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OPERATORS AND EQUATIONS: DISCRETE AND CONTINUOUS

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Abstract. We consider discrete pseudo-differential equations with elliptic symbols and the corresponding discrete boundary-value problems in special canonical domains of multidimensional spaces. The solvability of such equations and boundary-value problems in discrete analogs of Sobolev–Slobodetsky spaces is examined.

***Keywords and phrases*:** discrete pseudo-differential operator, discrete boundary-value problem, elliptic symbol, discrete equation, solvability.

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1. Introduction. The theory of pseudo-differential operators and equations exists for more than half a century. The theory of pseudo-differential operators and the theory of equations represent two different views on seemingly very similar objects. The operator theory is focused on the description of classes of symbols that provide the boundedness of a pseudo-differential operator in a suitable functional space, while the theory of equations is mainly concerned with the solvability of such equations and the qualitative description of the properties of their solutions and, and if possible, finding this solution (at least by approximate methods).

As it turned out (see [2, 10]), certain topological characteristics of the symbol of a given operator, which do not affect the boundedness of the operator, play a fundamental role since they completely determine the solvability of the corresponding pseudo-differential equation. Moreover, these characteristics provide an explicit description of the structure of general solutions or solvability conditions. This allows the researcher to choose boundary conditions that guarantee the unique solvability of the boundary-value problem for the pseudo-differential equation considered.

We will discuss some "discrete" aspects of the well-developed theory of elliptic pseudo-differential operators and equations (see [2, 4, 8, 9]), namely, the solvability of their discrete analogs. For constructing discrete approximations of pseudo-differential equations, the study of these issues will be one of important components. Some preliminary considerations of the foundations of this theory were given in [10–15].

We also note that in the theory of boundary-value problems for differential equations, various discretization schemes have long been developed (see, e.g., [6, 7]). Without disparaging all these studies, we should note that the methods mentioned are very specific and are suitable only for these situations.

2. Discrete spaces and operators. We will use the following notation. Let $\mathbb{T}^m$ be a m-dimensional cube $[-\pi,\pi]^m$, $h>0$, $\hbar=h^{-1}$. We consider functions defined on this cube as periodic functions on $\mathbb{R}^m$ with the basic period cube $\mathbb{T}^m$.

We will use the term "discrete function" for functions $u_d(\tilde{x})$ of the discrete variable $\tilde{x}\in h\mathbb{Z}^m$. For such functions, we introduce the discrete Fourier transform

$$(F_d u_d)(\xi)\equiv\tilde{u}_d(\xi)=\sum_{\tilde{x}\in h\mathbb{Z}^m}e^{-i\tilde{x}\cdot\xi}u_d(\tilde{x})h^m,\quad \xi\in\hbar\mathbb{T}^m;$$

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if the series converges, the function $\tilde{u}_d(\xi)$ is periodic on $\mathbb{R}^m$ with the basic period cube $\hbar\mathbb{T}^m$. This discrete Fourier transform possesses the fundamental properties of the integral Fourier transform; in particular, the inverse discrete Fourier transform is given by the formula

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}_d(\xi)d\xi, \quad \tilde{x} \in h\mathbb{Z}^m.$$

The discrete Fourier transform establishes a bijective correspondence between the spaces $L_2(h\mathbb{Z}^m)$ and $L_2(\hbar\mathbb{T}^m)$ with the norms

$$\|u_d\|_2 = \left(\sum_{\tilde{x}\in h\mathbb{Z}^m} |u_d(\tilde{x})|^2 h^m\right)^{1/2}, \quad \|\tilde{u}_d\|_2 = \left(\int\limits_{\xi\in\hbar\mathbb{T}^m} |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2},$$

respectively.

