

UNIVERSALITY OF THE RELAXATION STRUCTURE OF EQUATIONS FOR THE DYNAMICS OF CONTINUOUS MEDIA AND DISSIPATIVE POISSON BRACKETS

M. Yu. Kovalevskii,^{*†} V. T. Matskevich,^{*} and A. Ya. Razumnyi[‡]

We generalize the Hamilton equations for dynamical processes with relaxation. We introduce a dissipative Poisson bracket in terms of the dissipation function. We obtain the universal structure of the relaxation terms in the equations for the dynamics of condensed media and verify this result for structureless liquids, elastic solids, and quantum liquids. In the examples of the condensed media under consideration, we obtain expressions for the dissipative Poisson brackets for the complete set of dynamical parameters.

Keywords: Hamiltonian approach, entropy, dissipative Poisson bracket, kinetic coefficient, solid, quantum liquid, dissipation function

1. Variational principle and Hamilton equations

It is important to take relaxation phenomena into account in studying the dynamics of multiparticle states. This problem is solved both by methods of statistical mechanics [1]–[5] and in the framework of phenomenological approaches [6]–[11]. Using the representation of a reduced description of nonequilibrium processes in the microscopic approximation, we can find a “rough” statistical operator in the perturbation theory with respect to some small parameter (the density, the interparticle interaction, or spatial inhomogeneities). The dynamical equations obtained in the leading approximation of the perturbation theory describe adiabatic processes. The inclusion of the next approximation leads to the appearance of collision integrals or kinetic coefficients that describe relaxation at the kinetic or hydrodynamic stage of the evolution. The current state of this area of theoretical physics was described in [12], [13].

The Hamiltonian formalism is an effective phenomenological method for studying the dynamics of various physical systems. Such an approach was developed for normal liquids, liquid crystals, and solids [6], [14], [15]. It was used in quantum systems (superfluid liquids, magnets, and quantum crystals [16]–[21]) and also to describe dynamical processes in plasma physics [22] and the kinetics of spin wave interactions [23].

The definition of Poisson brackets for a complete set of dynamical variables plays a fundamental role in the Hamiltonian approach. Invertible dynamical equations are obtained using the variational principle, and the Poisson brackets are found using transformations leaving the kinematic part of the action unchanged [21]. In the framework of phenomenological methods for constructing dynamical equations, the dissipative processes are taken into account using the dissipation function and the general principles of thermodynamics. In the framework of the Hamiltonian formalism, relaxation processes were taken into account in [10] and [24]–[27]. The kinetic stage of the evolution was considered in [24]. The introduced

^{*}National Scientific Center “Kharkov Institute of Physics and Technology,” Kharkov, Ukraine, e-mail: mikov@kharkov.ua.

[†]Belgorod State University, Belgorod, Russia.

[‡]Karazin Kharkiv National University, Kharkov, Ukraine.

dissipative Poisson bracket led to the relaxation Lenard–Balescu equation. A similar approach [25] described the relaxation of three-wave dynamical processes. The relaxation kinetics of polymer liquid crystals was studied in [10], and the dissipative hydrodynamics of classical liquids was described in the framework of the Hamiltonian approach in [10], [26], and [27]. The authors of these papers introduced dissipative brackets and obtained noninvertible dynamical equations in the quasi-Hamiltonian form. The structure of the relaxation terms in these dynamical equations did not follow from the variational principle and was obtained using the Fokker–Planck equations or other noninvertible dynamical equations [28].

We direct attention to a significant difference between the constructions of the dynamical equations in the microscopic and phenomenological Hamiltonian approaches. A knowledge of the principles of statistical mechanics allows obtaining relaxation equations in the framework of the perturbation theory. In the Hamiltonian formalism, dynamical equations can be obtained without using small perturbation theory. The problem of deriving the general form of relaxation terms in dynamical equations without using perturbation theory methods in Hamiltonian mechanics remains open.

Here, we construct the Hamiltonian formalism taking relaxation processes for normal liquids and condensed media with broken symmetry into account. We introduce dissipative Poisson brackets in terms of the dissipation function together with the standard definition of the reactive Poisson bracket. According to the symmetry principle of the Onsager kinetic coefficients, the dissipative Poisson bracket is symmetric under permutation of its dynamical quantities in contrast to the reactive one. We clarify the coincidence condition for the transformation properties of the reactive and dissipative Poisson brackets under finite canonical transformations. We show that the structure of relaxation flows of a normal liquid takes the universal form if the dissipation function is a second-order homogeneous functional. We formulate a general assertion that the structure of relaxation terms in dynamical equations is universal and establish that this assertion holds for two examples of the condensed media under consideration: superfluid liquid He II and a solid. The formalism used to introduce relaxation into the Hamiltonian approach allows establishing a closer connection with available statistical methods for studying condensed media.

According to the approach of the mechanics of condensed media, the Lagrangian of an arbitrary physical system has the form

$$L = L_k(\varphi, \dot{\varphi}) - H(\varphi),$$

where

$$L_k(\varphi, \dot{\varphi}) = \int d^3x F_\alpha(x, \varphi(x')) \dot{\varphi}_\alpha(x)$$

is the kinematic part of the Lagrangian and

$$H(\varphi) = \int d^3x \varepsilon(x, \varphi)$$

is the Hamiltonian of the system. The energy density $\varepsilon(x, \varphi(x'))$ of the medium and the quantities $F_\alpha(x, \varphi(x'))$ are functionals of the dynamical variables $\varphi_\alpha(x)$. Using the variational procedure, we obtain the dynamical equations for these quantities [21]:

$$\dot{\varphi}_\alpha(x) = \int d^3x' J_{\alpha\beta}^{-1}(x, x', \varphi) \frac{\delta H(\varphi)}{\delta \varphi_\beta(x')} \equiv \{\varphi_\alpha(x), H\}_R, \quad (1.1)$$

where the matrix $J_{\alpha\beta}(x, x'; \varphi)$ is defined by the equality

$$J_{\alpha\beta}(x, x'; \varphi) = \frac{\delta F_\beta(x', \varphi)}{\delta \varphi_\alpha(x)} - \frac{\delta F_\alpha(x, \varphi)}{\delta \varphi_\beta(x')}. \quad (1.2)$$

