GITA^{-1}: A symbolic computing program for an inverse problem of the Birkhoff–Gustavson normal form expansion

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Abstract

We present briefly a symbolic computing program named GITA^{-1} working on REDUCE 3.3 or later, which solves an 'inverse problem' for the Birkhoff–Gustavson normal form (BGNF) expansion.

Keywords: Inverse problem; Normal form; Computer algebra; REDUCE

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1. Introduction: A brief history of GITA^{-1}

It is widely known that the Birkhoff–Gustavson normal form (BGNF) expansion works effectively to study a behavior of nonlinear dynamical systems; the Hénon–Heiles system is often taken as a typical example [1]. Since a major part of the BGNF expansion is made on the polynomial algebra [2], the BGNF expansion fits very well the symbolic computing on computers. Three of the authors (N.A.C., V.A.R., and S.I.V.) have proposed, with Basios and Markovski, the symbolic computing programs named GITA, which realizes the algebraic procedure of converting power-series Hamiltonians into their BGNF [3].

The rest of the authors (Y.U.) has posed an 'inverse problem' of the BGNF expansion with the aim of applying the results in a series of his quantum studies on the BGNF dynamical systems [4,5] to certain systems of physical interests. The inverse problem is stated as follows: '*Identify a class of dynamical systems which are reduced to the same BGNF up to a certain order*'.

To solve the inverse problem, the authors have joined to propose a symbolic computing program named GITA^{-1}. As a test case, GITA^{-1} succeeds to fix a class of integrable Hamiltonian systems, which shares the same BGNF Hamiltonian as the regularized system of the planar hydrogen atom with the linear Stark effect (PHALSE) [6].

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In this short paper, GITA^{-1} for the general n-degree of freedom case is presented very briefly. The inverse problem of PHALSE is taken as an example.

2. GITA^{-1}: mathematical preliminaries

Let $H_{\mathrm{IN}}(q, p)$ and $H_{\mathrm{OUT}}(q, p)$ be the *input* and the *output* Hamiltonians, which are expressed as

$$H_\lambda(q, p) = \sum_{h=2}^{\infty} H_\lambda^{(h)}(q, p) \quad \text{with} \quad H_\lambda^{(2)}(q, p) = \tfrac{1}{2}\sum_{k=1}^{n}\left(p_k^2 + q_k^2\right), \tag{1}$$

where $H_\lambda^{(h)}$ $(\lambda = \mathrm{IN}, \mathrm{OUT},\ h = 3, 4, \ldots)$ is a degree-h homogeneous polynomial in (q, p) expressed as

$$H_\lambda^{(h)}(q, p) = \sum_{|\alpha|+|\beta|=h} c_\lambda^{(h)}(\alpha, \beta) q^\alpha p^\beta \quad \text{with}$$
$$q^\alpha p^\beta = q_1^{\alpha_1} \cdot\cdot q_n^{\alpha_n} p_1^{\beta_1} \cdot\cdot p_n^{\beta_n}, \quad |\alpha| = \sum_{k=1}^{n}\alpha_k,\ |\beta| = \sum_{k=1}^{n}\beta_k. \tag{2}$$

Let $G_{\mathrm{IN}}(\xi, \eta)$ and $G_{\mathrm{OUT}}(\xi, \eta)$ be the *input* and the *output* BGNF Hamiltonian,

$$G_\lambda(\xi, \eta) = \sum_{j=1}^{\infty} G_\lambda^{(2j)}(\xi, \eta) \quad \text{with} \quad G_\lambda^{(2)}(\xi, \eta) = \tfrac{1}{2}\sum_{k=1}^{n}\left(\eta_k^2 + \xi_k^2\right), \tag{3}$$

where $G_\lambda^{(2j)}$ $(\lambda = \mathrm{IN}, \mathrm{OUT},\ j = 2, 3, \ldots)$ is a degree-$2j$ homogeneous polynomial in (ξ, η) expressed as

$$G_\lambda^{(2j)}(\xi, \eta) = \sum_{|\alpha|+|\beta|=2j} \gamma_\lambda^{(2j)}(\alpha, \beta)\xi^\alpha \eta^\beta \quad \text{with} \quad \left\{G_\lambda^{(2)}, G_\lambda^{(2j)}\right\} = 0. \tag{4}$$

In Eq. (4), α and β are multi-indices used in the same way as in Eq. (2), and $\{\cdot, \cdot\}$ is the canonical Poisson bracket associated with the position variables ξ and the momentum ones η.

