On Dirichlet-Type Problems for the Lavrent'ev–Bitsadze Equation

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Abstract—The existence and uniqueness issues are discussed for several boundary value problems with Dirichlet data for the Lavrent'ev–Bitsadze equation in a mixed domain. A general mixed problem (according to Bitsadze's terminology) is considered in which the Dirichlet data are relaxed on a hyperbolic region of the boundary inside a characteristic sector with vertex on the type-change interval. In particular, conditions are pointed out under which the problem is uniquely solvable for any choice of this vertex.

Let D be a domain of the complex z-plane, $z = x + iy$, that is bounded for $y > 0$ and $y < 0$ by Lyapunov arcs σ and γ with endpoints $z = 0$ and $z = 1$. Suppose that at their endpoints these arcs do not form cusps with the segment $J = [0, 1]$ of the real axis. In this domain we consider the Lavrent'ev–Bitsadze equation

$$(\operatorname{sgn} y)u_{xx} + u_{yy} = 0. \tag{1}$$

It is assumed that the angles $\theta_k^{\pm}$, $k = 0, 1$, of the domains $D^{\pm} = D \cap \{\pm y > 0\}$ at the points $z = k$ are positive and that the arc γ is not tangent to the characteristics $x \pm y = \text{const}$ of equation (1). In particular, $0 < \theta_k^+ \le \pi$, $0 < \theta_k^- < \pi/4$, and the domain D^- lies inside the characteristic triangle with base J.

By a solution to equation (1) in the domain D we mean a function u that is harmonic in D^+, admits, together with its harmonic conjugate v, the limits

$$u^+(x) = u(x, +0), v^+(x) = v(x, +0) \in C(0, 1), \tag{2}$$

and can be represented in the domain D^- by the d'Alembert formula

$$2u(x, y) = (v + u)^+(x + y) - (v - u)^+(x - y). \tag{3}$$

Under the assumption that $u^+, v^+ \in C^2(0, 1)$, the function u belongs to $C^1(D^-)$ and is a classical solution to the string equation in the domain D^-. Note that if the function u^+ is locally Hölder continuous on the interval $(0, 1)$, then it follows from the Privalov theorem [1] that the harmonic conjugate function v has boundary values (2) with the same property. Therefore, the condition on v in the above definition of a solution to equation (1) can be omitted.

Below, we consider solutions u to equation (1) in the class of functions that are locally Hölder continuous in $\overline{D} \setminus \{0, 1\}$ and exhibit the power behavior

$$u(z) = O(1)|z|^{\lambda_0}|1 - z|^{\lambda_1}, \qquad \lambda_k > -\delta_k. \tag{4}$$

Here $\delta = \delta_k$ is the first positive root of the equation $A(\delta\theta_k^+) + \delta A(\theta_k^-) = 0$, where $A(x) = \operatorname{arcoth}(\tan x)$. The function $A(x)$ is defined and π-periodic on the intervals $|x - \pi k| < \pi/4$, on each of which this function increases from $-\infty$ to $+\infty$. In particular, $3\pi/4 < \theta_k^+\delta_k < \pi$.

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In what follows, by $O(1)$ in (4) we will mean the class $H = H(\overline{D})$ of Hölder continuous functions. It is also convenient to use the class $\mathring{H} = \mathring{H}(\overline{D}; \tau_1, \dots, \tau_n)$ of functions $u \in H$ that vanish at the points τ_j. We also introduce a subspace $\mathring{H}^{(1)} \subseteq \mathring{H}$ distinguished by the conditions $\rho u_x, \rho u_y \in \mathring{H}$, where $\rho(z) = |z - \tau_1| \dots |z - \tau_n|$. Below, the role of τ_j is mainly played by the points $z = 0$, $z = 1$, and $z = \tau$, $0 < \tau < 1$.

For example, using this notation, we can express condition (4) for the function u by the formula $u(z) = u_0(z)|z|^{-\delta_0}|1 - z|^{-\delta_1}$, $u_0 \in \mathring{H}(\overline{D}; 0, 1)$. We also use similar classes for functions defined on curves.

We will say that a domain on the plane is *convex with respect to a family of curves* if each curve of this family intersects the domain along a connected set (which may be empty).