The relation between the Poisson brackets of the quantities $\varphi_\alpha(x)$ and this matrix is given by

$$\{\varphi_\alpha(x), \varphi_\beta(x')\}_R \equiv J_{\alpha\beta}^{-1}(x, x'; \varphi). \quad (1.3)$$

To distinguish the ordinary (reactive) Poisson bracket from the dissipative bracket, we label the reactive bracket with the subscript R. It can be seen from relation (1.2) that brackets (1.3) are antisymmetric with respect to the permutation $\alpha \leftrightarrow \beta$, $x \leftrightarrow x'$ and satisfy the Leibniz and Jacobi identities:

$$\begin{aligned} \{\varphi_\alpha(x), \varphi_\beta(x')\}_R &= -\{\varphi_\beta(x'), \varphi_\alpha(x)\}_R, \\ \{\varphi_\alpha(x)\varphi_\beta(x'), \varphi_\gamma(x'')\}_R &= \varphi_\alpha(x)\{\varphi_\beta(x'), \varphi_\gamma(x'')\}_R + \\ &\quad + \varphi_\beta(x')\{\varphi_\alpha(x), \varphi_\gamma(x'')\}_R, \\ \{\varphi_\alpha(x), \{\varphi_\beta(x'), \varphi_\gamma(x'')\}_R\}_R &+ \{\varphi_\beta(x'), \{\varphi_\gamma(x''), \varphi_\alpha(x)\}_R\}_R + \\ &\quad + \{\varphi_\gamma(x''), \{\varphi_\alpha(x), \varphi_\beta(x')\}_R\}_R = 0. \end{aligned} \quad (1.4)$$

For arbitrary functionals $A(\varphi)$ and $B(\varphi)$, we define the Poisson bracket by the equality

$$\{A, B\}_R \equiv \int d^3x \int d^3x' \frac{\delta A}{\delta \varphi_\alpha(x)} J_{\alpha\beta}^{-1}(x, x'; \varphi) \frac{\delta B}{\delta \varphi_\beta(x')}. \quad (1.5)$$

Using a functional expression for the energy density $\varepsilon(x, \varphi(x'))$, we can describe the dynamics of condensed media with arbitrary spatial inhomogeneities. Functional equations of motion that are very difficult to analyze appear in this case. Studying dynamical equations is considerably simplified in the case of a long-wavelength limit, where the characteristic dimensions of spatial inhomogeneities of the dynamical variables L are much greater than the mean free path l (the hydrodynamic approximation $L \gg l$). In this case, the expression for the energy density becomes a local function of the set of parameters of the reduced description, $\varepsilon(x, \varphi(x')) \approx \varepsilon(\varphi(x), \nabla\varphi(x))$, and the functional dynamical equations pass into the differential equations.

The finite transformations

$$\varphi_\alpha(x) \rightarrow \varphi'_\alpha(x) = \varphi_\alpha(x, \varphi(x')) \quad (1.6)$$

leaving the kinetic part of the action unchanged,

$$L_k(\varphi, \dot{\varphi}) = \int d^3x F_\alpha(x; \varphi) \dot{\varphi}_\alpha(x) = L_k(\varphi', \dot{\varphi}') = \int d^3x F_\alpha(x; \varphi') \dot{\varphi}'_\alpha(x),$$

are canonical if the relation

$$F_\beta(x'; \varphi) = \int d^3x F_\alpha(x; \varphi') \frac{\delta \varphi'_\alpha(x)}{\delta \varphi_\beta(x')} \quad (1.7)$$

holds. Taking (1.2) and (1.7) into account, we represent the canonicity condition in the form

$$J_{\alpha\beta}(x, x'; \varphi) = \int d^3x_1 d^3x_2 \frac{\delta \varphi'_\lambda(x_1)}{\delta \varphi_\alpha(x)} J_{\lambda\nu}(x_1, x_2; \varphi') \frac{\delta \varphi'_\nu(x_2)}{\delta \varphi_\beta(x')}, \quad (1.8)$$

whence we obtain the canonicity condition for the transformation of the inverse matrix

$$J_{\beta\gamma}^{-1}(x', x''; \varphi) = \int d^3y_1 d^3y_2 \frac{\delta \varphi_\beta(x')}{\delta \varphi'_\delta(y_1)} J_{\delta\rho}^{-1}(y_1, y_2; \varphi') \frac{\delta \varphi_\rho(y_2)}{\delta \varphi'_\gamma(y_2)}. \quad (1.9)$$

This relation leads to the invariance of the Poisson brackets under canonical transformations (1.6):

$$\{A, B\}_{\mathbb{R}} = \int d^3x \int d^3x' \frac{\delta A}{\delta \varphi'_\alpha(x)} J_{\alpha\beta}^{-1}(x, x'; \varphi') \frac{\delta B}{\delta \varphi'_\beta(x')}.$$

We now verify that the dynamical equations are covariant under canonical transformations (1.6). Indeed, in view of (1.9), it is easy to see that Eq. (1.1) implies that the dynamical equation

$$\dot{\varphi}'_\alpha(x) = \int d^3x' J_{\alpha\beta}^{-1}(x, x'; \varphi') \frac{\delta H'(\varphi')}{\delta \varphi'_\beta(x')}$$

holds, where $H'(\varphi') \equiv H(\varphi(\varphi'))$.

