Article

On Solvability Conditions for a Certain Conjugation Problem

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Abstract: We study a certain conjugation problem for a pair of elliptic pseudo-differential equations with homogeneous symbols inside and outside of a plane sector. The solution is sought in corresponding Sobolev–Slobodetskii spaces. Using the wave factorization concept for elliptic symbols, we derive a general solution of the conjugation problem. Adding some complementary conditions, we obtain a system of linear integral equations. If the symbols are homogeneous, then we can apply the Mellin transform to such a system to reduce it to a system of linear algebraic equations with respect to unknown functions.

Keywords: pseudo-differential equation; conjugation problem; wave factorization; solvability condition

1. Introduction

The theory of pseudo-differential equations on manifolds with a smooth boundary was systematically developed, starting from the papers of M.I. Vishik and G.I. Eskin [1,2] in the middle of the last century. After this start, L. Boutet de Monvel [3] published a paper in which he suggested an algebraic variant of the theory, including the index theorem. These studies were continued and refined by S. Rempel and B.-W. Schulze [4], and then such results have became useful for situations of manifolds with non-smooth boundaries [5–7].

The first author has started to develop a new approach for non-smooth situations in the middle of the last century [8], and general concepts of the approach are presented in the book and latest papers [9–11]. This paper is related to this approach, and it is devoted to some generalizations of classical results for the Riemann boundary value problem [12,13] in which we consider model pseudo-differential equations in canonical non-smooth domains instead of the Cauchy–Riemann operator. These studies were indicated in [14], and here we develop these results, obtaining more exact and refined solvability conditions. We formulate the solvability conditions in terms of a system of linear algebraic equations similar to well-known Shapiro–Lopatinskii conditions [2]. The Mellin transform [15] is used to reduce the problem for homogeneous elliptic symbols to the mentioned algebraic system.

2. Auxiliaries

A pseudo-differential operator A in a domain $D \subset \mathbf{R}^m$ is defined by its symbol $A(\xi)$ in the following way

$$u(x) \longmapsto \int\limits_{D} \int\limits_{\mathbf{R}^m} A(\xi) e^{i(y-x) \cdot \xi} u(y) dy d\xi, \quad x \in D,$$

where the function u is defined in the domain D. The symbol $A(\xi)$ is a certain measurable function defined in $\mathbf{R}^m$. The space $H^s(D)$ consists of functions from Sobolev–Slobodetskii space $H^s(\mathbf{R}^m)$ with supports in $\overline{D}$. The norm in $H^s(D)$ is induced by the H^s-norm

$$||u||_s = \left(\int\limits_{\mathbf{R}^m} |\tilde{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi \right)^{1/2},$$

where $\tilde{u}$ is the Fourier transform of u:

$$\tilde{u}(\xi) = \int\limits_{\mathbf{R}^m} e^{ix\cdot\xi}u(x)dx.$$

We start our considerations from measurable symbols $A(\xi)$, satisfying the condition

$$c_1(1+|\xi|)^\alpha \leq |A(\xi)| \leq c_2(1+|\xi|)^\alpha$$

with positive constants c_1, c_2, and the number α, we call an order of the pseudo-differential operator A. Such operators are linear bounded operators $H^s(D) \to H^{s-\alpha}(D)$ [2].

In this paper, we consider plane case $m = 2$ and canonical plane domain $D = C_+^a = \{x \in \mathbf{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\}$. For such domains, the key role for the solvability description for the pseudo-differential equation

$$(Au)(x) = v(x), \quad x \in C_+^a,$$

takes the wave factorization concept for the symbol $A(\xi)$ [9].

Let us reiterate that the radial tube domain $T(C_+^a)$ over the cone C_+^a is called the following domain $\mathbf{R}^2 + iC_+^a$ of a two-dimensional complex space $\mathbf{C}^2$ [9].

Definition 1. *By wave factorization of $A(\xi)$ with respect to cone $C_+^a = \{x = (x_1, x_2) \in \mathbf{R}^2 : x_2 > a|x_1|, a > 0\}$, we mean its representation in the form*

$$A(\xi) = A_{\neq}(\xi)A_=(\xi),$$

where the factors $A_{\neq}(\xi)$, $A_=(\xi)$ must satisfy the following conditions:
 (1) $A_{\neq}(\xi)$ is defined, generally speaking, on the set $\{x \in \mathbf{R}^2 : a^2x_2^2 \neq x_1^2\}$ only;