Theorem 1 (Soldatov [2–4]). *Suppose that the domain D^- is convex with respect to the pencil of straight lines passing through the point $z = 0$, $0 < \lambda_k < \delta_k$ are fixed for $k = 0, 1$, and a function f is defined such that*

$$f(t) = f_0(t)|t|^{\lambda_0}|1 - t|^{-\lambda_1}, \qquad f_0 \in \mathring{H}(\sigma \cup \gamma; 0, 1).$$

Then the Dirichlet problem

$$u\big|_{\sigma \cup \gamma} = f \tag{5}$$

for equation (1) *is uniquely solvable in the class of functions*

$$u(z) = u_0(z)|z|^{\lambda_0}|1 - z|^{-\lambda_1}, \qquad u_0 \in \mathring{H}(\overline{D}; 0, 1).$$

If, in addition, $f_0 \in \mathring{H}^{(1)}(\sigma \cup \gamma; 0, 1)$, then the function u_0 also belongs to $\mathring{H}^{(1)}(\overline{D}; 0, 1)$.

A similar assertion holds for the point $z = 1$ (in this case the signs of the exponents λ_0 and $-\lambda_1$ should be interchanged).

Note that the second assertion of the theorem turns into the first one under the change of variables $x' = 1 - x$, $y' = y$, which preserves equation (1).

We can reformulate problem (1), (5) in terms of the harmonic function u in D^+ by "carrying over" the boundary condition from γ to $J = [0, 1]$. To this end, we express the curve γ in the characteristic coordinates by the equation $x + y = \alpha(x - y)$. According to the assumptions made about γ, the function α is defined and continuously differentiable on the interval J; moreover, $0 < t < \alpha(t) < 1$ for $0 < t < 1$, $\alpha(0) = 0$, $\alpha(1) = 1$, and the derivative $\alpha'(t)$ is positive for $0 \le t \le 1$. The values $\alpha'(0) < 1$ and $\alpha'(1) > 1$ are related to the angles θ_0^- and θ_1^- of the domain D^-. Denote the inverse transformation α^{-1} by β.

Using the notation adopted and taking into account (2), we can rewrite the boundary condition (5) as

$$u\big|_{\sigma} = f, \tag{6}$$

$$(v + u)^+ \circ \alpha - (v - u)^+ = f_0, \tag{7_0}$$

where $f_0(x - y) = 2f(x + iy)$, $x + iy \in \gamma$. The latter boundary condition can also be written in the equivalent form

$$(v + u)^+ - (v - u)^+ \circ \beta = f_1, \tag{7_1}$$

where $f_1(x + y) = 2f(x + iy)$, $x + iy \in \gamma$.

It is easy to see that the domain D^- is convex with respect to the pencil of straight lines passing through the point $z = 0$ if and only if the following inequality holds:

$$\alpha'(t) \ge \frac{\alpha(t)}{t}, \qquad 0 \le t \le 1. \tag{8_0}$$

Indeed, in the characteristic variables $s = x + y$ and $t = x - y$, the domain D^- turns into $G = \{(s,t) \mid 0 < s < \alpha(t),\ 0 < t < 1\}$. Inequality (8_0) is equivalent to the fact that the function $\ln[\alpha(t)/t]$, as well as $\alpha(t)/t$, monotonically increases. In particular,

$$\frac{\alpha(t)}{t} \le \frac{\alpha(t_0)}{t_0}, \qquad t < t_0.$$

Geometrically, this fact means that the graph of the function $\alpha(t)$, $0 \le t \le t_0$, lies above the secant line passing through the points $(0,0)$ and $(t_0, \alpha(t_0))$. Therefore, the intersection of this line with G is connected.

A similar property of convexity with respect to the point $z = 1$ is expressed by the inequality

$$\alpha'(t) \le \frac{1-\alpha(t)}{1-t}, \qquad 0 \le t \le 1. \tag{8_1}$$

The convexity of the domain D^- in the conventional sense is equivalent to the fact that the function α' monotonically increases. In this case, both conditions (8) are satisfied automatically.

