi_a(\mathbf{x}) \to \phi'_a(\mathbf{x}) = \phi_a(\mathbf{x}) + \delta\phi_a(\mathbf{x}, \phi(\mathbf{x}'))$ we have

$$\delta\phi_a(\mathbf{x}) = \{\phi_a(\mathbf{x}), G\}, \quad G(\phi) \equiv \int d^3x\, F_a(\mathbf{x}, \phi)\delta\phi_a(\mathbf{x}), \tag{5}$$

where $G(\phi)$ is the generator of infinitesimal canonical transformations. The formulation of Hamiltonian approach requires a choice of the certain set of dynamic variables, for which the Poisson structure is introduced. In magnets under consideration we proceed from the following expression for the kinematic part of Lagrangian:

$$L_k(\mathbf{x}) = b_{\alpha\beta}(\mathbf{x})\dot a_{\beta\alpha}(\mathbf{x}) \equiv \operatorname{Sp} \hat b(\mathbf{x})\dot{\hat a}(\mathbf{x}). \tag{6}$$

Here $b_{\alpha\beta}(\mathbf{x})$ and $a_{\beta\alpha}(\mathbf{x})$ are Hermitian 3×3 matrices ($\hat a = \hat a^+$, $\hat b = \hat b^+$), both contains nine independent quantities. For convenience and abridgment we shall omit spin indices in matrix elements there, where it is possible. Below we shall connect matrices $(\hat a, \hat b)$ with physical dynamic variables, which are used in theory of magnetism.

Let us find infinitesimal canonical transformations $\delta\phi_a(\mathbf{x}, \phi)$ leaving invariant kinematic part of Lagrangian. Knowing, on the one hand, their explicit form, and, on the other hand, taking into account relation (5), it is easy to find the Poisson brackets of dynamic variables. In particular, it is easy to see, that variations $\delta b_{\alpha\beta}(\mathbf{x}) = 0$, $\delta a_{\alpha\beta}(\mathbf{x}) \neq 0$ (functions $\delta a_{\alpha\beta}(\mathbf{x})$ do not depend on $\hat a, \hat b$), leave invariant kinematic part of Lagrangian and in accordance with (5) can be represented as

$$\delta a_{\alpha\beta}(\mathbf{x}) = \{a_{\alpha\beta}(\mathbf{x}), G\}, \qquad \delta b_{\alpha\beta}(\mathbf{x}) = \{b_{\alpha\beta}(\mathbf{x}), G\}, \tag{7}$$

where the generator of transformations reads $G \equiv \int d^3x\, b_{\alpha\beta}(\mathbf{x}) \times \delta a_{\beta\alpha}(\mathbf{x})$. Comparing (5), (7), we obtain the set of the Poisson brackets for these matrices

$$\{b_{\alpha\beta}(\mathbf{x}), b_{\mu\nu}(\mathbf{x}')\} = 0,$$
$$\{b_{\alpha\beta}(\mathbf{x}), a_{\mu\nu}(\mathbf{x}')\} = -\delta_{\alpha\nu}\delta_{\beta\mu}\delta(\mathbf{x}-\mathbf{x}'). \tag{8}$$

For the kinematic part of Lagrangian, $L_k(\mathbf{x}) = -\dot b_{\alpha\beta}(\mathbf{x})a_{\beta\alpha}(\mathbf{x})$, which differs from (6) in the time derivative, we similarly find

$$\{a_{\alpha\beta}(\mathbf{x}), a_{\mu\nu}(\mathbf{x}')\} = 0,$$
$$\{a_{\alpha\beta}(\mathbf{x}), b_{\mu\nu}(\mathbf{x}')\} = \delta_{\alpha\nu}\delta_{\beta\mu}\delta(\mathbf{x}-\mathbf{x}'). \tag{9}$$

The obtained expressions for the Poisson brackets are consistent with hermiticity requirement for these matrices and satisfy the Jacobi identity. We see, that the matrices $\hat a$ and $\hat b$ represent the canonically conjugate quantities. Let us now introduce into consideration a matrix

$$\hat g(\mathbf{x}) \equiv i[\hat b(\mathbf{x}), \hat a(\mathbf{x})]. \tag{10}$$

