QUASI-AVERAGES IN THE SOLUTION OF THE CLASSIFICATION PROBLEM FOR EQUILIBRIUMS OF CONDENSED MEDIA WITH A SPONTANEOUSLY BROKEN SYMMETRY

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We develop a statistical approach for solving the classification problem for equilibriums of degenerate condensed media. We introduce generators of unbroken and spatial symmetries of an equilibrium and use them to derive the classification equations for the order parameter. We elucidate the mechanism of the appearance of additional thermodynamic parameters characterizing both homogeneous and inhomogeneous equilibriums. We solve the classification problem for equilibriums of various liquid crystals analytically.

Keywords: equilibrium, quasi-average, order parameter, classification equation, liquid crystal, thermodynamic parameter

1. Motivation of the investigation and the problem setting

The theoretical foundation of statistical physics that describes equilibriums of condensed matter with a spontaneously broken symmetry is the quasi-average concept [1], [2]. Constructively, it pertains to introducing an infinitesimal source into an equilibrium statistical operator, which reduces a symmetry of the statistical equilibrium as compared with the symmetry of the Hamiltonian and allows generalizing the Gibbs distribution to degenerate condensed media. The development of the quasi-average concept and its application to quantum liquids with scalar and tensor order parameters can be found in [3]–[6].

Equilibriums with a spontaneously broken symmetry appearing as a result of a second-order phase transition in condensed media can be described and classified with a phenomenological theory [7]. Media with various forms of the order parameter were investigated based on a model form of the free energy, in particular, superfluid systems [8] and liquid crystals [9]. Symmetry considerations impose restrictions on an explicit form of the free energy as a functional of the order parameter, but attempts to relate phenomenological expansion parameters to parameters of the interparticle interaction encounter substantial difficulties [10]. Another drawback of this method is that it is applicable only in the vicinity of the phase transition point where the order parameter is small. In the regime far from the critical temperature, the explicit form of the free energy as a functional of the order parameter becomes ambiguous, and solving the corresponding nonlinear equations for this functional becomes impossible [11]. The group theory approach [12], [13] interprets an unbroken symmetry of the degenerate state as a subgroup of a symmetry group of the normal phase. In [14]–[16], states were classified with this approach for superfluid ^{3}He, which is characterized by the tensor order parameter.

Reducing the symmetry of an equilibrium as the result of a phase transition calls for generalizing the thermodynamics of condensed media and introducing additional parameters that adequately describe degenerate states. Our main aim is to establish the character and form of these parameters. Here, we propose

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a method for classifying equilibriums of degenerate condensed media based on introduced representations of generators of unbroken and spatial symmetries. We demonstrate the role played by the Hamiltonian symmetry and the transformation properties of the order parameter with respect to global transformations, whose generators are integrals of motion, in the classification of such equilibriums. The solution of this problem relies on the conditions of unbroken and spatial symmetries for nonzero values of the order parameter. We study condensed media with spontaneously broken translational and rotational symmetries in detail. We establish the connection between the symmetry conditions with one- and two-axis structures of the order parameter in liquid crystals with the respective nematic and cholesteric orderings in the case of continuous symmetries of the equilibrium. In the case where the equilibrium has discrete symmetry properties, we find the relation between the structures of generators of unbroken and spatial symmetries with the smectic ordering.

2. Normal equilibrium for a condensed medium

The Gibbs distribution for a normal nonmagnetic condensed medium in the laboratory reference frame has the form $\widehat{w}(Y) = e^{\Omega(Y)-Y_a\hat{\gamma}_a}$ and depends on thermodynamic parameters Y_a related to the symmetry of a Hamiltonian. These quantities are the temperature $Y_0^{-1} \equiv T$, the macroscopic velocity $-Y_k/Y_0 \equiv v_k$, and the chemical potential $-Y_4/Y_0 \equiv \mu$; they are the conjugates of the respective additive integrals of motion $\hat{\gamma}_a = \widehat{H}$, $\widehat{P}_k$, and $\widehat{N}$. Here, $\widehat{H}$ is the Hamiltonian, $\widehat{P}_k$ is the momentum, and $\widehat{N}$ is the particle number operator. The symmetry relation

$$[\widehat{w}, \widehat{P}_i] = 0 \tag{2.1}$$

reflects the translational invariance, and the relation

$$[\widehat{w}, \widehat{N}] = 0 \tag{2.2}$$

reflects the phase invariance of the equilibrium. The equality

$$[\widehat{w}, \hat{L}_k(\mathbf{Y})] = 0 \tag{2.3}$$

describes the property of one-axis anisotropy of the equilibrium statistical operator under spatial rotations; it can be expressed in terms of the generalized orbital momentum operator $\hat{L}_i(\mathbf{Y}) \equiv \hat{L}_i + \hat{L}_i^{\mathbf{Y}}$, where $\hat{L}_i$ is the orbital momentum operator and $\hat{L}_i^{\mathbf{Y}} \equiv -\mathrm{i}\varepsilon_{ikl} Y_k\, \partial/\partial Y_l$, (here $Y_k/|\mathbf{Y}|$ is the axis of spatial anisotropy). It acts in the Hilbert space and in the space of thermodynamic functions. The laboratory and the proper reference frames are related by the Galilean transformation defined by the unitary operator

$$U_{\mathbf{u}} = \exp\left\{-\mathrm{i}u_i \int d^3x\, x_i m \hat{n}(\mathbf{x})\right\}, \tag{2.4}$$

where m is the particle mass and u_i is the transformation parameter. In the proper reference frame, the Gibbs statistical operator is

$$w(Y') = U_{\mathbf{u}}\widehat{w}(Y)U_{\mathbf{u}}^{+} \equiv e^{\Omega - Y'_a\hat{\gamma}_a}, \tag{2.5}$$

where

$$Y'_0 = Y_0, \qquad Y'_k = Y_k + Y_0 u_k, \qquad Y'_4 = Y_4 + Y_k u_k + \frac{Y_0 u^2}{2}.$$

At $u_k = -Y_k/Y_0$, we have $Y'_k = 0$. The normal condensed medium is isotropic in the proper reference frame, $[\widehat{w}(Y'), \hat{L}_k] = 0$.