The conservation laws closely related to the symmetry properties of the Hamiltonian play an important role in constructing the dynamical equations of the mechanics of condensed media. The collection of additive integrals of motion of a nonmagnetic medium consists of the Hamiltonian, momentum, and mass,

$$\gamma_a = H, P_k, M \equiv \int d^3x \zeta_a(x), \quad a = 0, k, 4, \quad k = 1, 2, 3,$$

and the conservation laws in the differential form are

$$\dot{\zeta}_a(x) = -\nabla_k \zeta_{ak}(x). \quad (1.10)$$

The flow densities of the additive integrals of motion in terms of the Poisson brackets of the corresponding densities ζ_a can be represented as [29]

$$\begin{aligned} \zeta_{ak}(x) &= -\delta_{ak} \varepsilon(x) + \int d^3x' x'_k \int_0^1 d\lambda \{\zeta_a(y), \varepsilon(y')\}_{\mathbb{R}}, \quad a \neq 0, \\ \zeta_{0k}(x) &= \frac{1}{2} \int d^3x' x'_k \int_0^1 d\lambda \{\varepsilon(y), \varepsilon(y')\}_{\mathbb{R}}, \\ y &\equiv x + \lambda x', \quad y' \equiv x - (1 - \lambda)x'. \end{aligned} \quad (1.11)$$

In deriving (1.11), we take the symmetry properties of the Hamiltonian into account:

$$\begin{aligned} \{M, \varepsilon(x)\}_{\mathbb{R}} &= 0, & M &\equiv \int d^3x \rho(x), \\ \{P_i, \varepsilon(x)\}_{\mathbb{R}} &= \nabla_i \varepsilon(x), & P_i &\equiv \int d^3x \pi_i(x), \\ \{L_i, \varepsilon(x)\}_{\mathbb{R}} &= \varepsilon_{ikl} x_k \nabla_l \varepsilon(x), & L_i &\equiv \varepsilon_{ikl} \int d^3x x_k \pi_l(x), \end{aligned} \quad (1.12)$$

where L_i is the orbital moment. The first relation reflects the property of the phase invariance of the Hamiltonian, the second reflects the invariance under translations, and the last describes the invariance of the Hamiltonian under rotations in the configuration space.

The condensed medium deforms as it evolves. The notion of Lagrangian and Eulerian coordinates are used to describe the deformation. The Eulerian coordinates $x_k = x_k(\xi, t)$ specify the position of a medium particle at the current instant, and the displacement vector $u_k(x)$ connects the Lagrangian coordinate ξ_k with the Eulerian coordinate x_k , $x_k \equiv \xi_k + u_k(x)$. We define the distortion tensor by the equality

$$b_{ik}(x) = \nabla_k \xi_i(x) = \delta_{ik} - \nabla_k u_i(x). \quad (1.13)$$

It is well known that the nonzero Poisson brackets for the momentum, mass, and entropy densities and the distortion tensor have the forms [14], [21]

$$\begin{aligned}\{\pi_i(x), \sigma(x')\}_{\mathbb{R}} &= -\sigma(x)\nabla_i\delta(x-x'), & \{\pi_i(x), \rho(x')\}_{\mathbb{R}} &= \rho(x)\nabla'_i\delta(x-x'), \\ \{\pi_i(x), \pi_j(x')\}_{\mathbb{R}} &= \pi_j(x)\nabla'_i\delta(x-x') - \pi_i(x')\nabla_j\delta(x-x'), & & \\ \{b_{ij}(x), \pi_k(x')\}_{\mathbb{R}} &= b_{ik}(x')\nabla'_j\delta(x-x').\end{aligned}\tag{1.14}$$

Poisson brackets (1.14) serve as a basis for constructing nonlinear equations for the dynamics of classical condensed media. The specific features of the dynamical description of liquids, crystals, and liquid crystals in the framework of the Hamiltonian approach are manifested in the different dependences of the energy density on the distortion tensor. For the condensed medium under study, the Hamiltonian is the functional of the set of hydrodynamical parameters that includes the entropy, momentum, and mass densities:

$$H = H_0(\pi_i(x), \rho(x)) + V(\sigma(x'), \rho(x'), b(x')).\tag{1.15}$$

Everywhere in what follows, we assume that the Hamiltonian of the system is Galilean-invariant, i.e., its dependence on the momentum density has the form

$$\begin{aligned}H_0 &= \int d^3x \frac{\pi^2(x)}{2\rho(x)}, \\ V &= \int d^3x' \Phi(\rho(x'), \sigma(x'), b_{ij}(x')).\end{aligned}$$

2. Dynamics of a normal liquid, Navier–Stokes equations, and kinetic coefficients

For the medium under consideration, we define the differential of the energy density and introduce the required thermodynamic quantities:

$$d\varepsilon = \frac{\partial\varepsilon}{\partial\sigma} d\sigma + \frac{\partial\varepsilon}{\partial\pi_i} d\pi_i + \frac{\partial\varepsilon}{\partial n} dn \equiv T d\sigma + \nu_k d\pi_k + \mu dn,\tag{2.1}$$

where $\rho(x) = mn(x)$, m is the particle mass, and $n(x)$ is the number of particles. Here, the temperature T , the macroscopic velocity ν_k , and the chemical potential μ are related to the thermodynamic forces $Y_a \equiv Y_0, Y_k, Y_4$ by the equalities $T \equiv Y_0^{-1}$, $\nu_k \equiv -Y_k/Y_0$, and $\mu \equiv -Y_4/Y_0$. We define the thermodynamic potential

$$\omega(Y_a) \equiv Y_a \zeta_a - \sigma\tag{2.2}$$

and write the second law of thermodynamics

$$d\omega = \zeta_a dY_a = \frac{\partial\omega(Y)}{\partial Y_a} dY_a.\tag{2.3}$$

The representation of the flow density in the form of Poisson brackets of the densities of additive integrals of motion (1.12) allows finding the expressions for the flow densities of the additive integrals of motion in terms of the thermodynamic potential. Taking expressions (1.11) for these flows and Poisson brackets (1.14) into account, we obtain (see [4])

$$\zeta_{ak}^{(0)} = -\frac{\partial}{\partial Y_a} \left(\frac{\omega Y_k}{Y_0} \right), \quad a = 0, k, 4.\tag{2.4}$$