If $f \in H(\sigma \cup \gamma)$, then, according to Theorem 1, there exists a unique solution $u \in C(\overline{D} \setminus \{1\})$ to problem (1), (5) that has a logarithmic singularity at the point $z = 1$, as well as a unique solution with a similar property with respect to the point $z = 0$. However, as was first shown by Bitsadze [5], this problem is strongly overdetermined in the class $C(\overline{D})$, and the Dirichlet data should be relaxed on a certain part of the arc γ.

Let us fix a point $0 < \tau < 1$ and emit two characteristics $x \pm y = \tau$ from it into the domain D^-. These characteristics partition D^- into subdomains D_0^-, D_1^-, and D_τ^-. Similarly γ is partitioned into arcs γ_0^-, γ_1^-, and γ_τ^-. The domains D_0^- and D_1^- are based on the segments $J_0 = [0,\tau]$ and $J_1 = [\tau, 1]$ of the real axis, respectively, and the boundary of the domain D_τ^- is composed of the arc γ_τ and the segments $l_k \subseteq \partial D_k^-$, $k = 0, 1$, of the characteristics. If we again emit the characteristics from the endpoints of these segments lying on γ and continue this process, then we obtain a polygonal chain $L \subseteq \overline{D^-}$ with an infinite number of segments that converge to the points $z = 0$ and $z = 1$. Let $n = (n_1, n_2)$ be the unit outward normal to the boundary of the domain D. By a *conormal* we mean a vector ν with components $\nu_1 = \operatorname{sgn} n_1$ and $\nu_2 = n_2$. Since the curve γ has no characteristic directions, the conormal is not tangent to γ.

Let us introduce a mixed domain $D(\tau)$ such that $D^+(\tau) = D^+$ and $D^-(\tau) = D_0^- \cup D_1^-$ and consider the following two Dirichlet problems for equation (1) in this domain:

$$u\big|_{\sigma \cup \gamma_0 \cup \gamma_1} = f, \tag{5^+}$$

$$u\big|_{\sigma \cup \gamma_0 \cup \gamma_1} = f, \qquad u\big|_{l_0 \cup l_1} = 0. \tag{5^-}$$

Of course, within the same class of functions problem (5^-) is overdetermined compared to (4), and the latter is overdetermined compared to (5^+). We seek a solution to problem (5^+) in the class H and a solution to problem (5^-) in the class of functions (4). In the case of the first problem, we can assume without loss of generality that the right-hand side $f \in H$ vanishes at the points $z = 0$ and $z = 1$; i.e., it belongs to $\overset{\circ}{H}(\overline{D}; 0, 1)$. As for the second problem, according to (5^-), the function f should vanish at the common endpoints z_k of the arcs γ_k and l_k.

Note that problems $(5^\pm)$ can also be considered in the entire mixed domain D, because the function u can always be extended from $D(\tau)$ to D as a solution to the Goursat problem in the domain D_τ^-. In the case of the lower sign, this function is naturally extended by zero.

Theorem 2 (Soldatov [2–4]). *Suppose that the domain D^- is convex with respect to the pencils of straight lines passing through the points $z = 0$ and $z = 1$. Then problem (5^-) is always solvable in the class (4), and the homogeneous problem has exactly one linearly independent solution $u_- \in C^1(\overline{D} \setminus L)$. Moreover, the product of the conormal derivative $\partial u_-/\partial \nu$ on $\sigma \cup \gamma_0 \cup \gamma_1$ and any function in the class $\overset{\circ}{H}(\sigma \cup \gamma; 0, 1)$ is integrable.*

Now we turn to problem (5^+) and consider it first in the codimension 1 subspace $\overset{\circ}{H}(\overline{D};0,1)$ distinguished by the condition

$$v(0) = v(1) = 0 \tag{9}$$

imposed on the harmonic conjugate v of the function u in the domain D^+ (obviously, v is Hölder continuous in $\overline{D^+}$).