3. Quasi-averages: The classification equation for the order parameter

In the case of second-order phase transitions, condensed matter passes from the normal state to a state with spontaneously broken symmetry. Additional thermodynamic parameters depend on the character of the broken symmetry and appear as a result of the presence of the residual symmetry of the equilibrium. In accordance with the concept of quasi-averages, the Gibbs distribution for degenerate media has the form $\widehat{w}_v \equiv e^{\Omega_v - Y_a \hat{\gamma}_a - v\widehat{F}}$. The source $\widehat{F}$ that breaks the symmetry of the normal equilibrium is a linear functional of the order parameter operator $\hat{\Delta}_a(\mathbf{x})$,

$$\widehat{F} = \int d^3x \big(f_a(\mathbf{x})\hat{\Delta}_a(\mathbf{x}) + \text{H.c.} \big).$$

In a degenerate state, the equilibrium mean of the order parameter is nonzero,

$$\Delta_a(\mathbf{x}, Y, f) = \operatorname{Sp} \widehat{w}\hat{\Delta}_a(\mathbf{x}) \equiv \lim_{v\to 0} \lim_{V\to\infty} \operatorname{Sp} \widehat{w}_v \hat{\Delta}_a(\mathbf{x}) \neq 0.$$

This formula relates the equilibrium mean of the order parameter to thermodynamic forces and to the function $f_a(\mathbf{x})$ in the source.

To obtain classification equations for equilibriums, we describe the transformation properties of the order parameter operator. Because the order parameter operators $\hat{\Delta}_a(\mathbf{x})$ are either linear or bilinear in the field operators, the right-hand sides of the quantum Poisson brackets between integrals of motion and order parameter operators must be linear functions of the order parameter operators. The condition of translational invariance is

$$\mathrm{i}[\widehat{P}_k, \hat{\Delta}_a(\mathbf{x})] = -\nabla_k \hat{\Delta}_a(\mathbf{x}). \tag{3.1}$$

The group of phase transformations is generated by the particle number operator, and the order parameter operator $\hat{\Delta}_a(\mathbf{x})$ then transforms in accordance with the relation

$$[\widehat{N}, \hat{\Delta}_a(\mathbf{x})] = -g\hat{\Delta}_a(\mathbf{x}), \tag{3.2}$$

where the constant quantity g depends on the tensorial dimension and on the internal structure of the order parameter operator. Under spatial rotations, the order parameter operators transform in accordance with the formula

$$\mathrm{i}[\hat{L}_l, \hat{\Delta}_a(\mathbf{x})] = -g_{lab}\hat{\Delta}_b(\mathbf{x}) - \varepsilon_{ljk} x_k \nabla_j \hat{\Delta}_a(\mathbf{x}), \tag{3.3}$$

where g_{lab} are constants.

For translational-invariant degenerate equilibriums satisfying (2.1), the condition of unbroken (residual) symmetry of the equilibrium is determined by the relation

$$[\widehat{w}, \widehat{T}(\xi, \mathbf{Y})] = 0. \tag{3.4}$$

Here, the generator of the unbroken symmetry

$$\widehat{T}(\xi, \mathbf{Y}) \equiv a_i \hat{L}_i(\mathbf{Y}) + c\widehat{N} \tag{3.5}$$

is a linear combination of integrals of motion with the real parameters $\mathbf{Y}$ and $\xi \equiv (a_i, c)$ and represents the residual symmetry remaining in the degenerate medium, which is less than that of a more symmetric normal equilibrium. Relations (2.1), (3.4), and (3.5) result in differential equations for the function $f_a(\mathbf{x})$, which are obtained from the condition $[\widehat{T},\widehat{F}]=0$, but obtaining classification equations directly for the equilibrium mean of the order parameter is more explicit. For this, we multiply relation (3.4) by the order parameter operator, take transformation properties (3.1)–(3.3) into account, and take the average. As a result, the property of unbroken symmetry in the presence of spatially homogeneous equilibriums results in a linear partial differential equation for the order parameter,

$$a_i\left(g_{iab}\Delta_b(\mathbf{x})+\varepsilon_{ikl}Y_k\frac{\partial\Delta_a(\mathbf{x})}{\partial Y_l}\right)+\mathrm{i}gc\Delta_a(\mathbf{x})=0. \tag{3.6}$$

Spatial symmetry condition (2.1) implies the equality $\nabla_k\Delta_a(\mathbf{x})=0$. The linearity of commutation relations (3.1)–(3.3) for the order parameter operator results in the equation for determining its equilibrium structure becoming linear. In the case $\mathbf{Y}=0$, Eq. (3.6) is drastically simplified and becomes a homogeneous linear algebraic equation,

$$T_{ab}(\xi,0)\Delta_b=0,\qquad T_{ab}(\xi,0)\equiv a_ig_{iab}+\mathrm{i}gc\delta_{ab}. \tag{3.7}$$

The nontriviality condition for a solution, $\Delta_a\neq0$, implies the equation $\det|T_{ab}(\xi,0)|=0$ for the admissible values of the parameters of the unbroken symmetry generator, which allows classifying equilibriums of degenerate condensed media. The Gibbs statistical operator $\widehat{w}=\widehat{w}(Y,\xi)$ then depends both on thermodynamic parameters Y and on the parameters ξ of the unbroken symmetry generator.