Relations (2.3) and (2.4) show that the densities of the additive integrals of motion and the corresponding flows are expressed in terms of the thermodynamic potential density. These formulas lead to the Euler equations of hydrodynamics and describe the flow of liquids in the adiabatic approximation. The total entropy $\Sigma \equiv \int d^3x \sigma(x)$ is conserved, i.e., $\dot{\Sigma} = 0$, and the dynamical equation for the entropy density has the form

$$\dot{\sigma} = -\nabla_k(\sigma\nu_k). \quad (2.5)$$

Formulas (1.14), (2.3), and (2.4) allow establishing the set of Poisson brackets for the densities $\zeta_a(x)$ of the additive integrals of motion:

$$\{\zeta_a(x), \zeta_b(x')\}_R = [\delta_{ai}\zeta_b(x) + \delta_{bi}\zeta_a(x') + \delta_{a0}\zeta_{bi}^{(0)}(x) + \delta_{b0}\zeta_{ai}^{(0)}(x')] \nabla'_i \delta(x-x').$$

The matrix $J_{ab}^{-1}(x, x'; \zeta)$ for the structureless condensed medium is thus obtained in explicit form. Substituting the last formula in (1.5), we obtain the Poisson bracket for arbitrary quantities,

$$\begin{aligned} \{A, B\}_R \equiv & \int d^3x \left[\frac{\delta B}{\delta \zeta_b(x)} \nabla_i \left(\frac{\delta A}{\delta \pi_i(x)} \zeta_b(x) + \frac{\delta A}{\delta \varepsilon(x)} \zeta_{bi}(x) \right) - \right. \\ & \left. - \frac{\delta A}{\delta \zeta_b(x)} \nabla_i \left(\frac{\delta B}{\delta \pi_i(x)} \zeta_b(x) + \frac{\delta B}{\delta \varepsilon(x)} \zeta_{bi}(x) \right) \right]. \end{aligned}$$

We find the relaxation terms for the hydrodynamic equations. In this case, the additional terms (the dissipative flows $\dot{\zeta}_{ak}^{(1)}(x)$)

$$\dot{\zeta}_a(x) = -\nabla_k(\zeta_{ak}^{(0)}(x) + \zeta_{ak}^{(1)}(x)) \quad (2.6)$$

appear in the right-hand side of Eqs. (1.10). These flows are linear functions of the gradients of the thermodynamic forces

$$\zeta_{ak}^{(1)}(x) = I_{ak,bl}(x) \nabla_l Y_b(x), \quad (2.7)$$

where the $I_{ak,bl}$ are generalized kinetic coefficients, which satisfy the symmetry principle for the Onsager kinetic coefficients

$$I_{ak,bl} = I_{bl,ak}. \quad (2.8)$$

For the isotropic equilibrium in which the macroscopic velocity is zero, only the quantities

$$I_{0k,0l} = T^2 \delta_{kl} \kappa, \quad I_{ij,lk} = T \zeta \delta_{jl} \delta_{ik} + T \eta \left(\delta_{il} \delta_{jk} + \delta_{lk} \delta_{ij} - \frac{2}{3} \delta_{ij} \delta_{lk} \right) \quad (2.9)$$

are nonzero, where κ is the heat conductivity coefficient, and ζ and η are the coefficients of the first and the second viscosity. The other kinetic coefficients vanish in Galilean-invariant media. The noninvertible dynamical equation for the entropy density

$$\dot{\sigma} = -\nabla_k(\nu_k \sigma + j_{\sigma k}^{(1)}) + I \quad (2.10)$$

is a consequence of Eqs. (2.6), (2.7), and (2.1). Here, the density $j_{\sigma k}^{(1)}$ of the dissipative entropy flow and the entropy production I in terms of the dissipation function have the forms

$$j_{\sigma k} = \frac{\partial \sigma}{\partial \zeta_a} \frac{\partial r}{\partial \nabla_k \partial \sigma / \partial \zeta_a}, \quad I = \nabla_k \frac{\partial \sigma}{\partial \zeta_a} \frac{\partial r}{\partial \nabla_k \partial \sigma / \partial \zeta_a} = 2r, \quad (2.11)$$

where

$$R \equiv \frac{1}{2} \int d^3x \nabla_k Y_a(x) I_{ak,bl}(x) \nabla_l Y_b(x) = \int d^3x r(x) \quad (2.12)$$

is the dissipation function. The entropy production in the isotropic state is guaranteed to be positive definite because the kinetic coefficients $\kappa > 0$, $\zeta > 0$, and $\eta > 0$ are positive.

We transform hydrodynamic equations (2.6) to a form reflecting that the structure of the dissipative terms of these equations is universal. It can be seen from definition (2.12) that the dissipation function is a quadratic functional of $\nabla_k Y$. By the Euler theorem, we then obtain the equality

$$\int d^3x Y_a(x) \frac{\delta R(Y)}{\delta Y_a(x)} \Big|_{I=\text{const}} = 2R(Y).$$

As a result, hydrodynamic equations (2.6) become

$$\dot{\zeta}_a(x) = -\nabla_k \zeta_{ak}^{(0)}(x) + \frac{\delta R}{\delta(\delta\Sigma/\delta\zeta_a(x))}, \quad (2.13)$$

where

$$\frac{\delta R}{\delta Y_a(x)} \Big|_{I=\text{const}} = \frac{\partial r(x)}{\partial Y_a(x)} \Big|_{I=\text{const}} - \nabla_k \frac{\partial r(x)}{\partial \nabla_k Y_a(x)}, \quad \frac{\delta \Sigma}{\delta \zeta_a(x)} = Y_a(x).$$

The second term in (2.13) has a universal form and is expressed in terms of the dissipation function.

