ON THE THEORY OF MIXED-TYPE EQUATIONS

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Abstract. We briefly review different well-posed boundary-value problems for mixed-type equations and their applications in transonic gas dynamics. We present barotropic relations for plane-parallel flow of a compressible gas that leads to mixed-type model equations on the hodograph plane. We also discuss the question on the existence of transonic solutions of profile flow problems, which is closely related to Dirichlet problems in mixed-type domains.

Mixed-type elliptic-hyperbolic equations naturally arise in gas dynamics. The first equation of such type appeared in Chaplygin's paper [6] on gas jets in 1902. This equation has the form

$$K(y)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and is called the *Chaplygin equation*. Here, the function $K(y)$ is positive for $y > 0$ and negative for $y < 0$; therefore, the equation is elliptic for $y > 0$ and hyperbolic for $y < 0$.

The first boundary-value problem for a mixed-type equation was considered by Tricomi without any connection with applications [32]. Consider the Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which corresponds to the case where $K(y) = y$, in a domain D bounded by a smooth arc σ with endpoints $A(0, 0)$ and $B(1, 0)$ (this arc lies in the half-plane $y > 0$ outside these endpoints) and two characteristics AC and BC of this equation (see Fig. 1).

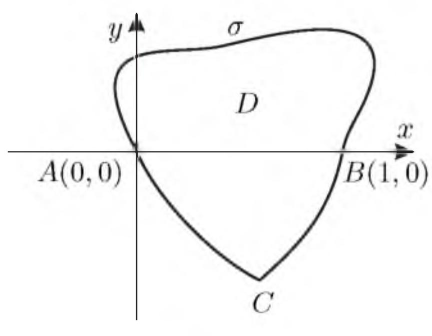

Fig. 1

The Tricomi problem T is to find a solution $u \in C(\overline{D})$ of this equation, which takes given values on σ and AC: $u\big|_{\sigma\cup AC} = \varphi$. Under certain assumptions on the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$, the smoothness of the function φ on σ and Γ_0, and properties of the arc σ, Tricomi proved that this problem has a unique solution.

The first investigations of Tricomi showed that boundary-value problems for mixed-type equations are very difficult to study. In particular, the proof of the existence of a solution is based on sophisticated methods of fractional differentiation, special functions, and one-dimensional singular equations, which were not well developed at that time.

In the 1930s, the Swedish mathematician Gellerstedt extended Tricomi's results to the case of the equation

$$y^{2m+1}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where m is a nonnegative integer number.

Beginning in the 1940s, problems for mixed-type equations had become the focus of many scientists (M. V. Keldysh, M. A. Lavrentiev, A. V. Bitsadze, K. I. Babenko, K. Friedrichs, K. Moravec, L. Bers, and others).

The topicality of these studies was stipulated by needs of high-speed aerodynamics. The first investigations in transonic dynamics that led to new boundary-value problems were made by Frankl and Guderley. In this connection, we emphasize the exceptional importance of the above-mentioned paper of Chaplygin, which was weakly understood for a long time. Only when the compressibility of air had been taken into account in aviation problems because of the high speeds in the 1930s, did this paper become the foundation of many works in gas dynamics.

As is known, a steady, plane-parallel flow of inviscid fluid (liquid or gas) is described by the continuity equation

$$\frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} = 0$$

and the Euler equations

$$\left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}\right) v_i + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2,$$

where the velocity vector $v = (v_1, v_2)$, the density ρ, and the pressure p are functions of x_1 and x_2. If the fluid is isentropic, we also have the equation of state $S(p, \rho) = 0$.

Assume that the fluid is barotropic, i.e., the equation of state $S(p, \rho) = 0$ can be uniquely solved with respect to p and determines a smooth, strictly increasing function $p = g(\rho)$.

We fix one of the antiderivatives of $g'(\rho)/\rho$ by setting $Q'(\rho) = g'(\rho)/\rho$. Obviously, the function $Q(\rho)$, called the enthalpy, monotonically increases in the variable ρ.