Example 2.1. Since the definition of Sobolev–Slobodetsky spaces contains the notion of partial derivatives, we will use their discrete analogs, i.e., the first-order divided differences

$$(\Delta_k^{(1)}u_d)(\tilde{x}) = \frac{1}{h}\big(u_d(x_1,\dots,x_k+h,\dots,x_m) - u_d(x_1,\dots,x_k,\dots,x_m)\big),$$

for which the discrete Fourier transform is as follows:

$$\widetilde{(\Delta_k^{(1)}u_d)}(\xi) = h^{-1}(e^{-ih\cdot\xi_k} - 1)\tilde{u}_d(\xi).$$

Further, for the second-order divided difference, we have

$$(\Delta_k^{(2)}u_d)(\tilde{x}) = \\ = \frac{1}{h^2}\Big(u_d(x_1,\dots,x_k+2h,\dots,x_m) - 2u_d(x_1,\dots,x_k+h,\dots,x_m) + u_d(x_1,\dots,x_k+h,\dots,x_m)\Big),$$

and the discrete Fourier transform has the form

$$\widetilde{(\Delta_k^{(2)}u_d)}(\xi) = \frac{1}{h^2}\big(e^{-ih\cdot\xi_k} - 1\big)^2\tilde{u}_d(\xi).$$

Thus, for the discrete Laplacian, we get the following expression:

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m}(\Delta_k^{(2)}u_d)(\tilde{x}),$$

so that

$$\widetilde{(\Delta_d u_d)}(\xi) = \frac{1}{h^2}\sum_{k=1}^{m}(e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

Using the discrete Fourier transform, we introduce discrete Sobolev–Slobodetsky spaces, which are convenient for studying discrete pseudo-differential equations. We introduce the notation

$$\zeta^2 = \frac{1}{h^2}\sum_{k=1}^{m}(e^{-ih\cdot\xi_k} - 1)^2.$$

Definition 2.1. The *space* $H^s(h\mathbb{Z}^m)$ consists of discrete functions $u_d(\tilde{x})$ for which the following norm is finite:

$$\|u_d\|_s = \left(\int\limits_{\hbar\mathbb{T}^m} (1+|\zeta^2|)^s|\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}$$

Let $D \subset \mathbb{R}^m$ be a domain and $D_d = D \cap h\mathbb{Z}^m$ be the corresponding discrete domain.

Definition 2.2. The *space* $H^s(D_d)$ consists of discrete functions of the space $H^s(h\mathbb{Z}^m)$ whose supports are contained in $\overline{D_d}$. The norm of the space $H^s(D_d)$ is induced by the norm of the space $H^s(h\mathbb{Z}^m)$. The *space* $H_0^s(D_d)$ consists of discrete functions u_d with supports in D_d that admit continuation to the whole space $H^s(h\mathbb{Z}^m)$. The norm of the space $H_0^s(D_d)$ is given by the formula

$$\|u_d\|_s^+ = \inf \|\ell u_d\|_s,$$

where inf is taken over all possible continuation of ℓ.

The Fourier transform of the space $H^s(D_d)$ is denoted by $\widetilde{H}^s(D_d)$.

Remark 2.1. These spaces were studied in detail in [11]. Obviously, the norms introduced are equivalent to the L_2-norm, but the equivalence constants depend on h. Note that in our further considerations all constants are independent of h.

Let $\widetilde{A}_d(\xi)$ be a periodic function on $\mathbb{R}^m$ with the basic period cube $\hbar\mathbb{T}^m$. Such functions are called *symbols*. As usually, we define a discrete pseudo-differential operator by its symbol.

Definition 2.3. A *discrete pseudo-differential operator* A_d in a discrete domain D_d is an operator of the following form:

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d.$$

An operator A_d is said to be *elliptic* if

$$\operatorname*{ess\,inf}_{\xi\in\hbar\mathbb{T}^m} |\widetilde{A}_d(\xi)| > 0.$$

Remark 2.2. One can introduce a symbol $\widetilde{A}_d(\tilde{x}, \xi)$ depending on the spatial variable $\tilde{x}$ and define a general discrete pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} \widetilde{A}_d(\tilde{x}, \xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d.$$

To study such operators and the corresponding equations, we need a rather complex and sophisticated technique.

Definition 2.4. The *class* E_α consists of symbols satisfying the condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \le |A_d(\xi)| \le c_2(1 + |\zeta^2|)^{\alpha/2} \tag{1}$$

with positive constants c_1 and c_2, independent of h. The number $\alpha \in \mathbb{R}$ is called the *order* of the discrete pseudo-differential operator A_d. Roughly speaking, the order of a discrete pseudo-differential operator is the power of h taken with the opposite sign.