We can analogously obtain classification equations for equilibriums in an inhomogeneous case. For spatially inhomogeneous states of degenerate condensed media, the generators of unbroken (residual) and spatial symmetries are determined by the respective equalities

$$\begin{aligned}&\widehat{T}(\xi,\mathbf{Y})\equiv a_i\hat{L}_i(\mathbf{Y})+c\widehat{N}+d_i\widehat{P}_i,\\&\widehat{P}_k(\eta,\mathbf{Y})\equiv\widehat{P}_k-p_k\widehat{N}-t_{kj}\hat{L}_j(\mathbf{Y})\end{aligned} \tag{3.8}$$

and result in the relations

$$\mathrm{Sp}[\widehat{w},\widehat{T}(\xi,\mathbf{Y})]\hat{\Delta}_a(\mathbf{x})=0,\qquad \mathrm{Sp}[\widehat{w},\widehat{P}_k(\eta,\mathbf{Y})]\hat{\Delta}_a(\mathbf{x})=0. \tag{3.9}$$

Here, $\xi\equiv(\mathbf{a},c,\mathbf{d})$ and $\eta\equiv(\mathbf{p},t_{kj})$ are real parameters characterizing the generators of unbroken and spatial symmetries. Relations (3.9) result in the order parameter becoming dependent on the coordinate and also on the parameters of the unbroken and spatial symmetries. To establish the structure of the order parameter in a homogeneous state and to determine the admissible form of symmetry generators, we must complete relations (3.9) with the equations

$$\mathrm{Sp}\big[\widehat{w},[\widehat{T}(\xi,\mathbf{Y}),\widehat{P}_k(\eta,\mathbf{Y})]\big]\hat{\Delta}_a(\mathbf{x})=0,\qquad \mathrm{Sp}\big[\widehat{w},[\widehat{P}_i(\eta,\mathbf{Y}),\widehat{P}_k(\eta,\mathbf{Y})]\big]\hat{\Delta}_a(\mathbf{x})=0, \tag{3.10}$$

which follow from the Jacoi identities for the respective operators $\widehat{w}$, $\widehat{T}(\xi,\mathbf{Y})$, $\widehat{P}_k(\eta,\mathbf{Y})$ and $\widehat{w}$, $\widehat{P}_i(\eta,\mathbf{Y})$, $\widehat{P}_k(\eta,\mathbf{Y})$. Relations (3.9) and (3.10) result in the relations between the parameters of the symmetry generators and allows solving the classification problem for equilibriums of nonhomogeneous condensed media. By virtue of this, the Gibbs statistical operator depends on the thermodynamic parameters: $\widehat{w}=\widehat{w}(Y,\xi,\eta)$. By virtue of algebra (3.1)–(3.3) at $\mathbf{Y}=0$, we obtain the system of equations

$$\begin{aligned}&\nabla_k\Delta_a(\mathbf{x})=G^k_{ab}\Delta_b(\mathbf{x}),\qquad T_{ab}(\xi,\eta)\Delta_b(\mathbf{x})=0,\qquad a_i\varepsilon_{ikl}G^l_{ab}\Delta_b(\mathbf{x})=0,\\&a_it_{kj}\varepsilon_{ij\lambda}\varepsilon_{\lambda uv}G^v_{ab}\Delta_b(\mathbf{x})=0,\qquad a_it_{kj}\varepsilon_{ij\lambda}g_{\lambda ab}\Delta_b(\mathbf{x})=0,\\&t_{ij}t_{k\lambda}\varepsilon_{j\lambda l}\varepsilon_{lsp}G^p_{ab}\Delta_b(\mathbf{x})=0,\qquad t_{kj}\varepsilon_{jsp}G^k_{ab}\Delta_b(\mathbf{x})=0,\qquad t_{ij}t_{k\lambda}\varepsilon_{j\lambda l}g_{lab}\Delta_b(\mathbf{x})=0,\end{aligned} \tag{3.11}$$

which establishes the structure of the order parameter in an inhomogeneous equilibrium. Here, we introduce the notation

$$G^k_{ab} \equiv \mathrm{i}p_k g\delta_{ab} + t_{kj}g_{jab}, \qquad T_{ab}(\xi, \eta) \equiv (a_i + d_l t_{li})g_{iab} + \mathrm{i}g(c + d_l p_l)\delta_{ab}.$$

We note that Eqs. (3.7) and (3.11) can also be useful when solving the classification problem under the condition of the coexistence of several nonzero equilibrium values of the order parameters.

4. Liquid crystals

We now use the formulated approach to classify equilibriums of liquid crystals with broken translational and rotational symmetries. The order parameter operator for such media is a symmetric traceless tensor [9], which we can define as

$$\widehat{Q}_{uv}(\mathbf{x}) \equiv \frac{1}{2}\bigg(\nabla_u\hat{\psi}^+(\mathbf{x})\nabla_v\hat{\psi}^+(\mathbf{x}) + \nabla_v\hat{\psi}^+(\mathbf{x})\nabla_u\hat{\psi}(\mathbf{x}) - \frac{2}{3}\delta_{uv}\nabla_j\hat{\psi}^+(\mathbf{x})\nabla_j\hat{\psi}(\mathbf{x})\bigg). \tag{4.1}$$

The quantum brackets of this operator with integrals of motion are

$$\begin{aligned} &[\widehat{N}, \widehat{Q}_{uv}(\mathbf{x})] = 0, \qquad \mathrm{i}[\widehat{P}_k, \widehat{Q}_{uv}(\mathbf{x})] = -\nabla_k\widehat{Q}_{uv}(\mathbf{x}), \\ &\mathrm{i}[\hat{L}_i, \widehat{Q}_{uv}(\mathbf{x})] = -\varepsilon_{ikl}x_k\nabla_l\widehat{Q}_{uv}(\mathbf{x}) - \varepsilon_{iuj}\widehat{Q}_{jv}(\mathbf{x}) - \varepsilon_{ivj}\widehat{Q}_{ju}(x). \end{aligned} \tag{4.2}$$

Because the order parameter is symmetric and traceless, its mean $Q_{uv}(\mathbf{x}, \hat{\rho}) = \mathrm{Sp}\,\hat{\rho}\widehat{Q}_{uv}(\mathbf{x})$ contains five independent quantities and can be represented in the form

$$Q_{ik}(\mathbf{x}) \equiv Q(\mathbf{x})\bigg(n_i(\mathbf{x})n_k(\mathbf{x}) - \frac{\delta_{ik}}{3}\bigg) + Q'(\mathbf{x})\bigg(m_i(\mathbf{x})m_k(\mathbf{x}) - \frac{\delta_{ik}}{3}\bigg), \tag{4.3}$$

where Q and Q' are the moduli of the order parameter and $\hat{\rho}$ is an arbitrary statistical operator. The mutually orthogonal unit vectors $\mathbf{n}$ and $\mathbf{m}$ indicate the anisotropy directions in liquid crystals. Because $m_u m_v + n_u n_v + l_u l_v = \delta_{uv}$, where $\mathbf{m}\times\mathbf{n} = \mathbf{l}$, we can have either one- or two-axis anisotropy of liquid-crystal equilibriums.