In the case of an ideal gas, we have

$$g(\rho) = A\rho^{\gamma}, \quad Q(\rho) = \frac{A\gamma}{\gamma - 1}\rho^{\gamma-1},$$

where $A > 0$ and $\gamma > 1$ are some constants.

Barotropic motions satisfy the Thompson theorem, which states that the circulation of a vector field v along a "fluid" contour consisting of particles and moving together with them is constant in time. Therefore, if there are no vortices at the initial instant, then this circulation vanishes and the Green formula implies the equation

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0,$$

which expresses the fact that v is a potential field. Hence,

$$\left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}\right) v_i = \frac{1}{2}\frac{\partial}{\partial x_i}\left(v_1^2 + v_2^2\right),$$

and the Euler equations imply the Bernoulli relation

$$\frac{r^2}{2} + Q(\rho) = C, \tag{1}$$

where C is a constant. This formula relates the speed $r = |v|$ with the density ρ.

Therefore, the pair of Euler equations is equivalent to the potentiality equation and the Bernoulli equation. Since the function Q strictly increases, the Bernoulli relation can be solved with respect to

ρ; thus, we obtain the monotonically decreasing function $\rho = \rho(r)$, $0 \le r \le r_0$. Finally, we obtain the following quasilinear, first-order system describing the motion:

$$\frac{\partial}{\partial x_1}[\rho(|v|)v_1] + \frac{\partial}{\partial x_2}[\rho(|v|)v_2] = 0, \quad \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0. \tag{2}$$

We introduce the following quantities:

$$c = \sqrt{g'(\rho)}, \quad M = \frac{|v|}{c}, \quad K = \frac{1 - M^2}{\rho^2}. \tag{3}$$

The first of them expresses the propagation speed of small perturbations (for fixed density ρ) and is called the (local) speed of sound. The dimensionless quantity M is called the Mach number and K is called the Chaplygin coefficient. Obviously, these quantities are functions of ρ (or $r = |v|$).

If the function $1 - M^2(\rho)$ changes its sign at a unique point ρ_*, then the subsonic and supersonic domains are determined by the inequalities $\rho(x, y) < \rho_*$ and $\rho(x, y) > \rho_*$, respectively.

By the Bernoulli relation, we have

$$r\,dr = -\frac{g'(\rho)}{\rho}d\rho = -\frac{c^2}{\rho}d\rho \tag{4}$$

and, hence,

$$1 - M^2 = \frac{1}{\rho}\frac{d(\rho r)}{dr}.$$

Therefore, the quantity $\rho r = \rho|v|$ (the mass flux) increases in the subsonic domain and decreases in the supersonic domain. The use of nozzles to obtain supersonic flows is based on this property. Consider a flow in a symmetric pipe having a narrowing (the Laval nozzle). Let the flow have sufficiently high speed at $-\infty$. The narrowing of the tube leads to the increasing of the speed. Having attained the value of the speed of sound, the speed of the flow continues to increase along the expanding part of the tube.

Let the function $1 - M^2(\rho)$ (or, equivalently, the Chaplygin coefficient $K(\rho)$) change its sign at a unique critical point ρ_*. Taking the Bernoulli equation into account, we obtain

$$K(\rho) = \frac{2\,[G(\rho) - C]}{\rho^2 g'(\rho)}, \quad G = \frac{g'}{2} + Q \tag{5}$$

and hence

$$G(\rho) < C \quad \text{for} \quad \rho < \rho_*, \qquad G(\rho) > C \quad \text{for} \quad \rho > \rho_*. \tag{6}$$

In particular, the point ρ_* is unique if the function G monotonically increases. For an ideal gas, we have

$$G(\rho) = \frac{A\gamma(\gamma+1)}{2(\gamma-1)}\rho^{\gamma-1}, \quad \rho_* = \left(\frac{2C}{A\gamma}\cdot\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}.$$

In the case where the function $g(\rho)$ is piecewise smooth, the functions c, M, and K are piecewise continuous and inequalities (6), determining the point ρ_* have sense even if ρ_* is a point of discontinuity for g'.