The square brackets hereinafter designate a commutator of two matrices. The Hermitian matrix $\hat g(\mathbf{x})$ contains eight independent quantities by virtue of a relation $\operatorname{Sp}\hat g(\mathbf{x}) = 0$. The densities of quadrupole matrix $q_{\alpha\beta}(\mathbf{x})$ and spin $s_\alpha(\mathbf{x})$ are connected with this matrix by a relation

$$g_{\alpha\beta}(\mathbf{x}) \equiv q_{\alpha\beta}(\mathbf{x}) - i\varepsilon_{\alpha\beta\gamma}s_\gamma(\mathbf{x})/2. \tag{11}$$

Quadrupole matrix $q_{\alpha\beta}$ is symmetric and traceless $q_{\alpha\beta} = q_{\beta\alpha}$, $q_{\alpha\alpha} = 0$. Five its independent components can be parameterized as

$$q_{\alpha\beta} \equiv q(e_\alpha e_\beta - \delta_{\alpha\beta}/3) + q'(f_\alpha f_\beta - \delta_{\alpha\beta}/3). \tag{12}$$

Here q and q' are the modules of this matrix. The vectors $d_\alpha, e_\alpha, f_\alpha = (\mathbf{d} \times \mathbf{e})_\alpha$ form an orthonormal basis.

The quantities $G_{\alpha\beta} = \int d^3x\, g_{\alpha\beta}(\mathbf{x})$ are additive integrals of motion

$$\{G_{\alpha\beta}, H\} = 0, \tag{13}$$

if we take into account only exchange magnetic interactions.

The formulas (8)–(10) allow us to find the Poisson brackets for $\hat a$ and $\hat g$. It is easy to see, that these expressions are of the form

$$i\{g_{\alpha\beta}(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\} = (g_{\gamma\beta}(\mathbf{x})\delta_{\alpha\rho} - g_{\alpha\rho}(\mathbf{x})\delta_{\gamma\beta})\delta(\mathbf{x}-\mathbf{x}'),$$
$$i\{a_{\alpha\beta}(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\} = (a_{\gamma\beta}(\mathbf{x})\delta_{\alpha\rho} - a_{\alpha\rho}(\mathbf{x})\delta_{\gamma\beta})\delta(\mathbf{x}-\mathbf{x}'). \tag{14}$$

These Poisson brackets are consistent with hermiticity requirement of matrices $\hat a$, $\hat g$ and satisfy the Jacobi identity. Notice that $\{\det\hat a(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\} = 0$. Therefore, without restriction of a generality, because of linearity of the right-hand side of the Poisson brackets (14), it is possible to consider $\det\hat a = 1$, so that the matrix $\hat a$ contains eight independent quantities. Using (14) the dynamic equations for non-equilibrium magnetic states with spin 1 can be obtained. We shall consider the case, when Hamiltonian $H(\hat g)$ and equilibrium state of magnetic media possess SU(3) symmetry. The matrices $g_{\alpha\beta}(\mathbf{x})$ form subalgebra of the Poisson brackets (14) and contain two Kazimir invariants

$$g_2(\mathbf{x}) \equiv \operatorname{Sp}\hat g^2(\mathbf{x}), \qquad g_3(\mathbf{x}) \equiv \operatorname{Sp}\hat g^3(\mathbf{x}),$$
$$\{g_3(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\} = 0, \qquad \{g_2(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\} = 0. \tag{15}$$

The presence of such invariants decreases a number of independent matrix elements $g_{\alpha\beta}$ up to six in biaxial case and up to four in uniaxial case. Other possible case, when the Hamiltonian $H(\hat g, \hat a)$ has SU(3) symmetry and the symmetry of equilibrium state is spontaneously broken with respect to generators $G_{\alpha\beta}$, is not considered in this Letter. The Poisson brackets (14) for sixteen quantities allow us to describe the dynamics of media with spin 1 taking into account spin and quadrupole degrees of freedom.