Theorem 3 (Soldatov [2–4]). *Suppose that the domain D^- is convex with respect to the pencil of straight lines passing through the points $z = 0$ and $z = 1$. Then the homogeneous problem (5^+) has only the zero solution in the class of functions $u \in \overset{\circ}{H}(\overline{D};0,1)$ satisfying condition (9), while the inhomogeneous problem is solvable in this class if and only if*

$$\int\limits_{\sigma\cup\gamma_0\cup\gamma_1} f\frac{\partial u_\tau}{\partial \nu}|dt| = 0. \tag{10}$$

In particular, there are only two possibilities for this problem in the entire class $\overset{\circ}{H}(\overline{D};0,1)$:

(i) *problem (5^+) is uniquely solvable*;

(ii) *the homogeneous problem has one linearly independent solution u_+, and condition (10) is necessary and sufficient for the solvability of the inhomogeneous problem.*

This theorem is, in a sense, of conditional nature, and the question of which of these two alternatives holds requires an additional analysis. Note also that one can apply Theorem 1 in order to describe the behavior of the solution u to problem (5^+) and its derivative in the neighborhood of the points $z = 0$ and $z = 1$.

The question of whether problem (5) is well posed in the class H was intensively discussed (especially among mechanicians) in the mid-1950s. As pointed out above, this question was solved by Bitsadze [5]. In that paper, Bitsadze established that under certain conditions of geometrical character imposed on the domain $D(\tau)$ and on the choice of the point τ, the Dirichlet problem (5) is overdetermined in the class H and problem (5^+) is uniquely solvable in this class. These conditions are as follows: the inequality

$$\mathrm{Im}[\, z(1-z)(\tau - z)\overline{z'(s)}] \geq 0 \tag{11}$$

should hold on the arc σ, where $z = z(s)$, $0 \leq s \leq l$, is the natural parametric equation of this arc ($z(0) = 1$, $z(l) = 0$), and there should exist a unique point $x_0 + iy_0 \in \gamma$ with the minimum ordinate such that, except for this point, the horizontal straight lines are not tangent to γ. In this case, one chooses the point τ inside the interval $(x_0 + y_0, x_0 - y_0)$, which guarantees the convexity of the domains D_0^- and D_1^- with respect to the horizontal lines.

Note that in terms of the translations α and $\beta = \alpha^{-1}$ appearing in (7), the convexity of the domains D_k^-, $k = 0, 1$, is equivalent to the inequalities

$$\alpha'(t) \leq 1, \quad 0 \leq t \leq \tau, \qquad \beta'(t) \leq 1, \quad \tau \leq t \leq 1. \tag{12}$$

Later, the Bitsadze requirements were slightly relaxed [6].

Theorem 4. *Suppose that the curve σ is such that*

$$0 \leq \arg z'(s) \leq 2\pi, \tag{13}$$

where a continuous branch of the argument is fixed by the condition $\arg z'(0) = \pi - \theta_1^+$ and the notation (11) is used. Suppose also that the domains D_0^- and D_1^- are convex with respect to the horizontal lines. Then problem (5^+) is uniquely solvable in the class H.

Proof. Since this result has not been published, we give its complete proof. By Theorem 3, it suffices to establish the uniqueness of a solution to the problem in question. Suppose that the homogeneous problem has a nonzero solution $u+iv \in H$, which belongs to the class $\mathring{H}^{(1)}(\overline{D^+};0,1)$ according to Theorem 3. The homogeneous boundary conditions (5^+) on γ_0 can be rewritten in the form (7_0) on the interval $J_0=[0,\tau]$, and those on γ_1 can be rewritten in the form (7_1) on the interval $J_1=[\tau,1]$. Thus,

$$u\big|_\sigma = 0, \tag{14}$$

$$\left[(v+u)^+\circ\alpha-(v-u)^+\right]\big|_{J_0}=0, \qquad \left[(v+u)^+-(v-u)^+\circ\beta\right]\big|_{J_1}=0. \tag{15}$$

As already mentioned, the behavior of the derivative of the analytic function $\phi=u+iv$ near the points $z=0$ and $z=1$ can be described by Theorem 1. In our case, for any $\varepsilon>0$ we have

$$\phi'(z)=O(1)|z-k|^{\delta_k-\varepsilon-1} \qquad \text{as} \quad z\to k.$$

To describe similar behavior of ϕ' in the neighborhood of the point $z=\tau$, we make use of the boundary condition (15). It shows that for a sufficiently small $c>0$

$$(v-u)^+(t)=g_0(t), \quad \tau-c\le t\le\tau, \qquad (v+u)^+(t)=g_1(t), \quad \tau\le t\le\tau+c,$$

with some functions g_k whose derivatives are H-continuous. As is well known [7], this implies

$$\phi'(z)=O(1)|z-\tau|^{-\varepsilon-\frac{1}{2}} \qquad \text{as} \quad z\to\tau.$$