Consider the situation where on some interval $\rho_1 < \rho < \rho_2$, the function G is constant. By the definitions of Q and G, this is equivalent to the relation

$$\frac{g''(\rho)}{2} + \frac{g'(\rho)}{\rho} \equiv 0, \quad \rho_1 < \rho < \rho_2.$$

The general solution of this differential equation is

$$g(\rho) = A - \frac{B}{\rho}, \quad \rho_1 < \rho < \rho_2,$$

where $B > 0$ (by the condition $g' > 0$).

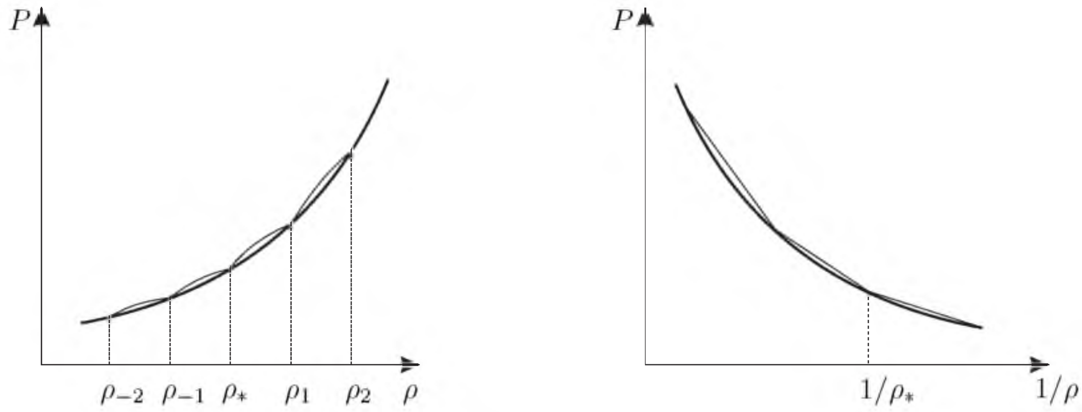

Fig. 2

The model corresponding to functions g of such type is called the Chaplygin gas. In this case, by (5), the function $K(\rho)$ is constant on the given interval:

$$K(\rho) \equiv \frac{2(C_1 - C)}{B}, \quad G(\rho) \equiv C_1, \quad \rho_1 < \rho < \rho_2.$$

The adiabat-approximation method is based on this fact. It consists of the change of the graph of the function g by the broken line composed of parts of hyperbolas $g(\rho) = A - B/\rho$ with different constants A and B (see Fig. 2).

These broken lines compose the graph of a piecewise-smooth function ρ_0 for which the function $K_0(\rho)$ is piecewise constant and changes sign at the point ρ_*. This approximative model of gas was proposed by Poritsky [22] and Lavrent'ev and Bitsadze [16]. In particular, in the case of a two-section broken line, the Chaplygin coefficient has the form $K_0(\rho) = \operatorname{sgn}(\rho - \rho_*)$.

Consider system (2). We can introduce the potential φ and the stream function ψ by the formulas $\operatorname{grad}\varphi = v$ and $\operatorname{grad}\psi = (-\rho v_2, \rho v_1)$ and reduce the problem to a single equation for φ or ψ. The functions φ and ψ are connected by the relations

$$\frac{\partial\varphi}{\partial x_1} = \frac{1}{\rho}\frac{\partial\psi}{\partial x_2}, \quad \frac{\partial\varphi}{\partial x_2} = -\frac{1}{\rho}\frac{\partial\psi}{\partial x_1}.$$

Substituting $v = \operatorname{grad}\varphi$ into the system, we obtain the following equation for the potential φ:

$$\frac{\partial}{\partial x_1}\left(\rho\frac{\partial\varphi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho\frac{\partial\varphi}{\partial x_2}\right) = 0.$$