In conclusion of this section we introduce the momentum of the considered magnets

$$P_k = \int d^3x\, \pi_k(\mathbf{x}), \quad \pi_k(\mathbf{x}) \equiv -\operatorname{Sp}\hat b(\mathbf{x})\nabla_k\hat a(\mathbf{x}). \tag{16}$$

It is evident that in virtue of (8), (9), (16) the following relations are true:

$$\{P_k, a_{\alpha\beta}(\mathbf{x})\} = \nabla_k a_{\alpha\beta}(\mathbf{x}), \quad \{P_k, b_{\alpha\beta}(\mathbf{x})\} = \nabla_k b_{\alpha\beta}(\mathbf{x}).$$

3. Differential conservation laws and models of exchange Hamiltonians

The basic interactions in magnetic systems carry exchange character. The consideration of dynamic processes in such media requires the formulation of conservation laws in differential form in view of Hamiltonian symmetry. In the case of SO(3) symmetry of Hamiltonian, $\{S_\alpha, H\} = 0$, the set of integrals of motion consists of Hamiltonian and spin moment $\gamma_a = H$, $S_\alpha = \int d^3x \zeta_a(\mathbf{x})$. Here $\zeta_a(\mathbf{x}) = e(\mathbf{x}), s_\alpha(\mathbf{x})$ are the densities of additive integrals of motion $(a = 0, \alpha)$. Using the representation of the flux densities of additive integrals of motion [14], we shall receive the dynamic equations reflecting conservation laws in differential form

$$\dot{e}(\mathbf{x}) = -\nabla_k q_k(\mathbf{x}),$$
$$q_k(\mathbf{x}) = \frac{1}{2}\int d^3x' x'_k \int_0^1 d\lambda \left\{e(\mathbf{x}+\lambda\mathbf{x}'), e(\mathbf{x}-(1-\lambda)\mathbf{x}')\right\},$$
$$\dot{s}_\alpha(\mathbf{x}) = -\nabla_k j_{\alpha k}(\mathbf{x}),$$
$$j_{\alpha k}(\mathbf{x}) = \int d^3x' x'_k \int_0^1 d\lambda \left\{s_\alpha(\mathbf{x}+\lambda\mathbf{x}'), e(\mathbf{x}-(1-\lambda)\mathbf{x}')\right\}, \quad (17)$$

where $q_k(\mathbf{x})$ is the energy flux density and $j_\alpha^k(\mathbf{x}) \equiv j_{\alpha k}(\mathbf{x})$ is the spin flux density. When obtaining the last equality we have used the symmetry of exchange energy density with respect to homogeneous spin rotations,

$$\left\{S_\alpha, e(\mathbf{x})\right\} = 0. \quad (18)$$

In the case of SU(3) symmetry of Hamiltonian, $\{G_{\alpha\beta}, H\} = 0$, the set of additive integrals of motion consists of Hamiltonian and matrix $G_{\alpha\beta}$: $\gamma_a = H$, $G_{\alpha\beta} = \int d^3x \zeta_a(\mathbf{x})$ and $\zeta_a(\mathbf{x}) = e(\mathbf{x}), g_{\alpha\beta}(\mathbf{x})$ are densities of additive integrals of motion $(a = 0, \alpha\beta)$. The equation of motion for the energy density and expression for its flux density (17) will not change. The energy density of exchange magnetic interactions is invariant with respect to homogeneous linear transformations

$$\left\{G_{\alpha\beta}, e(\mathbf{x})\right\} = 0. \quad (19)$$

Taking into account the last relation we obtain the differential conservation law

$$\dot{g}_{\alpha\beta}(\mathbf{x}) = -\nabla_k j^k_{\alpha\beta}(\mathbf{x}),$$
$$j^k_{\alpha\beta}(\mathbf{x}) = \int d^3x' x'_k \int_0^1 d\lambda \left\{g_{\alpha\beta}(\mathbf{x}+\lambda\mathbf{x}'), e(\mathbf{x}-(1-\lambda)\mathbf{x}')\right\}. \quad (20)$$

Here $j^k_{\alpha\beta}(\mathbf{x})$ is the flux density corresponding to conserved quantity $G_{\alpha\beta}$.