3. Relaxation processes and dissipative Poisson brackets

We introduce dissipative Poisson brackets and give the relaxation terms in Eqs. (2.13) a Hamiltonian form. For this, we define the dissipative Poisson brackets for structureless liquids by the equality

$$\{A, B\}_D \equiv \int d^3x d^3x' \frac{\delta A}{\delta \zeta_a(x)} \{\zeta_a(x), \zeta_b(x')\}_D \frac{\delta B}{\delta \zeta_b(x')}. \quad (3.1)$$

We use the subscript D to distinguish the dissipative bracket from the ordinary reactive Poisson bracket. We show that the dissipative Poisson brackets for the macroscopic parameters $\zeta_a(x)$ characterizing the structureless continuous medium have the form

$$\{\zeta_a(x), \zeta_b(x')\}_D \equiv -\frac{1}{T_0} \nabla_k \nabla'_l (I_{ak,bl}(x) \delta(x-x')), \quad (3.2)$$

where T_0 is a constant with the dimension of temperature. In accordance with definition (3.2), the dissipative bracket can be expressed in terms of dissipation function (2.12). We can easily see that

$$\frac{\delta^2 R}{\delta Y_a(x) \delta Y_b(x')} \Big|_{I=\text{const}} = \nabla_k \nabla'_j (\delta(x-x') I_{ak,bj}(x)) = -T_0 \{\zeta_a(x), \zeta_b(x')\}_D. \quad (3.3)$$

By the symmetry principle for Onsager kinetic coefficients (2.8), Poisson bracket (3.2) is symmetric:

$$\{\zeta_a(x), \zeta_b(x')\}_D = \{\zeta_b(x'), \zeta_a(x)\}_D. \quad (3.4)$$

In addition, the Leibniz rule

$$\{AB, C\}_D = A\{B, C\}_D + B\{A, C\}_D \quad (3.5)$$

holds for this bracket. We note that the relations

$$\{\zeta_a(x), \Sigma\}_R = 0, \quad \{\zeta_a(x), H\}_D = 0 \quad (3.6)$$

hold by virtue of formulas (1.14) and (3.5).

The quantity canonically conjugate to the entropy density is not contained in the Hamiltonian and is a cyclic variable (see [21], whence the first equality in (3.6) follows). The second relation in (3.6) is obvious by definition (3.3). If we introduce a generalized Poisson bracket using the equality

$$\{A, B\}_G \equiv \int d^3x \int d^3x' \frac{\delta A}{\delta \zeta_a(x)} [\{\zeta_a(x), \zeta_b(x')\}_R + \{\zeta_a(x), \zeta_b(x')\}_D] \frac{\delta B}{\delta \zeta_b(x')}, \quad (3.7)$$

then taking (3.6) into account, we can write the dynamical equations for the parameters of the reduced description including relaxation processes as

$$\dot{\zeta}_a(x) = \{\zeta_a(x), H - T_0 \Sigma\}_G. \quad (3.8)$$

Dissipative brackets (3.3) are similar to those introduced in [10], [26], and [27] and lead to the same dynamical equations for normal liquids.

We now consider an arbitrary condensed medium for which the collection of parameters of the reduced description is a set of quantities $\varphi_\alpha(x)$. In accordance with (2.12) and (3.3), we assume that the dissipation function and the dissipative Poisson brackets are related by the equalities

$$\begin{aligned} R &\equiv -\frac{T_0}{2} \int d^3x \int d^3x' \frac{\delta \Sigma}{\delta \varphi_\alpha(x)} \{\varphi_\alpha(x), \varphi_\beta(x')\}_D \frac{\delta \Sigma}{\delta \varphi_\beta(x')} = \int d^3x r(x), \\ \{\varphi_\alpha(x), \varphi_\beta(x')\}_D &\equiv -\frac{1}{T_0} \frac{\delta^2 R}{\delta(\delta \Sigma / \delta \varphi_\alpha(x)) \delta(\delta \Sigma / \delta \varphi_\beta(x'))} \Big|_{I=\text{const}} \equiv R_{\alpha\beta}(x, x'; \varphi). \end{aligned} \quad (3.9)$$

In the framework of the Hamiltonian formalism with dissipative processes taken into account, the equation of motion for $\varphi_\alpha(x)$ becomes

$$\dot{\varphi}_\alpha(x) \equiv \{\varphi_\alpha(x), H\}_R - T_0 \{\varphi_\alpha(x), \Sigma\}_D = L_{\varphi_\alpha}^R(x) + L_{\varphi_\alpha}^D(x), \quad (3.10)$$

where the first term has the ordinary Hamiltonian form corresponding to the reactive Poisson bracket and the second term takes the contribution of relaxation processes into account and is represented in the form of a dissipative Poisson bracket. According to (3.9) and (3.10), we obtain

$$\begin{aligned} L_{\varphi_\alpha}^D(x) &= -T_0 \int d^3x' \frac{\delta \Sigma}{\delta \varphi_\beta(x')} \{\varphi_\alpha(x), \varphi_\beta(x')\}_D = \\ &= \int d^3x' \frac{\delta \Sigma}{\delta \varphi_\beta(x')} \frac{\delta^2 R}{\delta(\delta \Sigma / \delta \varphi_\alpha(x)) \delta(\delta \Sigma / \delta \varphi_\beta(x'))}. \end{aligned} \quad (3.11)$$

This implies the universal structure of the dissipative terms in the equation for $\varphi_\alpha(x)$

$$L_{\varphi_\alpha}^D = \frac{\delta R}{\delta(\delta \Sigma / \delta \varphi_\alpha(x))} \quad (3.12)$$

because of the homogeneity of the second-degree dissipation functional, i.e., the relation

$$\int d^3x \frac{\delta \Sigma}{\delta \varphi_\beta(x)} \frac{\delta R}{\delta(\delta \Sigma / \delta \varphi_\beta(x))} = 2R \quad (3.13)$$

holds.