Taking (4) into account, we obtain

$$\frac{\partial[\rho(r)]}{\partial x_i} = -\frac{\rho r}{c^2}\left(\frac{\partial\varphi}{\partial x_1}\frac{\partial^2\varphi}{\partial x_1\partial x_i} + \frac{\partial\varphi}{\partial x_2}\frac{\partial^2\varphi}{\partial x_2\partial x_i}\right)\frac{1}{|\operatorname{grad}\varphi|}.$$

Therefore, the previous equation can be written in the form of the classical gas-dynamic equation:

$$\left[c^2 - \left(\frac{\partial\varphi}{\partial x_1}\right)^2\right]\frac{\partial^2\varphi}{\partial x_1^2} - 2\left(\frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial x_2}\right)\frac{\partial^2\varphi}{\partial x_1\partial x_2} + \left[c^2 - \left(\frac{\partial\varphi}{\partial x_2}\right)^2\right]\frac{\partial^2\varphi}{\partial x_2^2} = 0.$$

This equation is quasilinear; its discriminant equals

$$\left[c^2 - \left(\frac{\partial\varphi}{\partial x_1}\right)^2\right]\left[c^2 - \left(\frac{\partial\varphi}{\partial x_2}\right)^2\right] - \left(\frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial x_2}\right)^2 = c^4(1 - M^2).$$

Therefore, the equation (and the previous first-order system for the potential φ and the stream function ψ) is elliptic in the subsonic domain and hyperbolic in the supersonic domain, i.e., it is an equation of mixed type. We emphasize that c is considered as a function of $r = |\operatorname{grad} \varphi|$.

There is a method of linearization of this equation going back to Chaplygin. In this method, one uses the transformation $x \to v$ realized by the velocity vector $v = \operatorname{grad} \varphi$ as a change of variables for this equation. In other words, the variables v_1 and v_2 on the so-called hodograph plane are assumed to be independent and the variables x_1 and x_2 on the physical plane are considered as functions of v_1 and v_2. Relative to φ, this yields the Legendre transformation. The vector (x_1, x_2) is the gradient $\operatorname{grad} \chi$ of a certain function $\chi(v_1, v_2)$. Relative to the inverse Legendre transformation, the equation considered turns into the linear equation

$$(c^2 - v_1^2)\frac{\partial^2 \chi}{\partial v_1^2} - 2v_1 v_2 \frac{\partial^2 \psi}{\partial v_1 \partial v_2} + (c^2 - v_2^2)\frac{\partial^2 \chi}{\partial v_2^2} = 0.$$

The main obstruction in using the hodograph method is the possible ambiguity of the inverse transformation $v \to x$. In the subsonic domain of the hodograph plane this transformation admits isolated singularities (branching points); in the hyperbolic part, "folds" along certain curves may appear.

It is convenient to use the polar coordinates $v_1 = r\cos\theta$, $v_2 = r\sin\theta$ instead of the Cartesian coordinates v_1, v_2 and proceed with the system for the potential φ and the stream function ψ. Taking it in the form

$$d\varphi + \frac{i}{\rho}d\psi = (v_1 - iv_2)(dx_1 + idx_2)$$

or

$$d(x_1 + ix_2) = \frac{e^{-i\theta}}{r}\left(d\varphi + \frac{i}{\rho}d\psi\right),$$

we obtain the following relations for the first-order partial derivatives:

$$\frac{\partial(x_1 + ix_2)}{\partial r} = \frac{e^{-i\theta}}{r}\left(\frac{\partial \varphi}{\partial r} + \frac{i}{\rho}\frac{\partial \varphi}{\partial r}\right), \qquad \frac{\partial(x_1 + ix_2)}{\partial \theta} = \frac{e^{-i\theta}}{r}\left(\frac{\partial \varphi}{\partial \theta} + \frac{i}{\rho}\frac{\partial \psi}{\partial \theta}\right).$$

Equating the mixed second-order derivatives, we obtain

$$i\left(\frac{\partial \varphi}{\partial r} + \frac{i}{\rho}\frac{\partial \psi}{\partial r}\right) = -\frac{1}{r}\left(\frac{\partial \varphi}{\partial \theta} + \frac{i}{\rho}\frac{\partial \psi}{\partial \theta}\right) - \frac{i\rho'}{\rho^2}\frac{\partial \psi}{\partial \theta},$$

where ρ is considered as a function of r.