Recall that $\theta_k^+\delta_k>3\pi/2$ and, in particular, $2\delta_k-1>0$. Hence, the function $(\phi')^2$ admits only weak singularities at the points $z=0,\tau,1$. More precisely,

$$z(z-\tau)(z-1)[\phi'(z)]^2\in\mathring{H}(\overline{D};0,\tau,1). \tag{16}$$

Let us set $h(s)=\arg z'(s)$ for brevity and introduce the functions $h_1=\max(h-2\pi,-\pi)$ and $h_2=\min(h-\pi,0)$; these functions, as well as h, belong to the class $H[0,l]$. In view of (13) we have the inequality $h_1\le h_2$. Since $h(0)=\pi-\theta_1^+$ and $h(l)=\pi+\theta_0^+$, the values of the functions h_j at the endpoints of the interval are related by the inequalities $h_1(0)\le-\pi\le h_2(0)$ and $h_1(l)\le 0\le h_2(l)$. Therefore, there exists a function $\varphi\in H[0,l]$ such that

$$h_1\le\varphi\le h_2, \qquad \varphi(0)=-\pi, \qquad \varphi(l)=0. \tag{17}$$

So, if the imaginary part $\operatorname{Im} f$ of an analytic function $f(z)$ in D^+ solves the Dirichlet problem

$$\operatorname{Im} f\big|_\sigma=\varphi, \qquad \operatorname{Im} f\big|_{J_0}=0, \qquad \operatorname{Im} f\big|_{J_1}=-\pi, \tag{18}$$

then it is easy to see that $(z-\tau)^{-1}e^{f(z)}\in H(\overline{D^+})$. In view of (17), it follows that we can apply Cauchy's theorem to the function $[\phi'(z)]^2e^{f(z)}$, which yields

$$-\operatorname{Im}\left(\int_\sigma+\int_{J_0}+\int_{J_1}\right)[\phi'(z)]^2e^{f(z)}\,dz=I_\sigma+I_0+I_1=0. \tag{19}$$

Consider all three terms of this equality separately. In view of (14), on the arc σ we have $\phi[z(s)]=iv[z(s)]$. Differentiating this equality, we obtain $[\phi'(z)]^2z'=-|\phi'|^2e^{-ih}$, which, combined with (18), gives

$$I_\sigma=\int_0^l|\phi'|^2e^{\operatorname{Re} f}\sin[\varphi(s)-h(s)]\,ds.$$

According to (17), the function $e^{f(t)}$ is real on the intervals J_k, so that

$$I_k = -2\int\limits_{J_k} e^f u'v'\,dt = -\frac{1}{2}\int\limits_{J_k}\big[(v'+u')^2-(v'-u')^2\big]\,dt.$$

Substituting here the expression obtained by differentiating (15), we obtain

$$2I_0 = \int\limits_0^\tau \big(e^{f\circ\alpha}-e^f\alpha'\big)[(v'-u')^2\circ\alpha]\alpha'\,dt + \int\limits_{\alpha(\tau)}^\tau e^f(v'-u')^2\,dt,$$

$$2I_1 = \int\limits_\tau^1 \big(e^{\operatorname{Re} f\circ\beta}-e^{\operatorname{Re} f}\beta'\big)[(v'+u')^2\circ\beta]\beta'\,dt + \int\limits_\tau^{\beta(\tau)} e^{\operatorname{Re} f}(v'+u')^2\,dt.$$

Now, notice that by virtue of (17) the function φ is bounded between 0 and $-\pi$, so that $\operatorname{Re} f$ decreases on J_0 and increases on J_1. Therefore, in view of (12),

$$\big(e^{f\circ\alpha}-e^f\alpha'\big)\big|_{J_0}\le 0, \qquad \big(e^{\operatorname{Re} f\circ\beta}-e^{\operatorname{Re} f}\beta'\big)\big|_{J_1}\ge 0.$$

In addition, (17) implies the inequality $-2\pi\le\varphi-h\le-\pi$, according to which the function $\sin[\varphi(s)-h(s)]$ is nonnegative on $[0,l]$. Thus, all terms in (19) are represented as sums of positive definite integrals. Consequently, the integrands in all these integrals identically vanish; hence, $\phi=0$, which contradicts the assumption $\phi\ne 0$.