We next consider two condensed media (superfluid liquid He II and an elastic solid) and use these examples to show the validity of relations (3.12) and (3.13) and the equalities

$$\{\varphi_\alpha(x), \Sigma\}_R = 0, \quad \{\varphi_\alpha(x), H\}_D = 0. \quad (3.14)$$

The equations for the relaxation dynamics for the parameters of the reduced description hence become similar to formula (3.8),

$$\dot{\varphi}_\alpha(x) = \{\varphi_\alpha(x), H - T_0 \Sigma\}_G. \quad (3.15)$$

By definition (3.9), the dissipative Poisson bracket is invariant, and the dynamical equations taking relaxation processes into account are covariant under canonical transformations (1.6) if dissipative bracket (3.9) satisfies the relation

$$R_{\beta\gamma}(x', x''; \varphi) = \int d^3 y_1 d^3 y_2 \frac{\delta \varphi_\beta(x')}{\delta \varphi'_\delta(y_1)} R_{\delta\rho}(y_1, y_2; \varphi') \frac{\delta \varphi_\gamma(x'')}{\delta \varphi'_\rho(y_2)}.$$

4. Superfluid liquid He II, relaxation equations, and dissipative Poisson brackets

The considered superfluid liquid at the hydrodynamic stage of the evolution is characterized by an additional quantity, namely, the superfluid phase ϕ , which is canonically conjugate to the density of the number of particles. It can be shown that the Poisson brackets for this quantity and the momentum, mass, and entropy densities have the forms [21]

$$\begin{aligned} \{\pi_i(x), \phi(x')\}_R &= -\delta(x - x') \nabla_i \phi(x), \\ \{\rho(x), \phi(x')\}_R &= \delta(x - x'), \quad \{\sigma(x), \phi(x')\}_R = 0. \end{aligned} \quad (4.1)$$

Because of the phase invariance, the Hamiltonian depends only on the gradient of the superfluid phase: $H = H(\sigma(x'), \pi_i(x'), \rho(x'), p_k(x'))$, where $p_k = \nabla_k \phi$ is the superfluid momentum. For the thermodynamic potential $\omega(Y, \vec{p})$, which is defined by relation (2.2) as before, the main thermodynamic relation [13]

$$d\omega = \varepsilon dY_0 + \pi_k dY_k + n dY_4 + (Y_0 j_l + Y_l n) dp_l$$

holds, where j_k is the flux density of the number of particles. Using formulas (1.12), (1.14), and (4.1), we can express the flow densities of the additive integrals of motion in terms of the thermodynamic potential [13]:

$$\zeta_{ak}^{(0)} = -\frac{\partial}{\partial Y_a} \frac{\omega Y_k}{Y_0} + \frac{\partial \omega}{\partial p_k} \frac{\partial p_0}{\partial Y_a}, \quad p_0 \equiv \frac{Y_4 + Y_k p_k}{Y_0}. \quad (4.2)$$

Taking the dissipative processes into account, we can express the equation for superfluid liquids as

$$\dot{\zeta}_a = -\nabla_k (\zeta_{ak}^{(0)} + \zeta_{ak}^{(1)}), \quad \dot{\varphi} = p_0 + L_\varphi^{(1)}, \quad (4.3)$$

where the reactive flows $\zeta_{ak}^{(0)}$ and p_0 describe the dynamics of liquids in the adiabatic approximation. The dissipative flows $\zeta_{ak}^{(1)}$ and $L_\varphi^{(1)}$ are obtained in the microscopic consideration [13]:

$$\zeta_{ak}^{(1)} = \nabla_l Y_b I_{bl, ak} + \nabla_i \frac{\partial \omega}{\partial p_i} I_{ak}, \quad L_\varphi^{(1)} = \nabla_l Y_b I_{bl} + \nabla_i \frac{\partial \omega}{\partial p_i} \bar{I}. \quad (4.4)$$

The generalized kinetic coefficients $I_{bl,ak}$, I_{ak} , and \bar{I} describe relaxation processes in the superfluid liquid under study. They are related to the viscosity coefficients ζ_1 , ζ_2 , ζ_3 , and η and the heat conductivity coefficient κ by the equalities

$$\begin{aligned} I_{0k,0l} &= T^2 \delta_{kl} \kappa, & I_{bl} &= \delta_{bl} T \zeta_1, & \bar{I} &= T \zeta_3, \\ I_{lj,ik} &= T \zeta_2 \delta_{jl} \delta_{ik} + T \eta \left(\delta_{il} \delta_{jk} + \delta_{lk} \delta_{ij} - \frac{2}{3} \delta_{lj} \delta_{ik} \right). \end{aligned} \quad (4.5)$$

Equations (4.3) and (4.4) imply a noninvertible dynamical equation for entropy density (2.10), where

$$I = \nabla_k Y_a \zeta_{ak}^{(1)} + \nabla_k \frac{\partial \omega}{\partial p_k} L_\varphi^{(1)} \geq 0, \quad j_{\sigma k}^{(1)} = Y_a \zeta_{ak}^{(1)} + \frac{\partial \omega}{\partial p_k} L_\varphi^{(1)} \quad (4.6)$$

are the entropy production and the dissipative entropy flow density of superfluid liquid He II. Formulas (4.4) and (4.6) lead to the explicit form of the density of the dissipation function of the superfluid liquid

$$r = \frac{1}{2} \left(\nabla_k Y_a I_{ak,bl} \nabla_l Y_b + \nabla_k \frac{\partial \omega}{\partial p_k} \bar{I} \nabla_l \frac{\partial \omega}{\partial p_l} \right) + \nabla_k Y_a I_{ak} \nabla_l \frac{\partial \omega}{\partial p_l}. \quad (4.7)$$