Due to the last definition, one can easily prove the following property.

Lemma 2.1. *A discrete pseudo-differential operator $A_d \in E_\alpha$ is a linear bounded operator $H^s(h\mathbb{Z}^m) \to H^{s-\alpha}(h\mathbb{Z}^m)$ whose norm is independent of h.*

3. Discrete pseudo-differential equations. With an operator A_d, we associate the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d. \tag{2}$$

In this section, we consider this equation only in the half-space

$$D = \mathbb{R}_+^m \equiv \Big\{x \in \mathbb{R}^m : x = (x_1, \ldots, x_m),\ x_m > 0\Big\}.$$

3.1. Periodic factorization. We introduce the notation

$$\Pi_\pm = \Big\{ \big(\xi', \xi_m \pm i\tau\big),\ \tau > 0 \Big\}, \quad \xi = (\xi', \xi_m) \in \mathbb{T}^m.$$

Definition 3.1. A *periodic factorization* of an elliptic symbol $A_d(\xi) \in E_\alpha$ is the representation

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit analytic continuations to the half-bands $\hbar\Pi_\pm$ with respect to the last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$\big|A_{d,+}^{\pm 1}(\xi)\big| \le c_1 \big(1 + |\hat\zeta^2|\big)^{\pm\varkappa/2}, \quad \big|A_{d,-}^{\pm 1}(\xi)\big| \le c_2 \big(1 + |\hat\zeta^2|\big)^{\pm(\alpha-\varkappa)/2},$$

where the constants c_1 and c_2 are independent of h and

$$\hat\zeta^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number $\varkappa \in \mathbb{R}$ is called the *index* of the periodic factorization.

In some simple cases, we can use the topological formula

$$\varkappa = \frac{1}{2\pi} \int_{-\hbar\pi}^{\hbar\pi} d \arg A_d(\cdot, \xi_m),$$

where $A_d(\cdot, \xi_m)$ means that $\xi' \in \hbar\mathbb{T}^{m-1}$ is fixed, and the integral is understood in the sense of Stieltjes. In other words, it is necessary to calculate the increment of the argument of the symbol $A_d(\xi)$ when ξ_m changes from $-\hbar\pi$ to $\hbar\pi$ for fixed ξ' and divide it by 2π.

Theorem 3.1. *If an elliptic symbol $\tilde A_d(\xi) \in E_\alpha$ admits a periodic factorization with index $\varkappa$, so that $|\varkappa - s| < 1/2$, then Eq. (2) has a unique solution in the space $H^s(D_d)$ for any right-hand side $v_d \in H_0^{s-\alpha}(D_d)$,*

$$\tilde u_d(\xi) = \tilde A_{d,+}^{-1}(\xi) P_{\xi'}^{\mathrm{per}} \big(\tilde A_{d,-}^{-1}(\xi) \widetilde{\ell v_d}(\xi)\big),$$

$$\big(P_{\xi'}^{\mathrm{per}} \tilde u_d\big)(\xi) \equiv \frac{1}{2} \left(\tilde u_d(\xi) + \frac{1}{2\pi i}\, \mathrm{v.p.} \int_{-\hbar\pi}^{\hbar\pi} \tilde u_d(\xi', \eta_m) \cot \frac{h(\xi_m - \eta_m)}{2} d\eta_m \right).$$

Remark 3.1. It is easy to see that a solution is independent of the choice of the continuation ℓv_d.

Theorem 3.2. *Let $\varkappa - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$. Then the Fourier transform of the general solution of Eq. (2) has the form*

$$\tilde u_d(\xi) = \tilde A_{d,+}^{-1}(\xi) X_n(\xi) P_{\xi'}^{\mathrm{per}} \Big(X_n^{-1}(\xi) \tilde A_{d,-}^{-1}(\xi) \widetilde{\ell v_d}(\xi) \Big) + \tilde A_{d,+}^{-1}(\xi) \sum_{k=0}^{n-1} c_k(\xi') \hat\zeta_m^k,$$

where $X_n(\xi)$ is an arbitrary polynomial of degree of n of the variables

$$\hat\zeta_k = \hbar(e^{-ih\xi_k} - 1), \quad k = 1, \dots, m,$$

satisfying the condition (1) and $c_j(\xi')$, $j = 0, 1, \dots, n-1$, are arbitrary functions from $H_{s_j}(h\mathbb{T}^{m-1})$, $s_j = s - \varkappa + j - 1/2$.