Separating the real and imaginary parts and taking (4) into account, we obtain the system

$$\frac{\partial \varphi}{\partial r} = -\frac{\rho}{r}K\frac{\partial \psi}{\partial \theta}, \qquad \frac{\partial \psi}{\partial r} = \frac{\rho}{r}K\frac{\partial \varphi}{\partial \theta}$$

for the functions φ and ψ of the variables r and θ on the hodograph plane.

The change of variables

$$x = \theta, \qquad y = -\int_{\rho_*}^{\rho} \frac{\rho}{r}dr \tag{7}$$

transforms this system to the canonical form

$$\frac{\partial \varphi}{\partial y} = K(y)\frac{\partial \psi}{\partial x}, \qquad \frac{\partial \psi}{\partial y} = -\frac{\partial \varphi}{\partial x},$$

where $K(y)$ is the Chaplygin function, which changes sign at the point $y = 0$ of the y axis. Eliminating φ, we obtain the Chaplygin equation for the stream function $u = \psi$ in the variables x and y.

As was noted above, if the adiabat (the graph of the function $g(\rho)$) consists of two parts of hyperbolas of the form $\gamma(\rho) = A - B/\rho$ with the common point ρ_*, then the Chaplygin coefficient equals $K(y) = \operatorname{sgn} y$. In this case, the Chaplygin equation becomes the Lavrent'ev–Bitsadze equation

$$\operatorname{sgn} y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Along with the well-known Tricomi equation, it is a model equation of mixed type.

We have assumed above that there exists a unique ρ_* satisfying property (6). If the function $G = g'/2 + Q$ is not monotone, then there can exist several points ρ_* at which the function $G(\rho) - C$ changes sign. In this case, the Chaplygin function $K(y)$ changes its sign, for example, at the points $y = 0$ and $y = 1$. Boundary-value problems for the equation

$$y(y-1)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which simulates the situation of this type, were initially studied by Nakhushev [19].

Now we consider the Tricomi problem for the Chaplygin equation. It is closely related with the theory of transonic gas flows (see [9]). Frankl [11] also proved the uniqueness theorem for solutions of this problem under the condition $3(K')^2 \geq 2KK''$ for the Chaplygin coefficient.

As was noted by Friedrichs, Frankl's proof can be represented in terms of the "energy integral." We briefly present this method. If u is a solution of the Chaplygin equation in a domain D, then

$$\int\limits_D \left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu\right)\left(K\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy = 0$$

for any functions a, b, and c. Using the Green formula and the boundary conditions, we can represent the left-hand side as the integral over the boundary of a quadratic form of the variables $\partial u/\partial x$ and $\partial u/\partial y$. Therefore, the problem is reduced to the search for a, b, and c such that this quadratic form is positive definite. This method is called the *abc*-method and is widely used by many authors for proving the uniqueness theorems for different boundary-value problems for equations of mixed type and for obtaining a priori estimates.

The complete investigation of the Tricomi problem and the so-called problem T_* for the Lavrent'ev–Bitsadze equation was performed by Bitsadze [4]. The problem T_* is a generalization of the Tricomi problem; the support of the Dirichlet data in the hyperbolic part consists of two segments, AA' and BB', of characteristics whose endpoints A' and B' lie on characteristics beginning at some point $T(\tau, 0)$, $0 < \tau < 1$.