Theorem 4, just as Bitsadze's theorem, contains an implicit constraint on the choice of the point τ. The question arises as to whether it is possible to remove this constraint for some domains D. In the canonical case, when σ and γ are the arcs of a circle and a hyperbola, respectively, this question can indeed be answered positively with the use of the approach pointed out in [4]. The following theorem extends this class of domains.

Theorem 5. *Suppose that the arc γ is of class C^2, the domain D^- is convex, and the domain D^+ is convex with respect to the circles that are tangent to the real axis outside the interval $J=[0,1]$ (including the family of straight lines parallel to this axis). Then problem (5^+) is uniquely solvable in the class H for any choice of the point τ.*

Proof. We give a proof of this theorem separately for the cases of $\alpha'(\tau)=1$ and $\alpha'(\tau)\ne 1$. Let $\alpha'(\tau)=1$. As already pointed out above, the domain D^- is convex if and only if $\alpha''\ge 0$ (by assumption, the arc γ and, hence, the function α are of class C^2). Therefore, $\alpha'(t)\le 1$ for $t\le\tau$ and $\alpha'(t)\ge 1$ for $t\ge\tau$. Since

$$\beta'(t)=\frac{1}{\alpha'[\beta(t)]}\le 1, \qquad \beta(t)\ge\tau=\beta[\alpha(\tau)],$$

it follows that $\beta'(t)\le 1$ for $t\ge\tau$. Thus, condition (12) of Theorem 4 holds for the domains D_k^-. Let us verify that condition (13) of this theorem is also valid.

By assumption, the domain D^+ is convex with respect to the family of horizontal lines. Let us take a point z_0 on σ with the maximum ordinate y; this point divides σ into two arcs σ_k with the endpoints $z=z_0$ and $z=k$, $k=0,1$. Then the ordinate y as a function of the arc length s monotonically increases on the curve σ_1. Therefore, $y'(s)\ge 0$ or, which is the same, the unit vector $z'(s)$ lies in the upper half-plane. Similarly we can verify that on σ_0 the vector z' lies in the lower half-plane. In other words, $0\le\arg z'\le\pi$ on σ_1 and $\pi\le\arg z'\le 2\pi$ on σ_0. Thus, the conditions of Theorem 4 are completely satisfied, and hence problem (5^+) is uniquely solvable in the class H.

Let us turn to the case of $\alpha'(\tau) \neq 1$. Consider the function $f(a) = [a - \alpha(\tau)]^2(a-\tau)^{-2}$ on the real axis outside the interval $(0,1)$. This function strictly monotonically decreases from 1 to $f(0)$ on $(-\infty, 0]$ and from $f(1)$ to 1 on $[1,\infty)$. As pointed out above, under the assumption that the domain D^- is convex, both inequalities (8) hold; therefore, there exists a unique point a at which

$$\left[\frac{a-\alpha(\tau)}{a-\tau}\right]^2 = \alpha'(\tau). \tag{20}$$

Consider the linear-fractional transformation

$$\omega(z) = \frac{z}{b(1-z)+z}, \qquad b = \frac{a}{a+1} > 0, \tag{21}$$

which maps the upper half-plane, as well as the interval $[0,1]$, onto itself and leaves the points $z=0$ and $z=1$ fixed. Denote by $\widetilde{D}^+$, $\widetilde{\sigma}$, and $\widetilde{\tau}$ the images of D^+, σ, and τ, respectively, under this transformation and set $\widetilde{\alpha} = \omega \circ \alpha \circ \omega^{-1}$ and $\widetilde{\beta} = \omega \circ \beta \circ \omega^{-1}$. We will regard a solution u to problem (5^+) in the domain D^+ as a solution to the problem with the boundary conditions (6) and (7_0) and (7_1) on J_0 and J_1, respectively. Then the substitution $(\widetilde{u}+i\widetilde{v})[\omega(z)] = (u+iv)(z)$ leads to a similar problem in the domain $\widetilde{D}^+$ with respect to $\widetilde{\sigma}$ and to the translations $\widetilde{\alpha}$ and $\widetilde{\beta}$. Therefore, it remains to verify that the latter problem satisfies the conditions of Theorem 4.