Let us denote, by P_ℓ, the space of degree-ℓ homogeneous polynomials in $2n$-variables with real-coefficients, which can be identified with the vector space $\mathbb{R}^{N(n,\ell)}$, where $N(n, \ell)$ indicates the number of degree-ℓ monomials in $2n$-variables allowed to exist. Then, denoting such a correspondence by $\iota_\ell : P_\ell \to \mathbb{R}^{N(n,\ell)}$, we associate the vectors, $\vec{c}_\lambda^{(h)}$ and $\vec{\gamma}_\lambda^{(2j)}$, with H_λ and G_λ by

$$\vec{c}_\lambda^{(h)} = \iota_h(H_\lambda^{(h)}) \in \mathbb{R}^{N(n,h)} \quad \text{and} \quad \vec{\gamma}_\lambda^{(2j)} = \iota_{2j}(G_\lambda^{(2j)}) \in \mathbb{R}^{N(n,2j)}, \tag{5}$$

respectively. Further, using ι_ℓ, we express the differential operator,

$$D = \sum_{k=1}^{n}\left\{p_k(\partial/\partial q_k) - q_k(\partial/\partial p_k)\right\}, \tag{6}$$

restricted on P_ℓ by the matrix $M^{(\ell)}$ acting on $\mathbb{R}^{N(n,\ell)}$; $\iota_\ell \circ D = M^{(\ell)} \circ \iota_\ell$.

After the preparation above, what GITA solves is the series of equations,

$$\vec{\gamma}_\lambda^{(h)} = M^{(h)}\left\{\vec{c}_\lambda^{(h)} + \Phi^{(h)}(\vec{c}_\lambda^{(h-1)}, \cdot, \cdot, \vec{c}_\lambda^{(2)})\right\} \quad (h = 3, 4, \ldots) \tag{7}$$

for $\vec{\gamma}_\lambda^{(h)}$ $(\lambda = \mathrm{IN}, \mathrm{OUT})$ [2,6], where $\vec{\gamma}_\lambda^{(2j+1)}$ turns out to vanish [6].

We are now in a position to present what GITA^{-1} computes: Let us recall the inverse problem posed in Section 1, which is put in the following: '*For a given H_{IN}, identify all the possible (or a part of) H_{OUT} subject to $G_{\mathrm{IN}} = G_{\mathrm{OUT}}$*

up to a certain order'. Since $G_{\rm IN} = G_{\rm OUT}$ can read $\vec{\gamma}^{(h)}_{\rm IN} = \vec{\gamma}^{(h)}_{\rm OUT}$ $(h = 2, 3, \ldots)$, GITA^{-1} solves the series of equations,

$$M^{(h)}\vec{c}^{(h)}_{\rm OUT} = \vec{\gamma}^{(h)}_{\rm IN} - M^{(h)}\Phi^{(h)}(\vec{c}^{(h-1)}_{\rm OUT}, \cdot, \cdot, \vec{c}^{(2)}_{\rm OUT}) \quad (h = 3, 4, \ldots), \tag{8}$$

for $\vec{c}^{(h)}_{\rm OUT}$, where $\vec{\gamma}^{(h)}_{\rm IN}$ are determined beforehand from $H_{\rm IN}$ through GITA (i.e. (1)–(7)). In the next section, we demonstrate how GITA^{-1} works on REDUCE in the inverse problem of PHALSE.

3. Overlook on the GITA^{-1}: Program via a test example

GITA^{-1} consists of a core part and a subsidiary part. The core part is derived from GITA [3] and the subsidiary part contains the procedures characteristic of GITA^{-1}, both of which are put together in a single file. The procedure list and the input data (the input Hamiltonian) are loaded at the beginning of running GITA^{-1}.

As an example of program fulfillment, we take the inverse problem of PHALSE. Let the input Hamiltonian $H_{\rm IN}$ be

$$H_{\rm IN}(q, p) = \tfrac{1}{2}\big(p_1^2 + p_2^2 + q_1^2 + q_2^2\big) + \tfrac{8}{3}\varepsilon(q_1^4 - q_2^4), \tag{9}$$

the Hamiltonian of regularized system of PHALSE [6]. The input BGNF Hamiltonian, $G_{\rm IN}$, for $H_{\rm IN}$ is calculated, up to the fourth order, to be

$$G^{(4)}_{\rm IN}(\xi, \eta) = \varepsilon(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)(\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2), \tag{10}$$

where ε is a parameter. The vector $\vec{\gamma}^{(4)}_{\rm IN}$ in Eq. (8) is hence fixed by the coefficients, $\gamma^{(4)}_{\rm IN}(\alpha_1, \alpha_2, \beta_1, \beta_2)$, of $G^{(4)}_{\rm IN}$ through (5). It is worth noting that all these preliminary calculations can be made by GITA.