A solution of this problem is sought in the mixed domain D bounded by σ and the segments of characteristics AA', BB', TA', and TB' (see Fig. 3). Thus, this problem is determined by the boundary condition $u\big|_{\sigma\cup AA'\cup BB'} = \varphi$.

Another important generalization of the Tricomi problem, the so-called problem M, was proposed by Frankl [12] in the connection with the study of transonic flows of gas jets. In this problem, the characteristic AC (the support of the Dirichlet data in the hyperbolic part) is replaced by a curve $\Gamma = AF$ lying inside the characteristic triangle (Fig. 4). In other words, we have the problem with the boundary condition $u\big|_{\sigma\cup\Gamma} = \varphi$ in the domain bounded by the curves σ and Γ and the arc of the characteristic BF.

Frankl [10] studied this problem for the Tricomi equation in the case where σ is the normal contour $(x-1)^2/2 + 4y^3/9 = 1/4$, $y \geq 0$, and Γ in some neighborhood of its endpoint A coincides with the characteristic AC and weakly deviates from it in the whole.

As compared with the Tricomi problem, for this problem it is much more difficult to prove the uniqueness and the existence of a solution.

The first proof of the existence and uniqueness of a solution of this problem for the Lavrent'ev–Bitsadze equation was given by Bitsadze [4] under the assumption that the inequality $dy(x - x^2 - y^2) - ydx \geq 0$

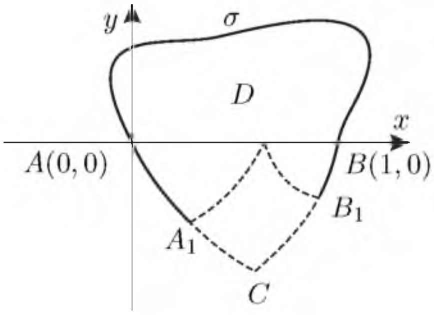

Fig. 3

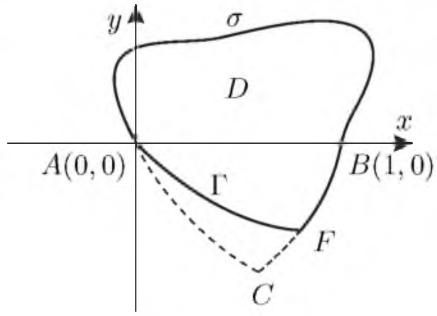

Fig. 4

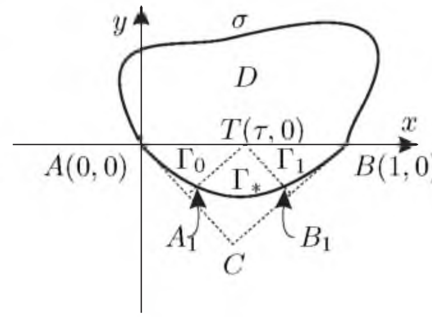

Fig. 5

holds on σ. This substantial progress invited Frankl to call problem M the Bitsadze problem (unpublished letter to Bitsadze). However, it is now usually called the generalized Tricomi problem.

For the Lavrent'ev–Bitsadze equation, Bitsadze considered the so-called generalized mixed problem, the problem M_*, corresponding to the removal from characteristics in the problem T_*. The support of the Dirichlet data in the hyperbolic part of this problem consists of two arcs Γ_0 and Γ_1 whose initial points lie on the characteristics AC and BC, respectively (see Fig. 5) and whose endpoints A_1 and B_1 lie on the characteristics beginning at some point $T(\tau, 0)$, $0 < \tau < 1$. We search for the solution in the mixed domain D bounded by the curves σ, Γ_0, and Γ_1 and segments of the corresponding characteristics. Thus, this problem is determined by the boundary condition $u\big|_{\sigma\cup\Gamma_0\cup\Gamma_1} = \varphi$. Bitsadze proved the existence and uniqueness of the solution of this problem in the class $C^1(D)$ under some geometric conditions on σ.