 (2) $A_{\neq}(\xi)$ admits an analytical continuation into radial tube domain $T(\overset{*}{C_+^a})$ over the cone $\overset{*}{C_+^a} = \{x \in \mathbf{R}^2 : ax_2 > |x_1|\}$, which satisfies the following estimate:

$$\left| A_{\neq}^{\pm 1}(\xi + i\tau) \right| \leq c(1 + |\xi| + |\tau|)^{\pm\varkappa}, \quad \forall \tau \in \overset{*}{C_+^a} .$$

The factor $A_=(\xi)$ has similar properties with $-\overset{*}{C_+^a}$ instead of $\overset{*}{C_+^a}$ and $\alpha - \varkappa$ instead of $\varkappa$.
The number $\varkappa$ is called index of wave factorization of $A(\xi)$ with respect to cone C_+^a.

Let us note that if the factors $A_{\neq}(\xi)$, $A_=(\xi)$ are homogeneous of order $\varkappa$ and $\alpha - \varkappa$, respectively, and then the symbol $A(\xi)$ is homogeneous of order α, then one can discusshomogeneous wave factorization. The corresponding definition is given in [9].

3. Statement of the Problem

Let us denote $\Gamma = \{x \in \mathbf{R}^2 : \mathbf{x_2 = a|x_1|, a > 0}\}$. We study here the following conjugation problem. Finding a function $U(x)$ which consists of two components

$$U(x) = \begin{cases} U_+(x), & x \in C_+^a \\ U_-(x), & x \in \mathbf{R}^2 \setminus \overline{C_+^a} \end{cases}$$

in the space $H^s(\mathbf{R}^2 \setminus \Gamma)$, and the function should satisfy the following conditions

$$\begin{cases} (AU)(x) = 0, \quad x \in \mathbf{R}^2 \setminus \Gamma \\ \int\limits_{-\infty}^{+\infty} U_+(x_1, x_2)dx_2 = g_0(x_1), \quad x_1 \in \mathbf{R} \\ \int\limits_{-\infty}^{+\infty} U_-(x_1, x_2)dx_2 = g_1(x_1), \quad x_1 \in \mathbf{R} \\ u_+(x) - u_-(x) = g_2(x), \quad x \in \Gamma, \end{cases} \tag{1}$$

where u_+, u_- are boundary values of U from C_+^a and $\mathbf{R}^2 \setminus \overline{C_+^a}$, respectively, and the functions $g_0, g_1 \in H^{s+1/2}(\mathbf{R})$ and $g_2 \in H^{s-1/2}(\Gamma)$ are given. Since we seek a solution in the space H^s, then such spaces $H^{s\pm1/2}$ are chosen according to the theorem on restriction on a hyper-plane [2].

If we consider the equation

$$(Au)(x) = 0, \quad x \in C_+^a, \tag{2}$$

separately, then we can use one of key results from the book [9], Theorem 8.1.2; more precisely, it is the following: if the symbol $A(\xi)$ admits the wave factorization with respect to the cone C_+^a with the index $\ae$ such that $\ae - s = n + \delta, n \in \mathbf{N}, |\text{ffi}| < \mathbf{1/2}$, then a general solution $u \in H^s(C_+^a)$ of Equation (2) has the following form

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi) \sum_{k=0}^{n-1} \left(\tilde{a}_k(\xi_1 - a\xi_2)(\xi_1 + a\xi_2)^k + \tilde{b}_k(\xi_1 + a\xi_2)(\xi_1 - a\xi_2)^k \right),$$

where a_k, b_k are arbitrary functions from $H^{s_k}(\mathbf{R})$, $s_k = s - \ae + k + 1/2, k = 0, 1, \ldots, n - 1$. Furthermore, we have a priori estimates

$$||u||_s \leq C \sum_{k=0}^{n-1} \left([a_k]_{s_k} + [b_k]_{s_k} \right),$$

where $[\cdot]_s$ denotes the $H^s(\mathbf{R})$-norm.

In this paper, we consider the case $n = 1$ so that we have the following formula for a general solution

$$\tilde{U}_+(\xi) = A_{\neq}^{-1}(\xi)(\tilde{a}_0(\xi_1 - a\xi_2) + \tilde{b}_0(\xi_1 + a\xi_2)).$$

For the second equation

$$(Au)(x) = 0, \quad x \in \mathbf{R}^2 \setminus \overline{C_+^a}. \tag{3}$$

we have an analogous formula for a general solution

$$\tilde{U}_-(\xi) = A_=^{-1}(\xi)(\tilde{c}_0(\xi_1 - a\xi_2) + \tilde{d}_0(\xi_1 + a\xi_2)),$$

where c_0, d_0 are a distinct pair of arbitrary functions.