In the normal state of a condensed medium, the spatial isotropy condition $[\widehat{w}(Y'), \hat{L}_k] = 0$, quantum bracket algebra (4.2), and the explicit form for mean (4.3) result in the equality

$$\mathrm{i}\,\mathrm{Sp}[\widehat{w}(Y'), \hat{L}_i]\widehat{Q}_{uv}(\mathbf{x}) = Q(Y')(\varepsilon_{iuj}n_jn_v + \varepsilon_{ivj}n_jn_u) + Q'(Y')(\varepsilon_{iuj}m_jm_v + \varepsilon_{ivj}m_jm_u) = 0,$$

whence it follows that the order parameter moduli are $Q(Y') = Q'(Y') = 0$ because there are no selected directions in this state. The mean of the order parameter for liquid crystals therefore vanishes, $\mathrm{Sp}\,\widehat{w}(Y')\widehat{Q}_{uv}(\mathbf{x}) = 0$, in the laboratory reference frame. In the laboratory reference frame in which the thermodynamic parameter $\mathbf{Y} \neq 0$, the order parameter is nonzero. From definition (4.1) and the relations $U_\mathbf{u}\hat{\psi}^+(\mathbf{x})U^+_\mathbf{u} = e^{\mathrm{i}m\mathbf{ux}}\hat{\psi}^+(\mathbf{x})$ and $U_\mathbf{u}\hat{\psi}(\mathbf{x})U^+_\mathbf{u} = e^{\mathrm{i}m\mathbf{ux}}\hat{\psi}(\mathbf{x})$, we obtain the transformation property for the order parameter under Galilean transformations,

$$U_\mathbf{u}\widehat{Q}_{ik}(\mathbf{x})U^+_\mathbf{u} = \widehat{Q}_{ik}(\mathbf{x}) + m\bigg(u_i\hat{\pi}_k(\mathbf{x}) + u_k\hat{\pi}_i(\mathbf{x}) - 2u_j\hat{\pi}_j(\mathbf{x})\frac{\delta_{ik}}{3}\bigg) + m^2\hat{n}(\mathbf{x})\bigg(u_iu_k - \frac{u^2\delta_{ik}}{3}\bigg) \tag{4.4}$$

(here $\hat{\pi}_k(\mathbf{x})$ is the momentum density operator). Further noting that we have the relations $\mathrm{Sp}\,w(Y')\hat{\pi}_k(\mathbf{x}) = 0$ and $\mathrm{Sp}\,w(Y')\widehat{Q}_{ik}(\mathbf{x}) = 0$ in the proper reference, we obtain the expression for the homogeneous value of the order parameter in the laboratory reference frame,

$$\mathrm{Sp}\,\widehat{w}(Y)\widehat{Q}_{ik}(\mathbf{x}) = m^2\,\mathrm{Sp}\,\widehat{w}\hat{n}(0)\cdot\bigg(u_iu_k - \frac{\delta_{ik}u^2}{3}\bigg) \neq 0. \tag{4.5}$$

One-axis nematics. We consider translational-invariant equilibriums of liquid-crystal media satisfying (2.1) and establish possible equilibrium structures of the order parameter. We analyze translational-invariant subgroups of the unbroken symmetry of equilibriums in the proper reference frame based on the relation

$$[\widehat{w}(Y'), \widehat{T}] = 0. \tag{4.6}$$

The unbroken-symmetry generator $\widehat{T}$ is a linear combination of integrals of motion. Because the phase invariance remains unbroken in liquid crystals, $[\widehat{w}(Y'), \widehat{N}] = 0$, we have

$$\widehat{T} \equiv a_i \hat{L}_i, \tag{4.7}$$

where a_i is a real vector characterizing spatial rotations. Without restricting the generality, we assume that the vector $\mathbf{a}$ is a unit vector, $a^2 = 1$. Unitary transformations $U = e^{\mathrm{i}\widehat{T}(\mathbf{a})}$ constitute continuous subgroups of an unbroken symmetry of an equilibrium, $U(\mathbf{a})U(\mathbf{a}') = U(\mathbf{a}'')$, $\mathbf{a}' = \mathbf{a}''(\mathbf{a}, \mathbf{a}')$. In accordance with (4.6) and (4.7), we have $\mathrm{Sp}[\widehat{w}(Y'), a_i\hat{L}_i]\widehat{Q}_{uv}(\mathbf{x}) = 0$. Therefore, taking algebra (4.2) into account, we obtain the equation

$$F^{uv}_{jk} Q_{jk} = 0, \qquad F^{uv}_{jk} \equiv a_i(\varepsilon_{iuj}\delta_{vk} + \varepsilon_{iuv}\delta_{uk}). \tag{4.8}$$

In this formula, we now pass from double to single summation, replacing the two pairs of indices u, v and j, k, each of which takes the three values 1, 2, and 3, with the indices $\alpha, \beta = 1, 2, \ldots, 9$ such that $(1, 1) \mapsto 1$, $(1, 2) \mapsto 2$, $\ldots$, $(3, 3) \mapsto 9$. We then obtain the system of linear algebraic equations $F^{\beta}_{\alpha} Q_{\alpha} = 0$. The existence condition for a nontrivial solution $Q_{\alpha}(\mathbf{x}) \neq 0$ of this system is $\det|F^{\beta}_{\alpha}| = 0$. It follows from the explicit form of F^{β}_{α} given by (4.8) that this relation is satisfied for any direction of the vector $\mathbf{a}$, which is easily seen if we write the determinant in the coordinate system in which this vector has the components $(0, 0, 1)$. The solution of Eq. (4.8) $Q_{ik}(\mathbf{a}) = Q(a_i a_k - \delta_{ik}/3)$ and the form of unbroken-symmetry generator (4.7) correspond to the one-axis nematic case. Reasoning analogously, we can show that the function f_{uv} in the expression for the source $\widehat{F}$ has the form $f_{uv} = a_u a_v - \delta_{uv}/3$. Hence, the liquid-crystal state under investigation is described by the statistical operator $\widehat{w} = \widehat{w}(Y, \mathbf{a})$, which depends on thermodynamic forces and on the vector of spatial anisotropy (the director), and the dependence of the equilibrium means on the latter argument survives the thermodynamic limit transition.