Under the transformation ω, the circles tangent to the real axis at the point a turn into straight lines parallel to this axis. Therefore, the domain $\widetilde{D}^+$ is convex with respect to these lines, and so, as shown above, the curve $\widetilde{\sigma}$ satisfies condition (13). Consider the translations $\widetilde{\alpha}$ and $\widetilde{\beta}$. The inverse transformation of (21) is obtained by replacing b with $1/b$; hence, we obtain the following expressions for the derivatives of the translations:

$$\widetilde{\alpha}'[\omega(t)] = \alpha'(t)\left[\frac{t-a}{\alpha(t)-a}\right]^2, \qquad \widetilde{\beta}'[\omega(t)] = \beta'(t)\left[\frac{t-a}{\beta(t)-a}\right]^2.$$

Therefore, conditions (12) for the translations $\widetilde{\alpha}$ and $\widetilde{\beta}$ and the point $\widetilde{\tau}$ reduce to the inequalities

$$\alpha'(t)\left[\frac{t-a}{\alpha(t)-a}\right]^2 \le 1, \quad 0 \le t \le \tau, \qquad \beta'(t)\left[\frac{t-a}{\beta(t)-a}\right]^2 \le 1, \quad \tau \le t \le 1.$$

Obviously, to prove these inequalities, it suffices to establish that

$$[f(t)-1](t-\tau) \ge 0, \qquad f(t) = \alpha'(t)\left[\frac{t-a}{\alpha(t)-a}\right]^2. \tag{22}$$

According to (20), the equation

$$f(t) = 1 \tag{23}$$

has a root τ. Let us verify that this equation has no other roots on the interval $[0,1]$. Indeed, suppose the contrary, and let $f(\tau_0) = 0$ and, say, $\tau_0 < \tau$. Then the function $(a-t)^2\alpha'(t) - [a-\alpha(t)]^2$ vanishes at the ends of the interval $[\tau_0, \tau]$. Therefore, there exists a point t_0 in the interval (τ_0, τ) at which the derivative of this function vanishes:

$$-2(a-t_0)\alpha'(t_0) + (a-t_0)^2\alpha''(t_0) + 2[a-\alpha(t_0)]\alpha'(t_0) = 0.$$

Hence, $(a-t_0)^2\alpha''(t_0) + 2[t_0 - \alpha(t_0)]\alpha'(t_0) = 0$, which is impossible because the first term here is nonnegative and the second is positive.

Now, consider the function

$$g(t) = \frac{1}{t-a} - \frac{1}{\alpha(t)-a}, \qquad 0 \le t \le 1,$$

which is nonpositive and vanishes at the ends of the interval $[0,1]$. Using the notation (22), we can write the derivative of this function as $g'(t) = (t-a)^{-2}[f(t)-1]$. In view of the above facts about equation (23), the derivative g' vanishes at the unique point τ, at which the function g attains its minimum. As a result, we arrive at inequality (22), which completes the proof of the theorem.

Suppose that under the conditions of Theorem 5 the function f belongs to $H(\sigma \cup \gamma)$ and $u_\tau(z)$ is a solution to problem (5^+) with the right-hand side $f|_{\sigma\cup\gamma_0\cup\gamma_1}$. As $\tau \to 1$, the arc γ_τ "vanishes," and it is natural to expect that u_τ tends to the solution u of the Dirichlet problem. As pointed out above, this solution has a logarithmic singularity at the point $z = 1$. Similarly, as $\tau \to 0$, the limit solution admits a logarithmic singularity at the point $z = 0$.

The question of whether alternative (ii) of Theorem 3 holds still remains open.

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