Now, our main goal is to describe the procedure to uniquely determine four arbitrary functions in general solutions of the Equations (2) and (3) using boundary and integral conditions.

4. A System of Linear Integral Equations

Using properties of the Fourier transform [2], we write integral conditions in the form

$$\tilde{U}_+(\xi_1, 0) = A_{\neq}^{-1}(\xi_1, 0)(\tilde{a}_0(\xi_1) + \tilde{b}_0(\xi_1)),$$

$$\tilde{U}_-(\xi_1, 0) = A_=^{-1}(\xi_1, 0)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_1)).$$

It gives the first two relations

$$\begin{aligned} A_{\neq}^{-1}(\xi_1, 0)(\tilde{a}_0(\xi_1) + \tilde{b}_0(\xi_1)) &= \tilde{g}_0(\xi_1) \\ A_=^{-1}(\xi_1, 0)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_1)) &= \tilde{g}_1(\xi_1). \end{aligned} \tag{4}$$

We introduce new variables

$$\begin{cases} \xi_1 - a\xi_2 = t_1 \\ \xi_1 + a\xi_2 = t_2 \end{cases}$$

and re-denote

$$\tilde{U}_{\pm}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a}\right) \equiv \tilde{V}_{\pm}(t_1, t_2),$$

$$A_{\neq}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a}\right) \equiv a_{\neq}(t_1, t_2), \quad A_{=}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2a}\right) \equiv a_{=}(t_1, t_2),$$

so that the boundary values $u_{\pm}$ will be boundary values $v_{\pm}$ for new variables t_1, t_2. Thus, general solutions of the Equations (2) and (3) take the form

$$\tilde{V}_{+}(t_1, t_2) = a_{\neq}^{-1}(t_1, t_2)(\tilde{a}_0(t_1) + \tilde{b}_0(t_2)),$$

$$\tilde{V}_{-}(t_1, t_2) = a_{=}^{-1}(t_1, t_2)(\tilde{c}_0(t_1) + \tilde{d}_0(t_2)).$$

Therefore, using properties of the Fourier transform [2] we obtain

$$\int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)(\tilde{a}_0(t_1) + \tilde{b}_0(t_2))dt_1 = \tilde{v}_{+}(0, t_2)$$

$$\int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)(\tilde{a}_0(t_1) + \tilde{b}_0(t_2))dt_2 = \tilde{v}_{+}(t_1, 0),$$

$$\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)(\tilde{c}_0(t_1) + \tilde{d}_0(t_2))dt_1 = \tilde{v}_{-}(0, t_2),$$

$$\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)(\tilde{c}_0(t_1) + \tilde{d}_0(t_2))dt_2 = \tilde{v}_{-}(t_1, 0).$$

Let us introduce new notations

$$r_1(t_2) \equiv \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)dt_1, \quad r_2(t_1) \equiv \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)dt_2,$$

$$r_3(t_2) \equiv \int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)dt_1, \quad r_4(t_1) \equiv \int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)dt_2.$$

We rewrite integral relations by using the above notations.

$$\int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)\tilde{a}_0(t_1)dt_1 + \tilde{b}_0(t_2)r_1(t_2) -$$

$$- \int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)\tilde{c}_0(t_1)dt_1 - \tilde{d}_0(t_2)r_3(t_2) = \tilde{g}_{21}(t_2),$$

$$r_2(t_1)\tilde{a}_0(t_1) + \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)\tilde{b}_0(t_2)dt_2 - r_4(t_1)\tilde{c}_0(t_1) -$$

$$- \int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)\tilde{d}_0(t_2))dt_2 = \tilde{g}_{22}(t_1),$$

where $\tilde{g}_{21}(t_2), \tilde{g}_{22}(t_1)$ are Fourier transforms of the function g_2, which is considered as two parts related to angle sides.