Besides the symmetry property (13), the exchange Hamiltonian is translationally invariant, $\{P_k, H\} = 0$, and invariant with respect to rotations in configuration space, $\{L_k, H\} = 0$, where L_k is the orbital momentum of the system. The relevant magnetic exchange energy density meets the following relationships:

$$\left\{P_k, e(\mathbf{x})\right\} = \nabla_k e(\mathbf{x}),$$
$$\left\{L_i(\mathbf{x}), e(\mathbf{x})\right\} = \varepsilon_{ikl} x_k \nabla_l e(\mathbf{x}). \quad (21)$$

The knowledge of relations (21) and flux densities in terms of work [14] allow us to formulate the conservation law of momentum density of magnons in differential form

$$\dot{\pi}_i(\mathbf{x}) = -\nabla_k t_{ik}(\mathbf{x}),$$
$$t_{ik}(\mathbf{x}) = -e(\mathbf{x})\delta_{ik} + \int d^3x' x'_k \int_0^1 d\lambda \left\{\pi_i(\mathbf{x}+\lambda\mathbf{x}'), e(\mathbf{x}-(1-\lambda)\mathbf{x}')\right\}. \quad (22)$$

The Heisenberg Hamiltonian has the form

$$H = \int d^3x\, e(\mathbf{x}) = -\int d^3x \int d^3x' J(|\mathbf{x}-\mathbf{x}'|) s_\alpha(\mathbf{x}) s_\alpha(\mathbf{x}'),$$

where $J(|\mathbf{x}-\mathbf{x}'|)$ is exchange integral of two-particle magnetic interaction. Accurate within terms square-law on spatial gradients of spin density we shall present expression of magnetic energy density corresponding to this Hamiltonian

$$e(\mathbf{x}) = -J s_\alpha(\mathbf{x}) s_\alpha(\mathbf{x}) + \frac{1}{2}\bar{J}\nabla_k s_\alpha(\mathbf{x})\nabla_k s_\alpha(\mathbf{x}), \quad (23)$$

where $J \equiv \int d^3x\, J(|\mathbf{x}|)$ and $\bar{J} \equiv \int d^3x\, x^2 J(|\mathbf{x}|)/3$, are effective exchange integrals of two-particle interaction. The first and second terms in (23) describe accordingly homogeneous and heterogeneous exchange interaction, and the functional kind of a homogeneous part of this energy is defined by Kazimir invariant.

Analytical form of SU(3) symmetric Hamiltonian we construct by analogy with Heisenberg Hamiltonian. Let's write magnetic Hamiltonian in such way that homogeneous part of energy density was expressed in terms of invariants g_2 and g_3:

$$H(g_2, g_3) = H(g_2) + H(g_3),$$
$$H(g_2) = 2\int d^3x \int d^3x' J(|\mathbf{x}-\mathbf{x}'|)\,\mathrm{Sp}\,\hat{g}(\mathbf{x})\hat{g}(\mathbf{x}'),$$
$$H(g_3) = -\int d^3x \int d^3x' \int d^3x'' I(|\mathbf{x}-\mathbf{x}'|, |\mathbf{x}-\mathbf{x}''|, |\mathbf{x}'-\mathbf{x}''|) \times \mathrm{Sp}\,\hat{g}(\mathbf{x})\hat{g}(\mathbf{x}')\hat{g}(\mathbf{x}''). \quad (24)$$

Here $J(|\mathbf{x}-\mathbf{x}'|)$ and $I(|\mathbf{x}-\mathbf{x}'|, |\mathbf{x}-\mathbf{x}''|, |\mathbf{x}'-\mathbf{x}''|)$ are exchange integrals of two- and three-particle magnetic interactions. The energy density corresponding to Hamiltonian $H(g_2)$ is of the form

$$e(\mathbf{x}, g_2) = 2Jg_2(\mathbf{x}) + \bar{J}\,\mathrm{Sp}\,\nabla_k\hat{g}(\mathbf{x})\nabla_k\hat{g}(\mathbf{x}). \quad (25)$$

