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Formalism of classical Cartan′ mechanics as a basis to construct unified field theory

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Eearlier the theory has been developed describing the motion of a particle and its interaction with physical fields on the basis of the symplectic metric of extended phase space-time [1, 2]. The corresponding field-equations are obtained in the framework of the formalism of the Kalutsa fifth-dimensional theory. It is evident that such a theory is not fully consecutive Here a more general approach is developed when not only equations of motion and interaction of particles but also the gravitational fields follow from the symplectic metric.

Let us consider a charged particle of the mass m in the curved space-time which interacts with the electromagnetic field. The phase space of such a particle is $R^4 \times R^4$. We write the symplectic metric of this space in the vector form

$$\delta\omega = (d\vec{p},\ \Lambda d\vec{r}) - dH \Lambda d\tau + \frac{e}{2c} F\,. \tag{1}$$

Here Λ is the outer product, e is the charge, c is the light speed. symbol (..., ...) denotes the arbitrary scalar product in a tangent stratification of the space-time, $H = (\vec{p},\ \vec{p})/2m$, $F = F_{ik} dx^i \Lambda dx^k$ is the form of elctromagnetic field which is rewritten as follows $F = (\vec{r},\ \Lambda \hat{F} d\vec{r})$, where $\hat{F}$ is the linear operator of the tangent stratification. We denote the value $\hat{F}$ as the operator of electromagnetic field.

With accordance to the Cartan ideas to give the geometry of manifold we introduce a space-time geometry, i.e. the curvature and torsion by using the equations of Cartan structure [4]

$$d\vec{r} = \vec{e}_i dx^i , \quad d\vec{e}_i = \omega_i^k \vec{e}_k . \tag{2}$$

Here ω_i^k is the conectedness one-form which gives translation in the space-time. Differentiating the form (2) yields the geometric characteristics of space-time, i.e. the curvature and torsion [4]

$$d^2\vec{r} = \Omega^i \vec{e}_i , \quad d^2\vec{e}_i = (d\omega_i^p - \omega_i^k \Lambda \omega_k^p)\vec{e}_p \equiv \Omega_i^j \vec{e}_j . \tag{3}$$

Here ω^i and ω_j^i are the forms of torsion and curvature of the space-time. On the basis of the symplectic metric (1) we obtain Hamilton equations

$$\frac{\partial \delta\omega}{\partial \delta\vec{p}} = 0 , \quad \frac{\partial \delta\omega}{\partial \delta\vec{r}} = 0 , \quad \frac{\partial \delta\omega}{\partial \delta\vec{\tau}} = 0 , \tag{4}$$

which, taking into account (1) are reduced to the form

$$d\vec{r} - \frac{\vec{p}}{m} d\tau = 0 , \quad -d\vec{p} + \frac{e}{2c}\frac{\partial F}{\partial d\vec{r}} = 0 , \quad dH = 0 . \tag{5}$$

where

$$\frac{\partial F}{\partial d\vec{r}} = \frac{\partial (d\vec{r}, \Lambda \hat{F} d\vec{r})}{\partial d\vec{r}} = 2\hat{F} d\vec{r} = 2\hat{F} e_i dx^i = 2F_k^i dx^k e_i ; \quad d\vec{p} = dp^i \vec{e}_i + p^i \omega_i{}^k \vec{e}_k .$$

Finally we obtain

$$dx^i - \frac{p^i}{m} d\tau = 0 , \quad dp^k + p^i \Gamma_{i\,l}^{\,k} dx^l = \frac{e}{c} F_l^k dx^l , \quad dH = 0 \tag{6}$$

or

$$p^i = m\frac{dx^i}{d\tau} , \quad m\left(\frac{d^2x^k}{d\tau^2} + \Gamma_{il}^k \frac{dx^i}{d\tau}\frac{dx^l}{d\tau}\right) = \frac{e}{c} F_l^k \frac{dx^l}{d\tau} , \quad dH = 0 . \tag{7}$$

Here Γ_{il}^k are the Christoffel symbols, i.e. the components of connectedness forms $\omega_i^k - \Gamma_{i\,l}^{\,k}$. As a result we have obtained the classical equations of the motion in the curved space-time for a particle interacting with the electromagnetic field [5].

Unlike previous works [1-3] we suppose that the space-time is not the Riemann space but a space with affine connectedness, i.e the geometry is given not by the metric tensor g_{ik} but the forms of curvature and torsion Ω_i^j and Ω^i.

Now, let us obtain the equations which extend well-known Einstein equations for the curvature of the space-time as the Riemann space. These equations are originated in our approach as the conditions for curvature and torsion which follow from that the form $d\omega$ is symplectic metric, and hence $d\delta\omega = 0$, from which it follows

$$\begin{aligned} d\delta\omega &= d(d\vec{p},\ \Lambda d\vec{r}) + \frac{e}{2c}dF = (d^2\vec{p},\ \Lambda d\vec{r}) - (d\vec{p}, \Lambda d^2\vec{r}) + \frac{e}{2x}dF = \\ &= (R(\vec{p}),\ \Lambda d\vec{r}) - (d\vec{p},\ \Omega^i\vec{e}_i) + \frac{e}{2c}dF = 0\,, \end{aligned} \tag{8}$$

where

$$R(\vec{p}) \equiv d^2\vec{p} = p^i d^2\vec{e}_i = \Omega^j{}_i p^i \vec{e}_j\,.$$

The equation (7) leads to the following equations for **the curvature** form Ω_i^j, torsion form ω^i and electromagnetic field form F

$$(R(\vec{p}),\ \Lambda d\vec{r}) = 0\,, \quad (d\vec{p},\ \ \Lambda\Omega^i\vec{e}_i) = 0\,,\ \ dF = 0\,. \tag{9}$$

therefore, $\Gamma_i{}^k{}_l dx^l \Lambda dx^i = 0$, and hence $\Gamma_i{}^k{}_l = \Gamma_l{}^k{}_i$.