So, we have the following relations for determining the unknown functions $\tilde{a}_0, \tilde{b}_0, \tilde{c}_0, \tilde{d}_0$. Of course, according to the equalities (4), we can write

$$\tilde{b}_0(\xi_1) = A_{\neq}(\xi_1, 0)\tilde{g}_0(\xi_1) - \tilde{a}_0(\xi_1),$$
$$\tilde{d}_0(\xi_1) = A_{=}(\xi_1, 0)\tilde{g}_1(\xi_1) - \tilde{c}_0(\xi_1),$$

and can obtain the following integral system with respect to unknowns $\tilde{a}_0, \tilde{c}_0$:

$$
\begin{cases}
\displaystyle\int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)\tilde{a}_0(t_1)dt_1 - \tilde{a}_0(t_2)r_1(t_2) - \\
\displaystyle\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)\tilde{c}_0(t_1)dt_1 + \tilde{c}_0(t_2)r_3(t_2) = \tilde{f}_1(t_2) \\
\displaystyle r_2(t_1)\tilde{a}_0(t_1) - \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)\tilde{a}_0(t_2)dt_2 - r_4(t_1)\tilde{c}_0(t_1) + \\
\displaystyle\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)\tilde{c}_0(t_2))dt_2 = \tilde{f}_2(t_1),
\end{cases}
\tag{5}
$$

where we have denoted

$$\tilde{f}_1(t_2) = \tilde{g}_{21}(t_2) - A_{\neq}(t_2, 0)\tilde{g}_0(t_2)r_1(t_2) - A_{=}(t_2, 0)\tilde{g}_1(t_2)r_3(t_2)$$

$$\tilde{f}_2(t_1) = \tilde{g}_{22}(t_1) - \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)A_{\neq}(t_2, 0)\tilde{g}_0(t_2)dt_2 + $$

$$\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)A_{=}(t_2, 0)\tilde{g}_1(t_2)dt_2.$$

Finally, we obtain the following assertion.

Theorem 1. *If the symbol $A(\xi)$ admits wave factorization with respect to the cone C_+^a with the index $\ae$ such that $\ae - s = 1 + \delta, |\delta| < 1/2$, then unique solvability of the problem (1) is equivalent to unique solvability of the system (5).*

The next section is devoted to study the system (5).

5. Homogeneous Symbols and Applying the Mellin Transform

We consider here the case when the symbol $A(\xi)$ is positively homogeneous of order α and the factors $A_{\neq}(\xi)$ and $A_{=}(\xi)$ are positively homogeneous of order $\ae$ and $\alpha - \ae$, respectively.

Lemma 1. *The functions r_1, r_2 are positively homogeneous function of order $1 - \ae$, and the functions r_3, r_4 are positively homogeneous functions of order $1 + \ae - \alpha$.*

Proof. Let us verify. Indeed, for $\lambda > 0$, we have

$$r_1(\lambda t_2) = \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, \lambda t_2)dt_1,$$

and after the change of variable $t_1 = \lambda t$ we obtain

$$r_1(\lambda t_2) = \lambda \int_{-\infty}^{+\infty} a_{\neq}^{-1}(\lambda t, \lambda t_2)dt = \lambda\lambda^{-\ae}r_1(t_2) = \lambda^{1-\ae}r_1(t_2).$$

Analogously,

$$r_3(\lambda t_2) = \int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, \lambda t_2)dt_1,$$

and after similar change we have

$$r_3(\lambda t_2) = \lambda \int_{-\infty}^{+\infty} a_{=}^{-1}(\lambda t, \lambda t_2)dt = \lambda\lambda^{-(\alpha-\ae)}r_3(t_2) = \lambda^{1+\ae-\alpha}r_3(t_2).$$

Similar conclusions are valid for r_2, r_4. □

Remark 1. *If $\ae = \alpha/2$, then all functions r_1, r_2, r_3, r_4 have the same order of homogeneity, which equals to $1 - \ae$.*

Lemma 2. *The functions $a_{\neq}^{-1}(t_1, t_2)r_1^{-1}(t_2), a_{\neq}^{-1}(t_1, t_2)r_2^{-1}(t_1)$ are homogeneous functions of order -1 with respect to variables t_1, t_2, and the functions $a_{=}^{-1}(t_1, t_2)r_3^{-1}(t_2), a_{=}^{-1}(t_1, t_2)r_4^{-1}(t_1)$ are homogeneous functions of order -1 too.*

Proof. According to Lemma 1, we have

$$a_{\neq}^{-1}(\lambda t_1, \lambda t_2)r_1^{-1}(\lambda t_2) = \lambda^{-\ae}a_{\neq}^{-1}(t_1, t_2)\lambda^{\ae-1}r_1^{-1}(t_2) =$$

$$\lambda^{-1}a_{\neq}^{-1}(t_1, t_2)r_1^{-1}(t_2).$$

Analogously,

$$a_{=}^{-1}(\lambda t_1, \lambda t_2)r_3^{-1}(\lambda t_2) = \lambda^{\ae-\alpha}a_{=}^{-1}(t_1, t_2)\lambda^{\alpha-\ae-1})r_3^{-1}(t_2) =$$