The general mixed problem for the Chaplygin equation was also investigated. In particular, Morawetz [18] proved by the abc-method the uniqueness of the solution of this problem in the case where the condition $(x - \tau)dy - ydx \geq 0$ holds on σ.

Soldatov [25] proved the uniqueness of the solution of problem M without any geometrical assumptions about the elliptic part σ of the boundary ∂D. We briefly discuss this proof.

A version of the abc-method proposed by Bitsadze consists of the reduction of problem M to an elliptic problem in the domain $D^+ = D \cap \{y > 0\}$ and of the appropriate choice of a function F analytic in D^+ such that the integrand in the relation

$$0 = \operatorname{Im} \int\limits_{\partial D^+} F(z) \left[\phi'(z)\right]^2 dz$$

is positive definite. Here ϕ is an analytic function whose real part is the solution of the considered homogeneous problem. Such a function F can be constructed for an arbitrary curve σ only under the assumption that F has a sufficient number of poles on the boundary; the number of poles depends on the variation of the argument of the tangent to σ. Therefore, if we know that there exists a sequence $z_n \to 0$, $z_n \in \sigma$, such that $\phi'(z_n) = 0$, then we can allow F to have poles in some of these points and then select this function appropriately.

On the other hand, if these points $z_n \in \sigma$ are isolated from $z = 0$, then we can select a subdomain D_0 in D bounded by the envelope of level curves of the solution $u(z)$ and the corresponding part of ∂D. For this subdomain, it is possible to choose a new function F_0 such that the integrand in the relation

$$0 = \operatorname{Im} \int\limits_{\partial D_0} F_0(z)\phi^2(z)dz$$

is positive definite.

In [23], the idea of using level curves was successfully applied for the proof of the uniqueness of the solution of problem M and of the general Chaplygin equation. This showed substantial progress in the investigation of this problem.

The condition for the elliptic part σ of the boundary ∂D in the general mixed problem M_* for the Lavrent'ev–Bitsadze equation imposed by Bitsadze was weakened by Soldatov [26]. It consists of the requirement that the continuous branch of the argument of the tangent vector takes values in the interval $[0, 2\pi]$. It can be proved that for an arbitrary curve σ, the homogeneous problem M_* has only a nonzero solution.

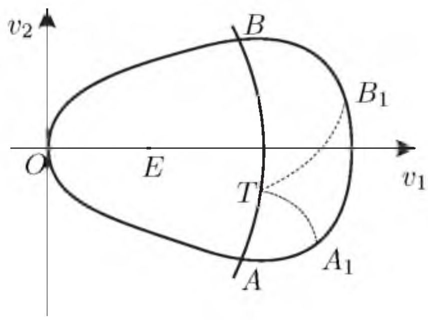

Fig. 6

However, it is unknown whether the uniqueness theorem holds for an arbitrary curve σ. A similar question is still open also for the general Chaplygin equation.

Some boundary-value problems for mixed-type equations related with direct problems for the Laval nozzle were considered by Kuz'min [15].

In conclusion, we discuss the so-called transonic polemic [1], which launched in the 1950s between mathematicians and mechanicians (Guderley, Buseman, Frankl, Schäffer, and others) in connection with the problem on existing smooth transonic solutions of the streamline problem for arbitrary profiles.

Let a given domain on the hodograph plane v_1, v_2 have the form shown in Fig. 6, where AB is an arc of the circle $v_1^1 + v_2^2 = r_*^2$ with critical speed of sound r_* corresponding to the value ρ_*, the arcs TB_1 and TA_1 are characteristics (Mach lines), and the arcs AA_1 and BB_1 are not of characteristic directions.

In the polar coordinates $v_1 = r\cos\theta$ and $v_2 = r\sin\theta$, after substitution (7) this domain turns (for the function $K(y) = \operatorname{sgn} y$) into the domain D shown in Fig. 5.