Cholesterics. We investigate the case where the spatial symmetry of the equilibrium is more complicated and is determined by the equality

$$\widehat{P}_k(t) \equiv \widehat{P}_k - t_{kj}\hat{L}_j, \tag{4.9}$$

where $\widehat{P}_k(t)$ is the spatial symmetry generator and t_{ik} is a numerical matrix with complex elements characterizing the spatial symmetry. The unbroken-symmetry generator for such inhomogeneous states is a linear combination of the momentum and orbital momentum operators,

$$\widehat{T}(\mathbf{a}, \mathbf{d}) \equiv a_i\hat{L}_i + d_i\widehat{P}_i, \tag{4.10}$$

where $\mathbf{a}$ and $\mathbf{d}$ are real-valued vector parameters and $a^2 = 1$. For such liquid crystals, the spatial symmetry condition for the equilibrium,

$$\left[\widehat{w}(Y'), \widehat{P}_k\right] = 0, \tag{4.11}$$

must be completed with unbroken-symmetry condition (4.6) for the equilibrium, where the generator $\widehat{T}$ is now defined by equality (4.10). In accordance with these symmetry conditions, we write the equalities

$$\mathrm{i}\,\mathrm{Sp}[\widehat{w}, \widehat{T}(\mathbf{a}, \mathbf{d})]\widehat{Q}_{uv}(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\widehat{w}, \widehat{P}_i(t)]\widehat{Q}_{uv}(\mathbf{x}) = 0. \tag{4.12}$$

Condition (4.11), the form of the generator $\widehat{P}_k$ given by (4.9), and algebra (4.2) result in the system of equations

$$\nabla_k Q_{uv}(\mathbf{x}) = t_{kj} Q^j_{uv}(\mathbf{x}), \qquad t_{kj}\varepsilon_{jsq}t_{qp}Q^p_{uv}(\mathbf{x}) = 0, \tag{4.13}$$

where we introduce the notation $Q^j_{uv}(\mathbf{x}) \equiv \varepsilon_{ju\lambda}Q_{\lambda v}(\mathbf{x}) + \varepsilon_{jv\lambda}Q_{\lambda u}(\mathbf{x})$. The second relation in (4.13) results from the condition that there is no term linear in the coordinate in spatial symmetry relation (4.12) and from the first relation in (4.12). The unbroken-symmetry condition in (4.12) by virtue of generator form (4.10) results in the relations

$$a_i\varepsilon_{ikl}t_{lj}Q^j_{uv}(\mathbf{x}) = 0, \qquad \underline{a}_i Q^i_{uv}(\mathbf{x}) = 0, \tag{4.14}$$

where $\underline{a}_i \equiv a_i + d_j t_{ji}$. We use the Jacobi identity to find additional constraints on the parameters of the symmetry generators introduced by relations (4.9) and (4.10). Taking symmetry conditions (4.6) and (4.11) into account, we obtain the equality $\mathrm{Sp}\big[\widehat{w}, [\widehat{T}, \widehat{P}_k]\big]\widehat{Q}_{uv}(\mathbf{x}) = 0$ for the operators $\widehat{w}$, $\widehat{T}$, and $\widehat{\mathbf{P}}$. Hence, taking (4.2), (4.9), and (4.10) into account, we obtain the relations

$$a_i t_{kj}\varepsilon_{ij\lambda}\varepsilon_{\lambda sl}t_{lp}Q^p_{uv}(\mathbf{x}) = 0, \qquad a_i t_{kj}\varepsilon_{ijp}Q^p_{uv}(\mathbf{x}) = 0. \tag{4.15}$$

Using the Jacobi identity for the operators $\widehat{w}$, $\widehat{P}_i$, and $\widehat{P}_k$ and taking spatial symmetry property (4.11) into account, we obtain the equality $\mathrm{Sp}\big(\big[\widehat{w}, [\widehat{P}_i, \widehat{P}_k]\big]\widehat{Q}_{uv}(\mathbf{x})\big) = 0$. Hence, we have the equations

$$t_{i\lambda}t_{kj}\varepsilon_{\lambda jq}\varepsilon_{qsl}t_{l\sigma}Q^\sigma_{uv}(\mathbf{x}) = 0, \qquad t_{i\lambda}t_{kj}\varepsilon_{\lambda jq}Q^q_{uv}(\mathbf{x}) = 0. \tag{4.16}$$

System of equations (4.13)–(4.16) completely determines the admissible structure of the parameters of the symmetry generator together with the form of the order parameter in the inhomogeneous equilibrium under investigation. We now demonstrate that a solution of this system of equations can be represented in the form

$$Q_{ik}(\mathbf{x}) = Q(Y)\Big\{\big(m_i\cos\varphi(\mathbf{x}) + n_i\sin\varphi(\mathbf{x})\big)\big(m_k\cos\varphi(\mathbf{x}) + n_k\sin\varphi(\mathbf{x})\big) - \frac{\delta_{ik}}{3}\Big\}. \tag{4.17}$$