The signs and coefficients in (25) are chosen so that in the absence of quadrupole degrees of freedom this expression has transformed to formula (23). The energy density corresponding to Hamiltonian $H(g_3)$ can be written in the form

$$e(\mathbf{x}, g_3) = -Ig_3(\mathbf{x}) + 2\bar{I}\,\mathrm{Sp}\,\hat{g}(\mathbf{x})\nabla_k\hat{g}(\mathbf{x})\nabla_k\hat{g}(\mathbf{x}). \quad (26)$$

It also meets the SU(3) symmetry relation (19). Here

$$I \equiv \int d^3x d^3x' I(|\mathbf{x}|, |\mathbf{x}'|, |\mathbf{x}-\mathbf{x}'|),$$
$$\bar{I} \equiv \int d^3x d^3x' (x^2 + x'^2 - \mathbf{x}\mathbf{x}') I(|\mathbf{x}|, |\mathbf{x}'|, |\mathbf{x}-\mathbf{x}'|)/3$$

are the effective exchange integrals of homogeneous and heterogeneous three-particle magnetic interaction.

4. Dynamics of spin-1 magnets with SU(3) symmetry

The considered magnetic media is characterized by exchange energy density, which is function of matrix $g_{\alpha\beta}$ and its gradient, $e(\mathbf{x}) = e(\hat{g}(\mathbf{x}), \nabla_k\hat{g}(\mathbf{x}))$. The basic thermodynamic relation is given by

$$de = \mathrm{Sp}\,\frac{\partial\hat{e}}{\partial g}d\hat{g} + \mathrm{Sp}\,\frac{\partial\hat{e}}{\partial\nabla_k g}d\nabla_k\hat{g} + \frac{\partial e}{\partial s}ds.$$

Here s is the entropy density. In accordance with formulas (14), the Poisson brackets of arbitrary functionals are determined by

$$\{A, B\} \equiv i \int d^3x\, d^3x' \operatorname{Sp} \hat{g}(\mathbf{x}) \left[\frac{\delta \hat{B}}{\delta g(\mathbf{x})}, \frac{\delta \hat{A}}{\delta g(\mathbf{x}')} \right]. \tag{27}$$

Using (1), (27) we obtain the dynamic equation of magnets with spin $s = 1$,

$$\dot{\hat{g}}(\mathbf{x}) \equiv i \left[\hat{g}(\mathbf{x}), \frac{\delta \hat{H}}{\delta g(\mathbf{x})} \right]. \tag{28}$$

It represents a generalization of Landau–Lifshitz equation on magnetic systems with SU(3) symmetry. Taking into account the symmetry property of exchange interactions for the energy density (19) we obtain a relation

$$\left[\frac{\partial \hat{e}}{\partial g}, \hat{g} \right] + \left[\frac{\partial \hat{e}}{\partial \nabla_k g}, \nabla_k \hat{g} \right] = 0.$$

Bearing in mind this equality and employing (17), (20), (14), one can easily derive the expressions for the flux densities of additive integrals of motion,

$$\hat{j}_k = i \left[\hat{g}, \frac{\partial \hat{e}}{\partial \nabla_k g} \right], \qquad q_k = \operatorname{Sp} \frac{\delta \hat{H}}{\delta g} \hat{j}_k. \tag{29}$$

Eqs. (17), (20) and the flux densities (29) describe the dynamics of spin-1 magnets in adiabatic approximation for Hamiltonian with SU(3) symmetry. It is easy to see that the structure of fluxes (see (29)) provides the validity of equations $\dot{g}_2 = \dot{g}_3 = 0$ for Kazimir invariants.