3.2. Discrete boundary-value problems. Theorem 3.2 shows that a solution of Eq. (2) is not unique. To get a unique solution, one should impose additional conditions that determine arbitrary functions $c_k(\xi')$, $k = 0, 1, \dots, n-1$. For simplicity, we consider the homogeneous equation (2), although all results can be easily extended to the nonhomogeneous case.

Let us consider the following boundary conditions:

$$(B_j u_d)(\tilde{x}', 0) = b_j(\tilde{x}'), \quad j = 0, 1, \dots, n-1, \tag{3}$$

where $B_{d,j}$ is a discrete pseudo-differential operator of order $\alpha_j \in \mathbb{R}$ with the symbols $\widetilde{B}_j(\xi) \in C(\hbar\mathbb{T}^m)$:

$$(B_{d,j} u_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} \sum_{\tilde{y} \in h\mathbb{Z}^m} e^{i\xi \cdot (\tilde{x} - \tilde{y})} \widetilde{B}_j(\xi) \tilde{u}_d(\xi) d\xi.$$

Introduce the Fourier transforms of the boundary conditions (3):

$$\int\limits_{-h^{-1}\pi}^{h^{-1}\pi} \widetilde{B}_j(\xi', \xi_m) \tilde{u}_d(\xi', \xi_m) d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1, \dots, n-1;$$

taking into account Lemma 2.1 and "trace" properties of discrete spaces H^s (see [3]), we must require that $b_j(\tilde{x}') \in H^{s-\alpha_j-1/2}(h\mathbf{Z}^{m-1})$.

We introduce the notation

$$s_{jk}(\xi') = \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}_{d,+}^{-1}(\xi) \widetilde{B}_j(\xi', \xi_m) \hat{\zeta}_m^k d\xi_m.$$

Theorem 3.3. *If $\varkappa - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$, then the discrete boundary-value problem (2), (3) has a unique solution in the space $H^s(D_d)$ for arbitrary boundary functions $b_j \in H^{s-\alpha_j-1/2}(h\mathbf{Z}^{m-1})$, $j = 0, \dots, n-1$, if and only if*

$$\det(s_{kj}(\xi'))_{k,j=0}^{\varkappa} \neq 0 \quad \forall \xi' \in \mathbb{T}^{m-1}.$$

Moreover, the following a priori estimate holds:

$$\|u_d\|_s \le c \sum_{j=0}^{n-1} [b_j]_{s-\alpha_j-1/2},$$

where the constant c is independent of h and $[\cdot]_s$ denotes the H^s-norm in the discrete space $H^s(h\mathbb{Z}^{m-1})$.

3.3. Representation of solution. Now we consider the remaining case $\varkappa - s = -n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$.

Lemma 3.1. *There is a unique set of functions $c_j(\xi') \in H^{s_j}(\hbar\mathbb{T}^{m-1})$, $s_j = s - \varkappa + j + 1/2$, $j = 0, 1, \dots, n$, for which the following representation holds*:

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \cot \frac{h(\eta_m - \xi_m)}{2} g(\xi', \eta_m) d\eta_m = \sum_{j=0}^{n} c_j(\xi') (e^{ih\xi_m} - 1)^{-j}$$
$$+ (e^{ih\xi_m} - 1)^{-n} \int\limits_{-\hbar\pi}^{\hbar\pi} \cot \frac{h(\eta_m - \xi_m)}{2} g(\xi', \eta_m)(e^{ih\eta_m} - 1)^n d\eta_m,$$

where

$$c_j(\xi') = ih \int\limits_{-\hbar\pi}^{\hbar\pi} (e^{ih\xi_m} - 1)^j g(\xi', \xi_m) d\xi_m, \quad j = 0, 1, \dots, n,$$

for all $g(\xi', \xi_m) \in H^{-n-\delta}(\hbar\mathbb{T}^m)$*,* $n \in \mathbb{N}$*, and* $|\delta| < 1/2$*.*