Indeed, if we substitute this expression for the order parameter in Eq. (4.13), then the equations are satisfied identically. Further, the first relation in (4.14) for $\mathbf{a}\|\mathbf{l}$ results in the equality $t = -l/\mathbf{dl}$. The matrix t_{ik} of spatial symmetry generator (4.9) is then $t_{ik} = tl_il_k$. We can analogously show that the other equations in (4.15) and (4.16) are satisfied under the same conditions imposed on the parameters of the symmetry generators. In the case under consideration, the inhomogeneous structure of the order parameter describes a cholesteric state and is characterized by the function $\varphi(\mathbf{x}) = \varphi - \mathbf{tx}$, where the vector of the cholesteric helix $\mathbf{t}$ is directed along the axis $\mathbf{l} = \mathbf{m}\times\mathbf{n}$ and the quantity $2\pi/t$ has the physical sense of the pitch of this helix. A solution of this type for the order parameter was previously obtained from the minimality condition for a model expression of the energy as a function of the director field [17]. The presence of two vectors in the spatial symmetry generator results in the possibility of the existence of straight and inclined cholesteric helices.

Two-axis nematics. We now consider homogeneous states of liquid crystals whose unbroken-symmetry generator is

$$\widehat{T}(\mathbf{a},\mathbf{m},\mathbf{n}) \equiv a_i\hat{L}_i(\mathbf{m},\mathbf{n}). \tag{4.18}$$

This form of the generator, as is seen below, corresponds to two-axis liquid-crystal states. Here, the generalized orbital momentum generator is defined as

$$\hat{L}_i(\mathbf{m},\mathbf{n}) \equiv \hat{L}_i + \hat{L}^{\mathbf{m}}_i + \hat{L}^{\mathbf{n}}_i, \qquad \hat{L}^{\mathbf{m}}_i \equiv -\mathrm{i}\varepsilon_{ikl}m_k\frac{\partial}{\partial m_l}, \qquad \hat{L}^{\mathbf{n}}_i \equiv -\mathrm{i}\varepsilon_{ikl}n_k\frac{\partial}{\partial m_l} \tag{4.19}$$

(the vectors $\mathbf{n}$ and $\mathbf{m}$ are unit vectors and are orthogonal, $\mathbf{nm} = 0$) and has the sense of the generator of a two-axis spatial symmetry. By virtue of definition (4.19), we have the relation $\mathrm{i}[\hat{L}_i(\mathbf{m},\mathbf{n}), \hat{L}_k(\mathbf{m},\mathbf{n})] = -\varepsilon_{ikl}\hat{L}_l(\mathbf{m},\mathbf{n})$. Using symmetry condition (4.6) at $Y_i = 0$ with the generator determined by relations (4.18) and (4.19), we obtain the equation

$$a_i\left(\varepsilon_{iuj}Q_{jv} + \varepsilon_{ivj}Q_{ju} + \varepsilon_{ikl}m_k\frac{\partial Q_{uv}}{\partial m_l} + \varepsilon_{ikl}n_k\frac{\partial Q_{uv}}{\partial n_l}\right) = 0. \tag{4.20}$$

We seek a solution of this equation in the form $Q_{uv} \equiv Q_1(e_u^{(1)}e_v^{(1)} - \delta_{uv}/3) + Q_2(e_u^{(2)}e_v^{(2)} - \delta_{uv}/3)$, where the unit vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are defined by the equalities $\mathbf{e}^{(1)} \equiv \mathbf{m}\cos\varphi + \mathbf{n}\sin\varphi$ and $\mathbf{e}^{(2)} \equiv -\mathbf{m}\sin\varphi + \mathbf{n}\cos\varphi$. Further, taking the formulas

$$\frac{\partial n_i}{\partial n_k} = \frac{\partial m_i}{\partial m_k} \equiv \delta_{ik} - n_i n_k - m_i m_k = l_i l_k$$

into account, we transform Eq. (4.20) into the form

$$\begin{aligned} &a_i\varepsilon_{iuj}(Q_1 e_j^{(1)}e_v^{(1)} + Q_2 e_j^{(1)}e_v^{(1)}) + a_i\varepsilon_{ivj}(Q_1 e_j^{(1)}e_u^{(1)} - + Q_2 e_j^{(1)}e_u^{(1)}) = \\ &\quad = Q_1(\mathbf{an}\cos\varphi - \mathbf{am}\sin\varphi)(l_u e_v^{(1)} + l_v e_u^{(1)}) - Q_2(\mathbf{an}\sin\varphi + \mathbf{am}\cos\varphi)(l_u e_v^{(2)} + l_v e_u^{(2)}). \end{aligned} \tag{4.21}$$

To determine admissible values of the vector $\mathbf{a}$, we seek it in the form of a component expansion in the orthonormal basis, $a_i = \alpha n_i + \beta m_i + \gamma l_i$, where the numbers α, β, and γ are related by $\alpha^2 + \beta^2 + \gamma^2 = 1$. As a result, we obtain the condition $\gamma(Q_1 - Q_2)(e_u^{(1)}e_v^{(2)} + e_v^{(1)}e_u^{(2)}) = 0$. It hence follows that the parameter $\gamma = 0$ for $Q_1 \neq 0$, $Q_2 \neq 0$, and $Q_1 \neq Q_2$. This solution describes a spatially homogeneous two-axis nematic:

$$Q_{uv} = Q_1\left(e_u^{(1)}e_v^{(1)} - \frac{\delta_{uv}}{3}\right) + Q_2\left(e_u^{(2)}e_v^{(2)} - \frac{\delta_{uv}}{3}\right). \tag{4.22}$$

If we set $Q_1 = Q_2$, then we obtain another solution. In this case, the vector $\mathbf{a}$ is arbitrary. By virtue of the relation $m_u m_v + n_u n_v + l_u l_v = \delta_{uv}$, the order parameter becomes $Q_{uv} = -Q(l_u l_v - \delta_{uv}/3)$ and corresponds to a one-axis nematic. The obtained classification results for homogeneous equilibriums of liquid crystals are presented in Table 1.