$$\lambda^{-1}a_{=}^{-1}(t_1, t_2)r_3^{-1}(t_2).$$

The same is valid for the left two functions. □

Let us note that Lemmas 1 and 2 are almost the same, as in [9].
Now, we divide by r_1 and r_2

$$\begin{cases} \displaystyle\int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)r_1^{-1}(t_2)\tilde{a}_0(t_1)dt_1 - \tilde{a}_0(t_2) - \\[3mm] \displaystyle\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)r_1^{-1}(t_2)\tilde{c}_0(t_1)dt_1 + \tilde{c}_0(t_2)r_1^{-1}(t_2)r_3(t_2) = \tilde{f}_1(t_2)r_1^{-1}(t_2) \\[3mm] \displaystyle\tilde{a}_0(t_1) - \int_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1, t_2)r_2^{-1}(t_1)\tilde{a}_0(t_2)dt_2 - r_4(t_1)r_2^{-1}(t_1)\tilde{c}_0(t_1) + \\[3mm] \displaystyle\int_{-\infty}^{+\infty} a_{=}^{-1}(t_1, t_2)r_2^{-1}(t_1)\tilde{c}_0(t_2))dt_2 = \tilde{f}_2(t_1)r_2^{-1}(t_1), \end{cases}$$

and we obtain the following system of two linear integral equations

$$
\left\{
\begin{aligned}
& \int_{-\infty}^{+\infty} K_1(t_1,t_2)\tilde{a}_0(t_1)dt_1 - \tilde{a}_0(t_2) - \\
& \int_{-\infty}^{+\infty} K_2(t_1,t_2)\tilde{c}_0(t_1)dt_1 + \tilde{c}_0(t_2)R(t_2) = F_1(t_2) \\
& \tilde{a}_0(t_1) - \int_{-\infty}^{+\infty} K_3(t_1,t_2)\tilde{a}_0(t_2)dt_2 - Q(t_1)\tilde{c}_0(t_1) + \\
& \int_{-\infty}^{+\infty} K_4(t_1,t_2)\tilde{c}_0(t_2))dt_2 = F_2(t_1),
\end{aligned}
\right.
\tag{6}
$$

after new notations with

$$
K_1(t_1,t_2) = a_{\neq}^{-1}(t_1,t_2)r_1^{-1}(t_2), K_2(t_1,t_2) = a_{=}^{-1}(t_1,t_2)r_1^{-1}(t_2),
$$

$$
K_3(t_1,t_2) = a_{\neq}^{-1}(t_1,t_2)r_2^{-1}(t_1), K_4(t_1,t_2) = a_{=}^{-1}(t_1,t_2)r_2^{-1}(t_1),
$$

$$
R(t_2) = r_1^{-1}(t_2)r_3(t_2), F_1(t_2) = \tilde{f}_1(t_2)r_1^{-1}(t_2),
$$

$$
Q(t_1) = r_4(t_1)r_2^{-1}(t_1), F_2(t_1) = \tilde{f}_2(t_1)r_2^{-1}(t_1).
$$

Lemma 3. *Let $æ = \alpha/2$. The kernels of integral operators K_1, K_2, K_3, K_4 are homogeneous of order -1, and the functions R, Q are homogeneous of order 0.*

Proof. Using Lemma 1 and Lemma 2, we obtain the required assertion.　□

Now, we will rewrite the system (6) as a system of integral equations on the positive half-axis to apply the Mellin transform.

$$
\left\{
\begin{aligned}
& \int_0^{+\infty} K_1(t_1,t_2)\tilde{a}_0(t_1)dt_1 + \int_{-\infty}^0 K_1(t_1,t_2)\tilde{a}_0(t_1)dt_1 - \tilde{a}_0(t_2) - \\
& -\int_0^{+\infty} K_2(t_1,t_2)\tilde{c}_0(t_1)dt_1 + \int_{-\infty}^0 K_2(t_1,t_2)\tilde{c}_0(t_1)dt_1 + \tilde{c}_0(t_2)R(t_2) = F_1(t_2) \\
& \tilde{a}_0(t_1) - \int_0^{+\infty} K_3(t_1,t_2)\tilde{a}_0(t_2)dt_2 - \int_{-\infty}^0 K_3(t_1,t_2)\tilde{a}_0(t_2)dt_2 - Q(t_1)\tilde{c}_0(t_1) + \\
& + \int_0^{+\infty} K_4(t_1,t_2)\tilde{c}_0(t_2))dt_2 + \int_{-\infty}^0 K_4(t_1,t_2)\tilde{c}_0(t_2))dt_2 = F_2(t_1),
\end{aligned}
\right.
$$

The next step is the following. We would like to transform the latter system to a 4×4 system on a positive half-axis. For this purpose, we introduce two additional unknown functions and new notations.