The GITA^{-1} starts with generating the field P_ℓ to describe the output Hamiltonian $H^{(4)}_{\rm OUT}$ all of whose coefficients $c^{(4)}_{\rm OUT}(\alpha, \beta)$ are unidentified. Next, Eqs. (1)–(7) with $\lambda = $ OUT are proceeded in GITA^{-1} to calculate $G^{(4)}_{\rm OUT}$, all of whose coefficients, $\gamma^{(4)}_{\rm OUT}(\alpha, \beta)$, expressed in terms of $c^{(4)}_{\rm OUT}(\alpha, \beta)$ unidentified. On equating $G_{\rm IN}$ with $G_{\rm OUT}$ in GITA^{-1}, Eq. (8) takes the following form:

$$\begin{aligned} &3c^{(4)}_{\rm OUT}(0,4,0,0) + c^{(4)}_{\rm OUT}(0,2,0,2) + 3c^{(4)}_{\rm OUT}(0,0,0,4) = -8\varepsilon, \\ &3c^{(4)}_{\rm OUT}(4,0,0,0) + c^{(4)}_{\rm OUT}(2,0,2,0) + 3c^{(4)}_{\rm OUT}(0,0,4,0) = 8\varepsilon, \end{aligned} \tag{11}$$

where the coefficients, $c^{(4)}_{\rm OUT}(k_1, k_2, \ell_1, \ell_2)$, not listed in (11) are set to zero.

Applying the subroutine SOLVE in REDUCE to Eq. (11), we have

$$\begin{aligned} &c^{(4)}_{\rm OUT}(0,0,4,0) = u(4), && c^{(4)}_{\rm OUT}(0,0,0,4) = u(2), \\ &c^{(4)}_{\rm OUT}(0,4,0,0) = \big(-3u(2) - u(1) - 8\varepsilon\big)/3, && c^{(4)}_{\rm OUT}(0,2,0,2) = u(1), \\ &c^{(4)}_{\rm OUT}(4,0,0,0) = \big(-3u(4) - u(3) + 8\varepsilon\big)/3, && c^{(4)}_{\rm OUT}(2,0,2,0) = u(3), \end{aligned} \tag{12}$$

as the solution of Eq. (11), where $u(i)$ $(i = 1, 2, 3, 4)$ denote the unidentified constants 'arbcomplex(i)' introduced automatically in SOLVE.

Finally, we identify the output Hamiltonian to be

$$\begin{aligned} H^{(4)}_{\rm OUT}(q, p) = {} & u(4)p_1^4 + u(2)p_2^4 + u(3)q_1^2p_1^2 + u(1)q_2^2p_2^2 \\ & + \tfrac{1}{3}\big\{-3u(4) - u(3) + 8\varepsilon\big\}q_1^4 + \tfrac{1}{3}\big\{-3u(2) - u(1) - 8\varepsilon\big\}q_2^4, \end{aligned} \tag{13}$$

which admits $G_{\rm IN}$ given by Eq. (10) as the BGNF. Note that if all the $u(i)$ vanish in Eq. (13), $H_{\rm OUT}$ becomes identical with $H_{\rm IN}$ given by (9).

On setting $u(2)=u(4)=0$ in Eq. (13), H_{OUT} becomes

$$
\begin{aligned}
H_{\mathrm{OUT}}(q,p) = {} & \frac{1+2u(3)q_1^2}{2}p_1^2 + \frac{1+2u(1)q_2^2}{2}p_2^2 \\
& + q_1^2 + \tfrac{1}{3}\{-u(3)+8\varepsilon\}q_1^4 + q_2^2 - \tfrac{1}{3}\{u(1)+8\varepsilon\}q_2^4.
\end{aligned} \tag{14}
$$

Surprisingly, H_{OUT} turns out to be a Hamiltonian admitting the separation of variables, which provides an integrable system accordingly. Although it seems to be incidental that we encounter such integrable systems after GITA^{-1}, one may expect to find a class of integrable systems whose Hamiltonians H_{OUT} reduce to a given BGNF Hamiltonian $G_{\mathrm{OUT}}=G_{\mathrm{IN}}$, which is generated by Hamiltonian H_{IN}.

From physical point of view one can expect that GITA^{-1} will give the opportunity to find an integrable prototype of Hamiltonian systems like hydrogen atom in external fields.

4. Concluding remarks

We would like to make a few remarks on the programming of GITA^{-1}.

(i) In generating the field P_ℓ and Eq. (8), we have used REDUCE to implement the combinatorial algorithms and the list processing.

(ii) As is seen from the algorithm (1)–(8) presented in Section 2, GITA^{-1} is proceeded by tracing back the procedures of GITA in principle. Hence we may expect that GITA and GITA^{-1} are put together to provide a unified symbolic computing program for various calculation around the BGNF expansion in future. The authors continue improving the program of GITA^{-1} to make it run faster, and wish to publish it like Long Write-up in Program Library CPC.

Note, a review of another symbolic computing algorithm named ANFER (Algorithm of Normal Form Expansion and Restoration) has been appeared recently [7]. We hope that GITA and GITA^{-1} can be put together with ANFER to create a unified symbolic computing program for calculation of BGNF from ordinary and inverse points of view.

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