The existence of a smooth solution of the streamline problem means that there exists a function ψ vanishing on the whole boundary of the domain shown in Fig. 5 (i.e., on the arcs OBB_1A_1AO and the segment OA) and having a singularity at the point E, which corresponds to the value of the complex potential $v_1 - iv_2$ at ∞.

Note that if the point T moves toward the point B, then the length of the arc A_1B_1 tends to zero and the general mixed problems become the Dirichlet problem. This led Busemann [5] and Guderley [8] to the conclusion that the considered problem has a solution if there is a singularity at the point A (although this has no physical meaning). Recent results concerning the Dirichlet problem for the Lavrent'ev–Bitsadze equation completely justify this point of view.

We consider this situation for the general mixed problem (M_*). Let a domain D be bounded by arcs $\sigma \subseteq \{y \geq 0\}$ and $\Gamma \subseteq \{y \leq 0\}$ with endpoints A and B. By the main result of Bitsadze, the Dirichlet problem $u\big|_{\sigma\cup\Gamma} = \varphi$ for the Lavrent'ev–Bitsadze equation in the class $C(\overline{D})\cap C^1(D)$ is overdetermined. The Dirichlet data φ must be disengaged on the arc Γ_*, which is the common part of Γ and the characteristic angle with vertex at the point $T(\tau, 0)$, $0 < \tau < 1$. In Bitsadze's posing of the problem, the point T is chosen in the segment $[0, 1]$ such that the arc Γ_* contains a point with the minimum ordinate (see Fig. 5). In effect, this restriction can be eliminated and a point $0 < \tau < 1$ can be chosen arbitrarily (perhaps arbitrarily close to the endpoints A and B).

If we reject the continuity of the solution u in the whole closed domain and allow it to have a power singularity of sufficiently small order at the point B, then the problem becomes uniquely solvable [28, 29]. The proof of the existence theorem requires the theory of singular integro-differential operators in the class of functions with condensing singularities on the endpoints of the segment $[0,1]$ (see [27]).

It seems sufficiently paradoxical that if we allow the solution to have a singularity at a unique point B of the boundary, then we must impose additional boundary data on the entire boundary arc Γ_*; this fact does not hold in the theory of elliptic boundary-value problems. The reason is that the considered Dirichlet problem is reduced to a nonlocal elliptic problem in the domain D^+, whose behavior substantially differs from the behavior of local problems.

Note that in the model case where σ and γ are arcs of a circle and a hyperbola, respectively, it is possible to obtain a direct proof of the existence and uniqueness of the Dirichlet problem in the class

$$u(z) = O(1)\left|\frac{z}{1-z}\right|^{\varepsilon}, \quad z = x + iy \in D^+,$$

where $\varepsilon > 0$ is sufficiently small (see [30]). The fact is that, in this case, an integro-differential equation on the segment $[0,1]$, to which the initial problem can be equivalently reduced, can be transformed to a convolution equation on the whole axis (in the sense of Hörmander) and the invertibility of this equation is determined by the condition that the Fourier image of its kernel does not vanish.

Note that the Dirichlet problem for mixed-type equations can also be posed for domains of other types [14]. Moreover, more general boundary conditions of Poincaré type can be imposed on the boundary of a mixed domain. Such problems are well posed in appropriate weighted spaces (see [31]).

We have discussed boundary-value problems in connection with their gas-dynamic applications. However, mixed-type equations play a significant role, for example, in the theory of bending of momentless thin shells (see [33]) and in different models of mathematical biology (see [20]).

For the Tricomi problem, important spectral-theoretic results were obtained [17, 21]. For the Lavrent'ev–Bitsadze, Tricomi, and Gellerstedt equation with spectral parameters, domains on the complex plane containing no points of spectra are described. The problem on the completeness of systems of eigenfunctions and joined functions of the Tricomi problem in some mixed domains is studied. Note that the existence of eigenvalues for this problem was proved by Kal'menov [13]. A detailed bibliography is contained in [24].

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