We denote for all $t_1 > 0$

$$
M_1(t_1,t_2) = K_1(-t_1,t_2), \quad M_2(t_1,t_2) = K_2(-t_1,t_2),
$$

and for all $t_2 > 0$

$$
M_3(t_1,t_2) = K_3(t_1,-t_2), \quad M_4(t_1,t_2) = K_4(t_1,-t_2),
$$

and we put also for $t > 0$

$$\tilde{a}_1(t) = \tilde{a}_0(-t), \quad \tilde{c}_1(t) = \tilde{c}_0(-t), \quad G_1(t) = F_1(-t), \quad G_2(t) = F_2(-t).$$

Thus, we have the following system of linear integral equations with respect to four unknown functions $\tilde{a}_0, \tilde{a}_1, \tilde{c}_0, \tilde{c}_1$ in which all kernel and functions are defined for positive t_1, t_2:

$$
\left\{
\begin{array}{l}
\displaystyle\int_0^{+\infty} K_1(t_1, t_2)\tilde{a}_0(t_1)dt_1 + \int_0^{+\infty} M_1(t_1, t_2)\tilde{a}_1(t_1)dt_1 - \tilde{a}_0(t_2) - \\[3mm]
\displaystyle - \int_0^{+\infty} K_2(t_1, t_2)\tilde{c}_0(t_1)dt_1 + \int_0^{+\infty} M_2(t_1, t_2)\tilde{c}_1(t_1)dt_1 + \tilde{c}_0(t_2)R(t_2) = F_1(t_2) \\[3mm]
\displaystyle\int_0^{+\infty} K_1(t_1, -t_2)\tilde{a}_0(t_1)dt_1 + \int_0^{+\infty} M_1(t_1, -t_2)\tilde{a}_1(t_1)dt_1 - \tilde{a}_1(t_2) - \\[3mm]
\displaystyle - \int_0^{+\infty} K_2(t_1, -t_2)\tilde{c}_0(t_1)dt_1 + \int_0^{+\infty} M_2(t_1, -t_2)\tilde{c}_1(t_1)dt_1 + \tilde{c}_1(t_2)R(-t_2) = G_1(t_2) \\[3mm]
\displaystyle\tilde{a}_0(t_1) - \int_0^{+\infty} K_3(t_1, t_2)\tilde{a}_0(t_2)dt_2 - \int_0^{+\infty} M_3(t_1, t_2)\tilde{a}_1(t_2)dt_2 - Q(t_1)\tilde{c}_0(t_1) + \\[3mm]
\displaystyle + \int_0^{+\infty} K_4(t_1, t_2)\tilde{c}_0(t_2))dt_2 + \int_0^{+\infty} M_4(t_1, t_2)\tilde{c}_1(t_2))dt_2 = F_2(t_1) \\[3mm]
\displaystyle\tilde{a}_1(t_1) - \int_0^{+\infty} K_3(-t_1, t_2)\tilde{a}_0(t_2)dt_2 - \int_0^{+\infty} M_3(-t_1, t_2)\tilde{a}_1(t_2)dt_2 - Q(-t_1)\tilde{c}_1(t_1) + \\[3mm]
\displaystyle + \int_0^{+\infty} K_4(-t_1, t_2)\tilde{c}_0(t_2))dt_2 + \int_0^{+\infty} M_4(-t_1, t_2)\tilde{c}_1(t_2))dt_2 = G_2(-t1).
\end{array}
\right.
$$

Further, we introduce notation:

$$
R(t_2) = \begin{cases} r_1, & t_2 > 0 \\ r_2, & t_2 < 0 \end{cases}, \quad Q(t_1) = \begin{cases} q_1, & t_1 > 0 \\ q_2, & t_1 < 0 \end{cases},
$$

$K_i(t_1, -t_2) = k_i(t_1, t_2), M_i(t_1, -t_2) = m_i(t_1, t_2), i = 1, 2$, and $K_j(-t_1, t_2) = k_j(t_1, t_2)$, $M_j(-t_1, t_2) = m_j(t_1, t_2), j = 3, 4$.