Table 1

Equilibrium	Unbroken-symmetry generator	Order parameter
One-axis nematic	$a_i\hat{L}_j$	$Q_{ik}(\mathbf{a}) = Q(a_i a_k - \delta_{ik}/3)$
Two-axis nematic	$(\alpha n_i + \beta m_i)\hat{L}_i(\mathbf{m},\mathbf{n})$, $\alpha^2 + \beta^2 = 1$	$Q_1(e_i^{(1)}e_k^{(1)} - \delta_{ik}/3) + Q_2(e_i^{(2)}e_k^{(2)} - \delta_{ik}/3)$, $e_i^{(1)} = m_i\cos\varphi + n_i\sin\varphi$, $e_i^{(2)} = -m_i\sin\varphi + n_i\cos\varphi$

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Double helix. This double helix case of liquid-crystal ordering can be characterized by an unbroken-symmetry generator of the form

$$\widehat{T}(\mathbf{a},\mathbf{d},\mathbf{m},\mathbf{n}) \equiv a_i\hat{L}_i(\mathbf{m},\mathbf{n}) + d_i\widehat{P}_i(t,\mathbf{m},\mathbf{n}). \tag{4.23}$$

Here, the generalized orbital momentum generator is defined by equality (4.19), and the spatial symmetry generator is defined by the relation

$$\widehat{P}_k(t, \mathbf{m}, \mathbf{n}) \equiv \widehat{P}_k - t_{kj}\hat{L}_j(\mathbf{m}, \mathbf{n}). \tag{4.24}$$

Admissible constraints on the parameters of the generators of the unbroken and spatial symmetries can be found from the relations

$$\mathrm{i}\,\mathrm{Sp}[\widehat{w}, \widehat{T}(\mathbf{a}, \mathbf{d}, \mathbf{m}, \mathbf{n})]\widehat{Q}_{uv}(\mathbf{x}) = 0, \qquad \mathrm{i}\,\mathrm{Sp}[\widehat{w}, \widehat{P}_i(t, \mathbf{m}, \mathbf{n})]\widehat{Q}_{uv}(\mathbf{x}) = 0. \tag{4.25}$$

Additional constraints appear when the Jacobi identities for the operators $\widehat{w}$, $\widehat{P}_i$, and $\widehat{P}_k$ and for the operators $\widehat{w}$, $\widehat{T}$, and $\widehat{P}_k$ are taken into account:

$$\mathrm{Sp}\big[\widehat{w}, [\widehat{T}, \widehat{P}_k]\big]\widehat{Q}_{uv}(\mathbf{x}) = 0, \qquad \mathrm{Sp}\big[\widehat{w}, [\widehat{P}_i, \widehat{P}_k]\big]\widehat{Q}_{uv}(\mathbf{x}) = 0. \tag{4.26}$$

From relations (4.25) with algebra (4.2) taken into account, we obtain the equations

$$\begin{aligned} &\nabla_k Q_{uv}(\mathbf{x}) = t_{kj} Q^j_{uv}(\mathbf{x}), \qquad t_{kj}\varepsilon_{jsq} t_{qp} Q^p_{uv}(\mathbf{x}) = 0, \\ &a_i \varepsilon_{ikl} t_{lj} Q^j_{uv}(\mathbf{x}) = 0, \qquad \underline{a}_i Q^j_{uv}(\mathbf{x}) = 0, \end{aligned} \tag{4.27}$$

where the quantity $Q^j_{uv}(\mathbf{x})$ is given by the equality

$$Q^j_{uv}(\mathbf{x}) \equiv \varepsilon_{ju\lambda} Q_{\lambda u}(\mathbf{x}) + \varepsilon_{jv\lambda} Q_{\lambda u}(\mathbf{x}) + \varepsilon_{jsl} n_s \frac{\partial Q_{uv}(\mathbf{x})}{\partial n_l} + \varepsilon_{jsl} m_s \frac{\partial Q_{uv}(\mathbf{x})}{\partial m_l}. \tag{4.28}$$

The Jacobi identities result in the equations

$$\begin{aligned} &a_i t_{kj} \varepsilon_{ij\lambda} \varepsilon_{\lambda sl} t_{lp} Q^p_{uv}(\mathbf{x}) = 0, \qquad a_i t_{kj} \varepsilon_{ijp} Q^p_{uv}(\mathbf{x}) = 0, \\ &t_{i\lambda} t_{kj} \varepsilon_{\lambda jq} \varepsilon_{qsl} t_{l\sigma} Q^\sigma_{uv}(\mathbf{x}) = 0, \qquad t_{i\lambda} t_{kj} \varepsilon_{\lambda jq} Q^q_{uv}(\mathbf{x}) = 0. \end{aligned} \tag{4.29}$$

Following the same line of reasoning as when considering the one-axis cholesteric case, we can easily show that for $\mathbf{a}\|\mathbf{l}$ and $t = -l/\mathbf{dl}$, Eqs. (4.27)–(4.29) are satisfied, and the order parameter in the equilibrium becomes

$$Q_{ik}(\mathbf{x}) = Q_1\left(e^{(1)}_i(\mathbf{x}) e^{(1)}_k(\mathbf{x}) - \frac{\delta_{ik}}{3}\right) + Q_2\left(e^{(2)}_i(\mathbf{x}) e^{(2)}_k(\mathbf{x}) - \frac{\delta_{ik}}{3}\right),$$

where $e^{(1)}_i(\mathbf{x}) = m_i \cos\varphi(\mathbf{x}) + n_i \sin\varphi(\mathbf{x})$ and $e^{(2)}_i(\mathbf{x}) = -m_i \sin\varphi(\mathbf{x}) + n_i \cos\varphi(\mathbf{x})$ are mutually orthogonal unit vectors and the phase depends on the coordinate as $\varphi(\mathbf{x}) = \varphi - \mathbf{lx}/\mathbf{dl}$. We note that the solution with the double helix spatial structure resembles the DNA molecular structure [18]. The complementary Watson–Crick pairs adenine–thymine and guanine–cytosine lie in the plane of the vectors $\mathbf{e}^{(1)}(\mathbf{x})$ and $\mathbf{e}^{(2)}(\mathbf{x})$ and rotate about the anisotropy axis $\mathbf{l}$ when the coordinate is shifted. The case of collinear vectors $\mathbf{l}$ and $\mathbf{d}$ corresponds to the DNA B-form. The complementary pairs are then in the plane orthogonal to the helix axis. If the vectors $\mathbf{l}$ and $\mathbf{d}$ are noncollinear, then such a spatial structure is an A-form, and we have the space structure ordering of the inclined helix form. The planes of complementary pairs are then situated at an angle to the spiral axis. The possibility of different signs of the scalar products of the vectors ($\mathbf{ld} > 0$ or $\mathbf{ld} < 0$) indicates the possibility of physically realizing right-handed and left-handed helices.