The symmetry properties (19), (21) allow us to formulate the dynamic equation for momentum density of magnons. According to (1), (8), (9) we come to the following equations of motion for canonically conjugate matrices $\hat{a}$ and $\hat{b}$:

$$\dot{\hat{a}}(\mathbf{x}) = \frac{\delta \hat{H}}{\delta b(\mathbf{x})}, \qquad \dot{\hat{b}}(\mathbf{x}) = -\frac{\delta \hat{H}}{\delta a(\mathbf{x})}.$$

The Hamiltonian of the states under consideration depends on these matrices by means of matrix $\hat{g}$: $H(\hat{b}, \hat{a}) = H(\hat{g}(\hat{b}, \hat{a}))$. Therefore, in virtue of relation (10), the last two equations can be written in the form:

$$\dot{\hat{a}}(\mathbf{x}) = i \left[\hat{a}(\mathbf{x}), \frac{\delta \hat{H}}{\delta g(\mathbf{x})} \right], \qquad \dot{\hat{b}}(\mathbf{x}) = i \left[\hat{b}(\mathbf{x}), \frac{\delta \hat{H}}{\delta g(\mathbf{x})} \right].$$

Bearing in mind the definition (16) and formula (22), we find the dynamic equation (22) for the momentum density of magnons, where the flux density of momentum takes the form

$$t_{ik} = \delta_{ik} \left(-e + \operatorname{Sp} \frac{\delta \hat{H}}{\delta g} \hat{g} \right) + \operatorname{Sp} \frac{\partial \hat{e}}{\partial \nabla_k g} \nabla_i \hat{g}.$$

The further analysis of Eqs. (20) we shall carry out using the model expression (25) for the energy density $e(g_2)$. It is easy to transform the nonlinear equation (28) to equation for quadrupole matrix and antisymmetric matrix $\varepsilon_{\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma} s_\gamma$,

$$\dot{\hat{q}} = \bar{J}[\Delta\hat{\varepsilon}, \hat{q}] + \bar{J}[\Delta\hat{q}, \hat{\varepsilon}], \qquad \dot{\hat{\varepsilon}} = 4\bar{J}[\hat{q}, \Delta\hat{q}] + \bar{J}[\Delta\hat{\varepsilon}, \hat{\varepsilon}]. \tag{30}$$

After linearization of these equations near equilibrium state $(\hat{\varepsilon}_0)_{\alpha\beta} = 0$, $(\hat{q}_0)_{\alpha\beta} \neq 0$ (T-even states, spin nematic) and Fourier transform we obtain the dispersion equation,

$$\det \hat{D}(\mathbf{k}, \omega) = 0,$$
$$D_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta}\left(\omega^2 - 8\bar{J}^2 k^4 \operatorname{Sp}(\hat{q}_0^2)\right) + 12\bar{J}^2 k^4 \left(\hat{q}_0^2\right)_{\alpha\beta}.$$

In the uniaxial case we find the solutions $\omega = 0$ and $\omega = \pm 2\bar{J}k^2 q_0$. For equilibrium biaxial quadrupole matrix the solution of dispersion equation leads to the following three spectra of quadrupole waves:

$$\omega_{\pm}^{(1)} = \pm 2\bar{J}k^2 q_0, \qquad \omega_{\pm}^{(2)} = \pm 2\bar{J}k^2 q_0',$$
$$\omega_{\pm}^{(3)} = \pm 2\bar{J}k^2 \left| q_0 - q_0' \right|.$$

Here q_0 and q_0' are modules of this quadrupole matrix in equilibrium state determined by relation (12). Linearization of Eqs. (20), (19) near equilibrium state $(\hat{\varepsilon}_0)_{\alpha\beta} \neq 0$, $(\hat{q}_0)_{\alpha\beta} \equiv 0$ (T-odd states, ferromagnetic) leads to the spectra of spin and quadrupole waves,

$$\omega_{\pm}^{(1)} = 0, \qquad \omega_{\pm}^{(2)} = \pm 2\bar{J}k^2 s_0, \qquad \omega_{\pm}^{(3)} = \pm \bar{J}k^2 s_0.$$

The parabolic dependence of frequency from wave vector in these spectra is in agreement with results of work [15].