Theorem 3.4. *Let* $\varkappa - s = -n + \delta$ *and* $|\delta| < 1/2$*. Equation* (2) *has a solution in the discrete space* $H^s(D_d)$ *if and only if*

$$c_j(\xi') = 0 \quad \text{for almost all} \quad \xi' \in \hbar\mathbb{T}^{m-1}, \quad j = 0, 1, \dots, n. \tag{4}$$

Remark 3.2. The conditions (4) can be written in terms of the original space $H^s(h\mathbb{Z}^m)$. It is convenient to use the operator of divided difference $\Delta_j^{(1)} : H^s(h\mathbb{Z}^m) \to H^{s-1}(h\mathbb{Z}^m)$ introduced above and its Fourier transform

$$\tilde{\Delta}_j^{(1)} : \tilde{u}_d(\xi) \longmapsto \frac{e^{-ih\xi_j} - 1}{h}\tilde{u}_d(\xi), \quad \xi \in \hbar\mathbb{T}^m.$$

There is a simple connection between the discrete Fourier transform and the operator of restriction to a discrete hyperplane. If we consider the operator of restriction to the discrete hyperplane $\tilde{x}_m = 0$, i.e., to $\mathbb{Z}^{m-1}$, then, in accordance with the properties of the inverse Fourier transform, we obtain

$$u_d(\tilde{x}', \tilde{x}_m) = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{T}^m} e^{i\tilde{x}'\cdot\xi'} e^{i\tilde{x}_m\cdot\xi_m} \tilde{u}_d(\xi', \xi_m) d\xi' d\xi_m;$$

therefore,

$$u_d(\tilde{x}', 0) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}'\cdot\xi'} \tilde{u}_d(\xi', \xi_m) d\xi' d\xi_m = \frac{1}{(2\pi)^{m-1}} \int\limits_{\hbar\mathbb{T}^{m-1}} e^{i\tilde{x}'\cdot\xi'} \left(\frac{1}{2\pi} \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi', \xi_m) d\xi_m \right) d\xi'.$$

This implies that the restriction to the hyperplane corresponds to the integration of the Fourier transform with respect to the last variable. Taking into account the definition of a discrete pseudo-differential operator in $H^s(h\mathbb{Z}^m)$, we can write the conditions (4) as follows:

$$(\Delta_m^{(j)} A_{d,-}^{-1}(\ell v_d)(\tilde{x}', 0) = 0 \quad \forall \tilde{x}' \in h\mathbb{Z}^{m-1}, \quad j = 0, 1, \dots, n, \tag{5}$$

where $A_{d,-}^{-1}$ is a discrete pseudo-differential operator with the symbol $A_{d,-}^{-1}(\xi)$.

3.4. Discrete problems with coboundary operators. Taking into account Lemma 3.1 and Theorem 3.4, we can consider equations of a more general form than (2), for example, the equation

$$(A_d u_d)(\tilde{x}) + \sum_{j=0}^{n} K_j \left(\tilde{b}_j(\tilde{x}') \otimes \delta(\tilde{x}_m) \right) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{6}$$

with unknown functions u_d and $\tilde{b}_j$, $j = 0, 1, \dots, n$, where K_j are given pseudo-differential operators with the symbols $K_j(\xi) \in E_{\alpha_j}$.

Remark 3.3. The operators K_j are called *coboundary operatorms* since they act as follows. Denoting by $\hat{K}_j(\tilde{x})$ the "kernel" of the pseudo-differential operator K_j, we get

$$K_j \left(\tilde{b}_j(\tilde{x}') \otimes \delta(\tilde{x}_m) \right) = \sum_{\tilde{y} \in h\mathbb{Z}^{m-1}} \hat{K}_j(\tilde{x}' - \tilde{y}', \tilde{x}_m) b_j(\tilde{y}') h^{m-1}.$$

The term "potential-type operator" is also acceptable.