Therefore, the condition of symplecticity of the metric $\delta\omega$ for electromagnetic field gives a pair of Maxwell equations $dF = 0$. As a solution of the equation (9) for the torsion form one can consider the absence of the space-time torsion $\Omega^i \equiv 0$ that gives the condition well-khown from the Einstein theory for the Christoffel symbols $\Gamma_i{}^k{}_l$ [5]. Indeed.

$$d^2\vec{r} = \Omega^i\vec{e}_i = 0 = d(\vec{e}_i dx^i) = d\vec{e}_i \Lambda dx^i = \omega_i{}^k \Lambda dx^i \vec{e}_k\,,$$

therefore, $\Gamma_i{}^k{}_l dx^l \Lambda dx^i = 0$ and hence $\Gamma_i{}^k{}_l = \Gamma_l{}^k{}_i$. At the same time, the equation for the curvature form Ω_i^j gives the searching extension of the Einstein equation for the tensor of space-time curvature. Indeed,

$$(R(\vec{p}),\ \Lambda d\vec{r}) = 0 = (R(\vec{p}) - \frac{1}{4}T(\vec{p}, e_s)dx^s \Lambda d\vec{r}. \Lambda d\vec{r})\,.$$

Here the following equality is taken into account

$$(d\vec{r}.\ \Lambda d\vec{r})(\vec{e}_i, \vec{e}_k)dx^i \lambda dx^k \equiv 0$$

Taking into account that the vector $\vec{p}$ is arbitrary yields the final condition for the torsion form

$$R(\ \ .) = \frac{1}{4}T(..\ \ , e_s)dx^s \Lambda d\vec{r} = \frac{1}{4}T(..\ \ , d\vec{r})\Lambda d\vec{r}\,. \tag{10}$$

Here the tensor T is arbitrary. The transition to components of tensor R and T leads to

$$R^p_{\ ksn} = p^k dx^s \Lambda dx^n = \frac{1}{4} T_{ks} p^k \delta^p_{\ n} dx^s \Lambda dx^n .$$

Hence

$$R^p_{\ ksn} dx^s \Lambda dx^n = \frac{1}{4} T_{ks} \delta^p_{\ n} dx^s \Lambda dx^n$$

or

$$R^p_{ksn} = \frac{1}{4} T_{ks} \delta^p_n . \tag{11}$$

Here R^p_{ksn} are the components of the curvature vector $\Omega^p_k = R^p_{ksn} dx^s \Lambda dx^n$, i.e. the tensor of one-form of the Riemann curvature. It gives directly the equation for the curvature form

$$\Omega^p_k = \frac{1}{4} T_{ks} \delta^p_n dx^s \Lambda dx^n . \tag{12}$$

The equations (10), (11) are the extension of Einstein equation. Indeed, it follows from (10)

$$R^p_{ksp} = T_{ks} , \tag{13}$$

i.e. the Einstein equation [5]. The concrete form of the tensor T_{ks} for different forms of the matter in the framework of the theory presented is not determined.

It follows from above that the closed system of equations of classical physics can be derived from the condition of symplecticity of the metric $\delta\omega$: $\delta\omega = 0$ and the additional condition on the symplectic metric $^*d^*$: $\delta\omega\big|_{R^4} = \frac{4p}{c}$ (where J is the one-form of electric current δ, * is the operator of form conjugation), which gives the second pair of the Maxwell equations. In this case the demand of invariance of the metric $\delta\omega$ under the action of one-parameter group of diffeomorphysm gives the Hamilton equations of motion (4).

Finally the full system of the equations of classical physics is of the form

$$d\delta\omega = 0 , \qquad \frac{\partial \delta\omega}{\partial d\vec{p}} ,$$

$$^*d^*\delta\omega\big|_{R^4} = \frac{4p}{c} - J , \qquad \frac{\partial \delta\omega}{\partial d\vec{p}} ,$$

$$\delta\omega = (d\vec{p},\ \Lambda d\vec{r}) + \frac{e}{2c} \boldsymbol{F} - dH \Lambda d\tau$$

References

[1] MENDYGULOV JU.D., MOSKOVKIN U.M. & SELEZOV I.T. *Some examples of applications of geometric Cartan principles in classical physics* (In Russian). - Asymptotic Integration of Nonlinear Equations, Kiev: Institute of Mathematics, NANU, 1992. – P. 81-86.

[2] MENDYGULOV JU.D., MOSKOVKIN V.M. & SELEZOV I.T. *On some extensions of the conceptions of physical properties of time* (In Russian). – Hydromechanics, 1994, 68. – P. 8-12.

[3] MENDYGULOV JU.D., MOSKOVKIN V.M. & SELEZOV I.T. *Physical-theoretical interpretation of phenomena in extended phase space* (In Russian). – Hydromechanics, 1994, 69. – P. 42-46.

[4] CARTAN E. *Geometry of Riemann spaces (Russian edition)* - Moscow-Leningrad, USSR, 1936.

[5] EINSTEIN A. *Collection of scientific papers.* Vol.1., (Russian edition) – M.: Nauka, 1965.