Similarly we can consider the dynamic processes with magnetic Hamiltonian $H(g_3)$. Then we obtain dispersion equation when $(\hat{\varepsilon}_0) = 0$, $(\hat{q}_0) \neq 0$:

$$\delta s_\beta(\mathbf{k}, \omega) D_{\beta\alpha}(\mathbf{k}, \omega) = 0,$$
$$D_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta}\left(\omega^2 - 4\bar{I}^2 k^4 \left(2\operatorname{Sp}(\hat{q}_0^4) - \left(\operatorname{Sp}(\hat{q}_0^2)\right)^2\right)\right) + 4\bar{I}^2 k^4 \left[3(\hat{q}_0^4) - 2(\hat{q}_0^2)\operatorname{Sp}(\hat{q}_0^2)\right]_{\alpha\beta}.$$

If the state of spin nematic is uniaxial, then we come to two spectra of quadrupole waves: $\omega = 0$, $\omega = \pm(2/3)\bar{I}k^2 q_0^2$. In biaxial case of quadrupole matrix the excitation spectra are given by

$$\omega_{\pm}^{(1)} = \pm 2\bar{I}k^2 \sqrt{\left(q_0^2 - q_0'^2\right)^2 + 4q_0 q_0' \left(q_0^2 + q_0'^2\right)}/3,$$
$$\omega_{\pm}^{(2)} = \pm 2\bar{I}k^2 \sqrt{q_0' \left(q_0'^3 + 4q_0^3 + 4q_0^2 q_0'\right)}/3,$$
$$\omega_{\pm}^{(3)} = \pm 2\bar{I}k^2 \sqrt{q_0 \left(q_0^3 + 4q_0'^3 + 4q_0'^2 q_0\right)}/3.$$

Qualitatively they are similar to the case with Hamiltonian $H(g_2)$ but the spectral dependence of modules of quadrupole matrix is more complicated.

In conclusion, using the variational principle, we have obtained the Poisson brackets and nonlinear equations of motion for the set of parameters, which macroscopically complete characterize the local equilibrium state of magnets with spin $s = 1$. We have introduced two types of magnetic exchange Hamiltonians corresponding to two Kazimir invariants of SU(3) group. The spectra of spin and quadrupole waves are found and their explicit form depending on property of invariance of equilibrium state relatively to time reversion is established.

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References

[1] L.D. Landau, E.M. Lifshitz, Phys. Z. Sowietunion 8 (1935) 153.
[2] V.S. Ostrovskii, Zh. Eksp. Teor. Fiz. 91 (1986) 1690, Sov. Phys. JETP 64 (1986) 999.
[3] V.M. Loktev, V.S. Ostrovskii, Fiz. Nizk. Temp. 20 (1994) 983, Low Temp. Phys. 20 (1994) 775.
[4] B.A. Ivanov, A.K. Kolezhuk, Phys. Rev. B 68 (2003) 052401.
[5] A.A. Isayev, M.Yu. Kovalevskii, S.V. Peletminskii, Fiz. Met. Metalloved. 77 (1994) 20 (in Russian).
[6] M. Nauciel-Bloch, G. Sarma, A. Castets, Phys. Rev. B 5 (1972) 4603.

[7] E.L. Nagaev, Usp. Fiz. Nauk 136 (1982) 61, Sov. Phys. Usp. 25 (1982) 31.
[8] A.F. Andreev, I.A. Grishchuk, Zh. Eksp. Teor. Fiz. 87 (1984) 467, Sov. Phys. JETP 60 (1984) 267.
[9] N. Papanicolaou, Nucl. Phys. B 305 (1988) 367.
[10] G. Fath, J. Solyom, Phys. Rev. B 51 (1995) 3620.
[11] O.R. Baran, R.R. Levitskii, Phys. Rev. B 65 (2002) 172407.
[12] M. Keskin, A. Erdinc, J. Magn. Magn. Mater. 283 (2004) 392.
[13] T. Grover, T. Senthil, Phys. Rev. Lett. 98 (2007) 247202.
[14] A.A. Isayev, M.Yu. Kovalevskii, S.V. Peletminskii, Fiz. Elem. Chastits At. Yadra 27 (1996) 431, Phys. Part. Nucl. 27 (1996) 179.
[15] V.I. Butrim, B.A. Ivanov, A.S. Kuznetsov, R.S. Khymyn, Fiz. Nizk. Temp. 34 (2008) 1266, Low Temp. Phys. 34 (2008) 997.