Continuing the right-hand side of the equation to the whole space $H^{s-\alpha}(h\mathbb{Z}^m)$ (we denote this continuation by ℓv_d) and applying the discrete Fourier transform, we obtain the system of linear algebraic equations

$$\sum_{j=0}^{n} t_{kj}(\xi') \tilde{b}_j(\xi') = f_k(\xi'), \quad k = 0, 1, \dots, n,$$

where

$$t_{kj}(\xi') = \frac{1}{2\pi} \int\limits_{-\hbar\pi}^{\hbar\pi} \left(\frac{e^{ih\xi_m} - 1}{h} \right)^k \frac{K_j(\xi', \xi_m)}{A_{d,-}(\xi', \xi_m)} d\xi_m,$$

$$f_k(\xi') = \frac{1}{2\pi} \int\limits_{-\hbar\pi}^{\hbar\pi} \left(\frac{e^{ih\xi_m} - 1}{h} \right)^k A^{-1}_{d,-}(\xi', \xi_m) \widetilde{(\ell v_d)}(\xi', \xi_m) d\xi_m.$$

Theorem 3.5. *Let $\varkappa - s = -n + \delta$, $n \in \mathbb{N}$, and $|\delta| < 1/2$. Equation (6) has a unique solution $u_d \in H^s(D_d)$, $c_j \in H^{s_j}(h\mathbb{Z}^{m-1})$, $s_j = s - \alpha + \alpha_j + 1/2$, $j = 0, 1, \ldots, n$, if and only if*

$$\operatorname*{ess\,inf}_{\xi' \in \hbar\mathbb{T}^{m-1}} |\det(t_{kj}(\xi'))^n_{k,j=0} > 0.$$

The following a priori estimate holds:

$$\|u_d\|_s \leq a\|v_d\|^+_{s-\alpha}, \quad \|b_j\|_{s_j} \leq a_j\|v_d\|^+_{s-\alpha}, \quad j = 0, 1, \ldots, n,$$

where the constants $a, a_1, \ldots, a_n$, are independent of h.

4. Discrete cones and complex variables. In this section, as a domain D we take a sharp convex cone in $\mathbb{R}^m$ that does not contain a whole straight line.

We denote by P_{D_d} the operator of restriction to D_d, $P_{D_d} : L_2(h\mathbb{Z}^m) \to L_2(D_d)$, so that for an arbitrary function $u_d \in L_2(h\mathbb{Z}^m)$ we have

$$(P_{D_d} u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}) & \text{if } \tilde{x} \in D_d, \\ 0 & \text{otherwise.} \end{cases}$$

4.1. Half-space and periodic Cauchy kernel. If we choose a half-space as D, the Fourier transform of the operator P_{D_d} can be calculated (see [12, 13]). Consider the following example.

Example 4.1. If $D = \mathbb{R}^m_+$, then

$$(F_d P_{D_d} u_d)(\xi', \xi_m) = \frac{1}{4\pi i} \lim_{\tau \to 0+} \int\limits_{-\hbar\pi}^{\hbar\pi} u_d(\xi', \eta_m) \cot \frac{h(\xi_m - \eta_m + i\tau)}{2} d\eta_m.$$

In fact, this property was used in the previous sections. Namely, we used the theory of the one-dimensional periodic Riemann boundary-value problem with the parameter $\xi' \in \hbar\mathbb{T}^{m-1}$, which is formulated as follows: Find a pair of functions $\Phi^{\pm}(\xi', \xi_m)$, which are the boundary values of analytic functions in the half-bands $\hbar\Pi_{\pm}$, $\Pi_{\pm} = \{z \in \mathbb{C} : z = \xi_m \pm i\tau, \ \tau > 0\}$, satisfying the linear relation

$$\Phi^+(\xi)(\xi', \xi_m) = G(\xi', \xi_m)\Phi^-(\xi)(\xi', \xi_m) + g(\xi), \quad \xi \in \hbar\mathbb{T}^m,$$

for almost all $\xi' \in \hbar\mathbb{T}^{m-1}$, where $G(\xi)$ and $g(\xi)$ are defined periodic functions. The problem is similar to the classical parobplem.