Now we can rewrite our system as follows.

$$\begin{cases} \displaystyle\int_0^{+\infty} K_1(t_1,t_2)\tilde{a}_0(t_1)dt_1 + \int_0^{+\infty} M_1(t_1,t_2)\tilde{a}_1(t_1)dt_1 - \tilde{a}_0(t_2) - \\[2mm] \displaystyle - \int_0^{+\infty} K_2(t_1,t_2)\tilde{c}_0(t_1)dt_1 + \int_0^{+\infty} M_2(t_1,t_2)\tilde{c}_1(t_1)dt_1 + r_1\tilde{c}_0(t_2) = F_1(t_2) \\[2mm] \displaystyle\int_0^{+\infty} k_1(t_1,t_2)\tilde{a}_0(t_1)dt_1 + \int_0^{+\infty} m_1(t_1,t_2)\tilde{a}_1(t_1)dt_1 - \tilde{a}_1(t_2) - \\[2mm] \displaystyle - \int_0^{+\infty} k_2(t_1,t_2)\tilde{c}_0(t_1)dt_1 + \int_0^{+\infty} m_2(t_1,t_2)\tilde{c}_1(t_1)dt_1 + r_2\tilde{c}_1(t_2) = G_1(t_2) \\[2mm] \displaystyle\tilde{a}_0(t_1) - \int_0^{+\infty} K_3(t_1,t_2)\tilde{a}_0(t_2)dt_2 - \int_0^{+\infty} M_3(t_1,t_2)\tilde{a}_1(t_2)dt_2 - q_1\tilde{c}_0(t_1) + \\[2mm] \displaystyle + \int_0^{+\infty} K_4(t_1,t_2)\tilde{c}_0(t_2))dt_2 + \int_0^{+\infty} M_4(t_1,t_2)\tilde{c}_1(t_2))dt_2 = F_2(t_1) \\[2mm] \displaystyle\tilde{a}_1(t_1) - \int_0^{+\infty} k_3(t_1,t_2)\tilde{a}_0(t_2)dt_2 - \int_0^{+\infty} m_3(t_1,t_2)\tilde{a}_1(t_2)dt_2 - q_2\tilde{c}_1(t_1) + \\[2mm] \displaystyle + \int_0^{+\infty} k_4(t_1,t_2)\tilde{c}_0(t_2))dt_2 + \int_0^{+\infty} m_4(t_1,t_2)\tilde{c}_1(t_2))dt_2 = G_2(t_1). \end{cases} \tag{7}$$

Now, we can apply the Mellin transform to the system (7). Let us restate that the Mellin transform for the function f of one real variable is the following [15]

$$\hat{f}(\mu) = \int_0^{+\infty} x^{\mu-1}f(x)dx,$$

and the function $\hat{f}$ exists for a wide class of functions.

We will use the following notations for the Mellin transforms. For $K_i(t_1,t_2), k_i(t_1,t_2)$, $M_i(t_1,t_2), m_i(t_1,t_2), i = 1,2$, the notation $\hat{K}_i(\mu), \hat{k}_i(\mu), \hat{M}_i(\mu), \hat{m}_i(\mu)$ denotes the Mellin transform of the functions $K_i(1,t), k_i(1,t), M_i(1,t), m_i(1,t)$, respectively. For $K_j(t_1,t_2)$, $k_j(t_1,t_2), M_j(t_1,t_2), m_j(t_1,t_2), j = 3,4$, the notation $\hat{K}_j(\mu), \hat{k}_j(\mu), \hat{M}_j(\mu), \hat{m}_j(\mu)$ denotes the Mellin transform of the functions $K_j(t,1).k_j(t,1), M_j(t,1), m_j(t,1)$, respectively.

Applying the Mellin transform to the system (7), we obtain at least formally the following system of linear algebraic equations

$$\begin{cases} (\hat{K}_1(\mu) - 1)\hat{a}_0(\mu) + \hat{M}_1(\mu)\hat{a}_1(\mu) + \\ (\hat{K}_2(\mu) + r_1)\hat{c}_0(\mu) + \hat{M}_2(\mu)\hat{c}_1(\mu) = \hat{F}_1(\mu) \\ \hat{k}_1(\mu)\hat{a}_0(\mu) + (\hat{m}_1(\mu) - 1)\hat{a}_1(\mu) - \\ \hat{k}_2(\mu)\hat{c}_0(\mu) + (\hat{m}_2(\mu) + r_2)\hat{c}_1(\mu) = \hat{G}_1(\mu) \\ (1 - \hat{K}_3(\mu))\hat{a}_0(\mu) - \hat{M}_3(\mu)\hat{a}_1(\mu) + \\ (\hat{K}_4(\mu) - q_1)\hat{c}_0(\mu) + \hat{M}_4(\mu)\hat{c}_1(\mu) = \hat{F}_2(\mu) \\ -\hat{k}_3(\mu)\hat{a}_0(\mu) + (1 - \hat{m}_3(\mu))\hat{a}_1(\mu) + \\ \hat{k}_4(\mu)\hat{c}_0(\mu) + (\hat{m}_4(\mu) - q_2)\hat{c}_1(\mu) = \hat{G}_2(\mu). \end{cases} \tag{8}$$