The entropy production in expression (4.6) for the isotropic case (for $Y_k = p_k = 0$) is guaranteed to be positive by the inequalities for the kinetic coefficients $\kappa \geq 0$, $\zeta_3 \geq 0$, $\eta \geq 0$, and $\zeta_1^2 \leq \zeta_2 \zeta_3$. Taking into account that

$$\left. \frac{\partial \omega}{\partial p_k} \right|_Y = - \left. \frac{\partial \sigma}{\partial p_k} \right|_\zeta$$

and using formula (4.7), we obtain the equations of superfluid hydrodynamics

$$\dot{\zeta}_a(x) = -\nabla_k \zeta_{ak}^{(0)}(x) + \frac{\delta R}{\delta(\delta \Sigma / \delta \zeta_a(x))}, \quad \dot{\varphi}(x) = p_0(x) + \frac{\delta R}{\delta(\delta \Sigma / \delta \varphi(x))}, \quad (4.8)$$

which have the universal form of the relaxation terms (see (3.9)). From the explicit form of dissipation function (4.7), we can also easily verify that relation (3.13) holds for the system under consideration. Knowing dissipation function (4.7) and using formula (3.9), we can find the dissipative Poisson brackets for the superfluid liquid under study:

$$\begin{aligned} \{\zeta_a(x), \zeta_b(x')\}_D &= -\frac{1}{T_0} \nabla_k \nabla'_j (\delta(x-x') I_{ak,bj}(x)), \\ \{\zeta_a(x), \phi(x')\}_D &= \frac{1}{T_0} \nabla_k (\delta(x-x') I_{ak}(x)), \quad \{\phi(x), \phi(x')\}_D = -\frac{1}{T_0} (\delta(x-x') \bar{I}(x)). \end{aligned}$$

5. Relaxation elasticity theory and dissipative Poisson brackets

To construct a relaxation elasticity theory, we consider the deformation tensor and the vectors of the reciprocal lattice in terms of the deformation tensor. Because we use the Eulerian description everywhere, these additional macroscopic parameters are also introduced in terms of the Eulerian variables. We define the Cauchy–Green deformation tensor by the equality

$$u_{ij}(x) \equiv \frac{1}{2} (\delta_{ij} - b_{\lambda i}(x) b_{\lambda j}(x)), \quad (5.1)$$

where the convolution in the right-hand side is taken with respect to the Lagrangian index. Taking formulas (1.14) into account, we write the unique nontrivial Poisson bracket for this quantity

$$\begin{aligned} \{u_{ij}(x), \pi_k(x')\}_R &= u_{ijk\lambda}(x) \nabla_\lambda \delta(x-x') - \delta(x-x') \nabla_k u_{ij}(x), \\ u_{ijk\lambda}(x) &\equiv \left(\frac{1}{2} \delta_{jk} - u_{jk}(x) \right) \delta_{i\lambda} + \left(\frac{1}{2} \delta_{ik} - u_{ik}(x) \right) \delta_{j\lambda}. \end{aligned} \quad (5.2)$$

We define the Eulerian vectors of the reciprocal crystal lattice in the deformed state by the equality

$$b_{\alpha i}(x) \equiv \underline{b}_{\alpha j} b_{ji}(x), \quad (5.3)$$

where $\underline{b}_{\alpha j}$ is the Lagrangian vector of the reciprocal crystal lattice in the undeformed state. According to (1.14) and (5.3), the nontrivial Poisson bracket for $b_{\alpha i}(x)$ with the momentum density has the form

$$\{b_{\alpha i}(x), \pi_k(x')\}_R = -\nabla_i (b_{\alpha k}(x) \delta(x-x')). \quad (5.4)$$

The energy density of solids is a function of the parameters, $\varepsilon(x) = \varepsilon(\sigma(x), \pi_i(x), u_k(x), u_{ik}(x), b_{\alpha k}(x))$. In view of the spatial homogeneity of energy density (1.12), we obtain the equality $\partial \varepsilon(x) / \partial u_k(x) = 0$. Therefore, the energy density and the thermodynamic potential are independent of the displacement vector. The thermodynamic potential is a function of the thermodynamic forces, the vectors of the reciprocal lattice, and the deformation tensor: $\omega = \omega(Y_a, b_{\alpha k}, u_{ik})$. Taking formulas (1.11) and Poisson brackets (1.14), (5.2), and (5.4) into account, we can easily show that the flow densities of the additive integrals of motion in the considered case become

$$\begin{aligned} \zeta_{ak}^{(0)} &= -\frac{\partial}{\partial Y_a} \frac{\omega Y_k}{Y_0} + \frac{\partial \omega}{\partial u_{ik}} \frac{\partial g_l}{\partial Y_a} + \frac{\partial \omega}{\partial b_{\alpha k}} \frac{\partial b_{0\alpha}}{\partial Y_a}, \\ g_l &= \frac{Y_i (2u_{il} - \delta_{il})}{Y_0}, \quad b_{0\alpha} = \frac{Y_i b_{\alpha i}}{Y_0}, \end{aligned} \quad (5.5)$$

where $a = 0, k$. In deriving expression (5.5), we take the theorem on the relation between the flow densities of the additive integrals of motion in local equilibrium [13] into account:

$$Y_a (Y_k \zeta_a + Y_0 \zeta_{ak}) = 0.$$

Taking Poisson brackets (5.2) and (5.4) for the deformation tensor and the vectors of the lattice into account, by virtue of (1.1), we obtain the dynamical equations

$$\dot{u}_{ik} = u_{ik\lambda l} \nabla_\lambda \nu_l - \nu_l \nabla_l u_{ik}, \quad \dot{b}_{\alpha i} = -\nabla_i (\nu_k b_{\alpha k}). \quad (5.6)$$

Precisely these equations, together with (1.10) and expression (5.5) for the flows, are the elasticity equations in the adiabatic approximation. Equation (2.5) holds for the entropy density.