4.2. Cone and periodic Bochner kernel. Let D be a sharp convex cone that does not contain a whole straight line and let D^* be a conjugate cone, that is,

$$D^* = \{x \in \mathbb{R}^m : x \cdot y > 0, \ y \in D\}.$$

We denote by $T(D^*) \subset \mathbb{C}^m$ the set $\hbar\mathbb{T}^m + iD^*$. In the case where $\hbar\mathbb{T}^m \equiv \mathbb{R}^m$ (this corresponds to the case of $h \to 0$), this set is called the *multidimensional tubular domain over the cone* D^* (see [1, 10, 16]). We introduce the function

$$B_d(z) = \sum_{\tilde{x} \in D_d} e^{i\tilde{x} \cdot z}, \quad z = \xi + i\tau, \quad \xi \in \hbar\mathbb{T}^m, \quad \tau \in D^*,$$

and define the operator

$$(B_d u)(\xi) = \lim_{\tau\to 0} \int\limits_{\hbar\mathbb{T}^m} B_d(z-\eta) u_d(\eta) d\eta.$$

Lemma 4.1. *For an arbitrary function $u_d \in L_2(h\mathbb{Z}^m)$, the following equality holds*:

$$F_d P_{D_d} u_d = B_d F_d u_d.$$

Next, we define the subspace $A(\hbar\mathbb{T}^m) \subset L_2(\hbar\mathbb{T}^m)$ consisting of functions that admit an analytic continuation to $T(D^*)$ satisfying the condition

$$\sup_{\tau\in D^*} \int\limits_{\mathbb{T}^m} \left|\tilde{u}_d(\xi + i\tau)\right|^2 d\xi < +\infty.$$

In other words, the space $A(\hbar\mathbb{T}^m) \subset L_2(\hbar\mathbb{T}^m)$ is the subspace of boundary values of analytic functions in $T(D^*)$.

We introduce the notation

$$B(\hbar\mathbb{T}^m) = L_2(\hbar\mathbb{T}^m) \ominus A(\hbar\mathbb{T}^m),$$

so that $B(\hbar\mathbb{T}^m)$ is the direct (and orthogonal) complement of the subspace $A(\hbar\mathbb{T}^m)$ in $L_2(\hbar\mathbb{T}^m)$.

The *jump problem* is stated as follows: Find a pair of functions $\Phi^{\pm}$, $\Phi^+ \in A(\hbar\mathbb{T}^m)$ and $\Phi^- \in B(\hbar\mathbb{T}^m)$, such that

$$\Phi^+(\xi) - \Phi^-(\xi) = g(\xi), \quad \xi \in \hbar\mathbb{T}^m, \tag{7}$$

where $g(\xi) \in L_2(\hbar\mathbb{T}^m)$ is a given function.

Lemma 4.2. *The operator $B_d : L_2(\hbar\mathbb{T}^m) \to A(\hbar\mathbb{T}^m)$ is a projector. Moreover, $u_d \in L_2(D_d)$ if and only if $\tilde{u}_d \in A(\hbar\mathbb{T}^m)$.*

Theorem 4.1. *The jump problem* (7) *is uniquely solvable for any right-hand side from $L_2(\hbar\mathbb{T}^m)$.*

Example 4.2. If $m = 2$ and D is the first quadrant on the plane, then the solution of the jump problem is given by the formulas

$$\Phi^+(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau\to 0} \int\limits_{-\hbar\pi}^{\hbar\pi} \int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 + i\tau_1 - t_1)}{2} \cot\frac{h(\xi_2 + i\tau_2 - t_2)}{2} g(t_1, t_2) dt_1 dt_2,$$

$$\Phi^-(\xi) = \Phi^+(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in D.$$

We consider the multidimensional periodic Riemann problem in the following formulation: Find a pair of functions $\Phi^{\pm}$, $\Phi^+ \in A(\hbar\mathbb{T}^m)$ and $\Phi^- \in B(\hbar\mathbb{T}^m)$, such that

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \quad \xi \in \hbar\mathbb{T}^m, \tag{8}$$

where $G(\xi)$ and $g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$, we return to the jump problem (7).

As in the classical case, we need a special representation for periodic elliptic symbols in order to obtain a solution to the problem (8).