A matrix of the (4×4)-system (8) is the following

$$
\mathcal{A}(\mu) = \begin{pmatrix} \hat{K}_1(\mu) - 1) & \hat{M}_1(\mu) & \hat{K}_2(\mu) + r_1 & \hat{M}_2(\mu) \\ \hat{k}_1(\mu) & \hat{m}_1(\mu) - 1 & \hat{k}_2(\mu) & \hat{m}_2(\mu) + r_2 \\ 1 - \hat{K}_3(\mu) & -\hat{M}_3(\mu) & \hat{K}_4(\mu) - q_1 & \hat{M}_4(\mu) \\ -\hat{k}_3(\mu) & 1 - \hat{m}_3(\mu) & \hat{k}_4(\mu) & \hat{m}_4(\mu) - q_2 \end{pmatrix}.
$$

6. Solvability Conditions

Here, we can formulate the following assertion on the solvability of system (5) for homogeneous kernels (see also [9]).

Theorem 2. *Let $A_{\neq}(\xi)$ and $A_{=}(\xi)$ be homogeneous non-vanishing functions of order $\alpha/2$ and $-\alpha/2$, respectively, and differentiable away from the origin, $r_1(t_2) \neq 0, \forall t_2 \neq 0, r_2(t_1) \neq 0$, $\forall t_1 \neq 0$. The system of linear integral Equation (5) is uniquely solvable if, and only if, the condition*

$$
\inf |\det \mathcal{A}(\mu)| \neq 0, \quad \Re\mu = 1/2 \tag{9}
$$

holds.

Proof. Basic elements of the proof were given in the above considerations and Lemmas 1–3. The condition (9) is related to properties of the Mellin transform [2,9,15].

Nevertheless, we will give some explanations. If we have the wave factorization, then we obtain the system (5). For homogeneous factors $A_{\neq}(\xi)$ and $A_{=}(\xi)$, the system (5) transforms into the system (7). The latter system of linear integral equations has kernels which are homogeneous of order -1. That is why we can apply the Mellin transform. If we have the expression

$$
\int_0^{+\infty} K(t_1, t_2) u(t_1) dt_1,
$$

in which the kernel $K(t_1, t_2)$ is a homogeneous function of order -1, then after applying the Mellin transform we obtain the following expression

$$
\int_0^{+\infty} t_2^{\mu-1} \left(\int_0^{+\infty} K(t_1, t_2) u(t_1) dt_1 \right) dt_2.
$$

The change of variable in the inner integral $t_2 = xt_1$ leads to the following integral

$$
\int_0^{+\infty} t_1^{\mu-1} x^{\mu-1} \left(\int_0^{+\infty} t_1 K(t_1, xt_1) u(t_1) dt_1 \right) dx,
$$

and after rearrangements of integrals we obtain the following product

$$
\int_0^{+\infty} t_1^{\mu-1} u(t_1) dt_1 \int_0^{+\infty} x^{\mu-1} K(1, x) dx = \hat{u}(\mu) \hat{K}(\mu),
$$

where $\hat{u}$ denotes the Mellin transform of u.

So, using the Mellin transform, we can obtain the system of linear algebraic Equation (8), which is equivalent to the system (7). Lemma 3 is needed for this purpose. The condition (9) is a necessary and sufficient condition for the unique solvability of such systems and the applicability of the inverse Mellin transform.

Since we suppose the factors $A_{\neq}, A_{=}$ are differentiable, then the Mellin transform is applicable for the kernels K_j. The functions under the integral can be assumed to be smooth enough, taking into account further approximation in H^s-spaces. □

Remark 2. *A priori estimates for a solution of the problem* (1) *can be obtained by the methods described in [9]. We will give these estimates in next papers.*

7. Conclusions

In this paper, we have shown that a certain conjugation problem can be reduced to a system of linear algebraic equations. One can consider other conjugation problems for homogeneous elliptic symbols using this approach. Perhaps it is reasonable to consider different boundary conditions which are local, such as Dirichlet and Neumann conditions.

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