We can take relaxation processes into account if we add the terms corresponding to the dissipative flows, i.e.,

$$\begin{aligned} \dot{\zeta}_a &= -\nabla_k (\zeta_{ak}^{(0)} + \zeta_{ak}^{(1)}) \equiv L_a^{(1)} + L_a^{(2)}, \\ \dot{b}_{\alpha i} &= -\nabla_i ((\nu_k + \nu_k^{(1)}) b_{\alpha k}) \equiv L_{\alpha i}^{(1)} + L_{\alpha i}^{(2)}, \\ \dot{u}_{ik} &= u_{ik\lambda l} \nabla_\lambda (\nu_l + \nu_l^{(1)}) - (\nu_l + \nu_l^{(1)}) \nabla_l u_{ik} \equiv L_{ik}^{(1)} + L_{ik}^{(2)}, \end{aligned} \quad (5.7)$$

to the reactive flows in the right-hand sides of Eqs. (1.10) and (5.6). Using (2.2) and (5.7), we obtain equation of motion (2.10) for the entropy density, where

$$\begin{aligned} I &= \nabla_k Y_a \zeta_{ak}^{(1)} + D_l \nu_l^{(1)} \geq 0, \\ j_{\sigma k}^{(1)} &= Y_a \zeta_{ak}^{(1)} - \frac{\partial \omega}{\partial u_{kj}} (2u_{jl} - \delta_{jl}) \nu_l^{(1)} - \frac{\partial \omega}{\partial b_{\alpha k}} b_{\alpha l} \nu_l^{(1)}, \\ D_l &\equiv -\frac{\partial \sigma}{\partial u_{ik}} \nabla_l u_{ik} + (2u_{kl} - \delta_{kl}) \nabla_i \frac{\partial \sigma}{\partial u_{ik}} + 2 \frac{\partial \sigma}{\partial u_{ik}} \nabla_i u_{lk} + b_{\alpha l} \nabla_i \frac{\partial \sigma}{\partial b_{\alpha i}}. \end{aligned} \quad (5.8)$$

The dissipative flows $\zeta_{ak}^{(1)}$ and $\nu_l^{(1)}$ are linear functions of the gradients of the thermodynamic forces $\nabla_k Y_a$ and the vector D_l ,

$$\zeta_{ak}^{(1)} = \nabla_l Y_b I_{bl,ak} + D_l I_{ak,l}, \quad \nu_j^{(1)} = \nabla_l Y_b I_{j,bl} + D_l \bar{I}_{j,l}, \quad (5.9)$$

and the kinetic coefficients $I_{bl,ak}$, $I_{ak,l}$, $I_{l,ak}$, and $\bar{I}_{j,l}$ satisfy the Onsager symmetry principle,

$$I_{ak,bl} = I_{bl,ak}, \quad I_{ak,l} = I_{l,ak}, \quad \bar{I}_{j,l} = \bar{I}_{l,j}. \quad (5.10)$$

In the considered case, the density of the dissipation function has the form

$$r = \frac{1}{2} (\nabla_k Y_a I_{ak,bl} \nabla_l Y_b + D_k \bar{I}_{k,l} D_l) + \nabla_k Y_a I_{ak,l} D_l. \quad (5.11)$$

Using expression (5.11) for the dissipation function, we obtain the universal structure of the dissipative flows (see (3.9)) in the relaxation equations of elasticity theory:

$$L_a^{(2)}(x) = \frac{\delta R}{\delta(\delta \Sigma / \delta \zeta_a(x))}, \quad L_{\alpha i}^{(2)}(x) = \frac{\delta R}{\delta(\delta \Sigma / \delta b_{\alpha i}(x))}, \quad L_{ik}^{(2)}(x) = \frac{\delta R}{\delta(\delta \Sigma / \delta u_{ik}(x))}.$$

In the case of isotropic solids where $b_{\alpha k} = Y_k = 0$ in equilibrium, the medium is characterized by five kinetic coefficients. Three of them have forms (2.9), and two new kinetic coefficients are defined by the equalities $\bar{I}_{ij} \equiv \delta_{ij} B$ and $I_{ak,l} \equiv \delta_{a0} \delta_{kl} A$. The kinetic viscoelasticity coefficients A and B for positive entropy production satisfy the inequalities $B \geq 0$ and $T^2 \kappa B \geq A^2$. In conclusion, we present the dissipative Poisson brackets for the elastic medium. Taking definition (3.9) and the explicit form of dissipation function density (5.11) into account, we obtain

$$\begin{aligned} \{\zeta_a(x), \zeta_b(x')\}_D &= -\frac{1}{T_0} \nabla_k \nabla'_j (\delta(x-x') I_{ak,bj}(x)), \\ \{\zeta_a(x), \zeta_{\beta l}(x')\}_D &= -\frac{1}{T_0} \nabla_k \nabla'_l (\delta(x-x') I_{ak,j}(x) b_{\beta j}(x)), \\ \{b_{\alpha k}(x), b_{\beta l}(x')\}_D &= -\frac{1}{T_0} \nabla_k \nabla'_l (\delta(x-x') b_{\alpha i}(x) \bar{I}_{i,j}(x) b_{\beta j}(x)), \\ \{b_{\alpha k}(x), u_{ij}(x')\}_D &= -\frac{1}{T_0} \nabla_k \{b_{\alpha l}(x) \bar{I}_{l,n}(x) [\delta(x-x') \nabla_n u_{ij}(x) + \\ &\quad + \nabla_p (\delta(x-x') u_{ijnp}(x))]\}, \\ \{\zeta_a(x), u_{ij}(x')\}_D &= -\frac{1}{T_0} \nabla_k \{I_{ak,l}(x) [\delta(x-x') \nabla_l u_{ij}(x) + \nabla_p (\delta(x-x') u_{ijlp}(x))]\}, \\ \{u_{ik}(x), u_{lj}(x')\}_D &= -\frac{1}{T_0} \{ \bar{I}_{n,m}(x) (\delta(x-x') \Gamma_{m,jl}(x) - u_{jlmq}(x) \nabla_q \delta(x-x')) \Gamma_{n,ik}(x) + \\ &\quad + \nabla_p [\bar{I}_{n,m}(x) (\delta(x-x') \Gamma_{m,jl}(x) - \\ &\quad - u_{jlmq}(x) \nabla_q \delta(x-x')) u_{iknp}(x)] \}, \end{aligned}$$

where $\Gamma_{\lambda,lj} \equiv \nabla_{\lambda} u_{lj} + \nabla_p u_{lj\lambda p}$ is the Christoffel symbol of the first kind. It follows from the form of these dissipative Poisson brackets that equality (3.14) holds and the relaxation structure of the equations of elasticity in form (3.15) is universal.

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