We denote by $H^s(D_d)$ the subspace of the space $H^s(\mathbb{Z}^m)$ consisting of functions of the discrete argument $\tilde{x}$ whose supports are contained in $\overline{D_d}$. We also denote by $\widetilde{H}^s(D_d)$ and $\widetilde{H}^s(\mathbb{Z}^m)$ the Fourier transforms of the corresponding spaces.

Lemma 4.3. *For $|s| < 1/2$, the operator $B_d : \widetilde{H}^s(\mathbb{Z}^m) \to \widetilde{H}^s(D_d)$ is a projector, and the jump problem has a unique solution $\Phi^+ \in \widetilde{H}^s(D_d)$, $\Phi^- \in \widetilde{H}^s(\mathbb{Z}^m \setminus D_d)$ for arbitrary $g \in \widetilde{H}^s(\mathbb{Z}^m)$.*

Definition 4.1. A *periodic wave factorization* of an elliptic symbol $A_d(\xi)$ is a representation of the form

$$A_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi),$$

where the factors $A_{\neq}^{\pm 1}(\xi)$ and $A_{=}^{\pm 1}(\xi)$ admit bounded analytic continuations in the domain $T(\pm D^*)$.

Theorem 4.2. *If* $|s| < 1/2$ *and an elliptic symbol* $A_d(\xi) \in E_\alpha$ *admits a periodic wave factorization, then the operator* A_d *is invertible in the space* $H^s(D_d)$.

Remark 4.1. The definition 4.1 of the periodic wave factorization corresponds to the zero index of the periodic wave factorization. Surely, this concept requires expansion in the context of [10] taking into account the results of the previous section.

Recent considerations may be useful in formulating boundary value problems for discrete elliptic pseudo-differential equations in canonical nonsmooth domains. Such boundary problems arise in cases where, roughly speaking, the periodic wave factorization index is nonzero. We also hope to establish a certain correspondence between the discrete and continuous cases (see [10]) and describe the transition from the discrete case to continuous.

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REFERENCES

1. S. Bochner and W. T. Martin, *Several Complex Variables*, Princeton Univ. Press, Princeton (1948).
2. G. I. Eskin, *Boundary-Value Problems for Elliptic Pseudodifferential Equations* [in Russian], Nauka, Moscow (1973).
3. L. S. Frank, "Spaces of grid functions," *Mat. Sb.*, **86**, No. 2, 187–233 (1971).
4. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer-Verlag, Berlin etc. (1985).
5. S. Rempel and B.-W. Schulze, *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin (1982).
6. V. S. Ryaben'ky, *Method of Difference Potentials for Some Problems of Continuum Mechanics* [in Russian], Nauka, Moscow (1987).
7. A. A. Samarsky, *Theory of Difference Schemes* [in Russian], Nauka, Moscow (1971).
8. M. E. Taylor, *Pseudodifferential Operators*, Princeton Univ. Press, Princeton, New Jersey (1981).
9. F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators*, Plenum Press, New York–London (1982).
10. V. B. Vasilyev, *Wave Factorization of Elliptic Symbols: Theory and Applications. Introduction to the Theory of Boundary-Value Problems in Nonsmooth Domains*, Kluwer Academic, Dordrecht–Boston–London (2000).
11. V. B. Vasilyev, *Multipliers of Fourier Integrals, Pseudo-Differential Equations, Wave Factorization, and Boundary-Value Problems* [in Russian], URSS, Moscow (2010).
12. V. B. Vasilyev, "Discreteness, periodicity, holomorphy, and factorization," in: *Integral Methods in Science and Engineering. Vol. 1. Theoretical Technique* (C. Constanda, M. Dalla Riva, P. D. Lamberti, and P. Musolino, eds.), Birkhäuser (2017), pp. 315–324.
13. V. B. Vasilyev, "The periodic Cauchy kernel, the periodic Bochner kernel, and discrete pseudo-differential operators," *AIP Conf. Proc.*, **1863**, 140014 (2017).
14. V. B. Vasilyev, "On discrete boundary-value problems," *AIP Conf. Proc.*, **1880**, 050010 (2017).

15. V. B. Vasilyev, "Discrete operators in canonical domains," *WSEAS Trans. Math.*, **16**, 197–201 (2017).
16. V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables* [in Russian], Nauka, Moscow (1964).

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