Table 2

Equilibrium	Unbroken-symmetry generator	Spatial symmetry generator	Order parameter
cholesteric	$l_i\hat{L}_i + d_i\widehat{P}_i$	$\widehat{P}_k + \dfrac{l_k}{\mathbf{dl}} l_i\hat{L}_i$	$Q\big(e_i^{(1)}(\mathbf{x})e_k^{(1)}(\mathbf{x}) - \delta_{ik}/3\big)$, $e_i^{(1)}(x) = m_i\cos\varphi(\mathbf{x}) + n_i\sin\varphi(\mathbf{x})$, $\varphi(\mathbf{x}) = \varphi - \mathbf{lx}/\mathbf{dl}$
double helix	$l_i\hat{L}_i(\mathbf{m},\mathbf{n}) + d_i\widehat{P}_i$,	$\widehat{P}_k + \dfrac{l_k}{\mathbf{dl}} l_i\hat{L}_i(\mathbf{m},\mathbf{n})$	$Q_1\big(e_i^{(1)}(\mathbf{x})e_k^{(1)}(\mathbf{x}) - \delta_{ik}/3\big)+$ $+Q_2\big(e_i^{(2)}(\mathbf{x})e_k^{(2)}(\mathbf{x}) - \delta_{ik}/3\big)$, $e_i^{(1)}(\mathbf{x}) = m_i\cos\varphi(\mathbf{x}) + n_i\sin\varphi(\mathbf{x})$, $e_i^{(2)}(\mathbf{x}) = -m_i\sin\varphi(\mathbf{x}) + n_i\cos\varphi(\mathbf{x})$

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A-smectic. For the A-smectic equilibrium of liquid crystals, the spatial symmetry conditions are

$$[\widehat{w}, l_i\hat{L}_i] = 0, \qquad \varepsilon_{ikl}l_k[\widehat{w}, \widehat{P}_l] = 0, \qquad e^{\mathrm{i}\widehat{P}_k l_k b}\widehat{w}e^{-\mathrm{i}\widehat{P}_k l_k b} = \widehat{w}. \tag{4.30}$$

They show that the medium has a one-axis spatial anisotropy. We have a periodic crystal structure along the anisotropy axis $\mathbf{l}$ (one-dimensional crystal), and a two-dimensional homogeneous and isotropic condensed medium state is realized (liquid phase) in the plane orthogonal to this axis. The first relation in (4.30) and operator algebra (4.2) result in the equilibrium structure of the order parameter,

$$Q_{uv}(\mathbf{x}, Y, \mathbf{l}, b) = Q(\mathbf{x}, Y, \mathbf{l}, b)\left(l_u l_v - \frac{1}{3}\delta_{uv}\right). \tag{4.31}$$

In the case under consideration, the absolute value of the order parameter depends on the space coordinate. We seek this dependence in the form

$$Q(\mathbf{x}, Y, \mathbf{l}, b) = Q\big(\varphi(\mathbf{x}, \mathbf{l}, b, Y)\big), \qquad \varphi(\mathbf{x}, \mathbf{l}, b) = \varphi(\mathbf{x} + k\mathbf{l}b). \tag{4.32}$$

The second condition in (4.30) results in the equality $[\mathbf{l}\times\nabla\varphi] = 0$, whence we find $\varphi(\mathbf{x}) = \varphi(0) + p\mathbf{lx}$. Because the last relation in (4.30) implies that the phase function is periodic, $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + b\mathbf{l})$, we find the values of the parameter $p = 2\pi/b$ from its expansion in the Fourier series

$$\varphi(\mathbf{x}) = \sum_k \varphi_k e^{\mathrm{i}kp\mathbf{lx}} = \sum_k \varphi_k e^{\mathrm{i}kp\mathbf{l}(\mathbf{x}+b\mathbf{l})}$$

(where k is a natural number). Hence, the spatial dependence of the function $\varphi(\mathbf{x})$ is $\varphi(\mathbf{x}) = \sum_k \varphi_k e^{2\pi\mathrm{i}k\mathbf{lx}/b}$.

C-smectic. The spatial symmetry conditions for the C-smectic type of liquid crystal are

$$[\widehat{w}, m_i\hat{L}_i] = 0, \qquad \varepsilon_{ikl}l_k[\widehat{w}, \widehat{P}_l] = 0, \qquad e^{\mathrm{i}\widehat{P}_k l_k b}\widehat{w}e^{-\mathrm{i}\widehat{P}_k l_k b} = \widehat{w}, \tag{4.33}$$

where the vectors $\mathbf{l}$ and $\mathbf{m}$ satisfy the relations $l^2 = m^2 = 1$ and $\mathbf{lm} = \cos\theta$. In every plane, the long axes of molecules (the optical axis $\mathbf{m}$) are inclined by the angle θ with respect to the crystal axis $\mathbf{l}$. Symmetry conditions (4.33) result in the equilibrium structure of the order parameter of the form

$$Q_{uv}(\mathbf{x}, Y, \mathbf{l}, \mathbf{m}) + Q_{uv}\big(\varphi(\mathbf{x}), Y\big)\left(m_u m_v - \frac{1}{3}\delta_{uv}\right), \qquad \varphi(\mathbf{x}) = \sum_k \varphi_k e^{2\pi\mathrm{i}k\mathbf{lx}/b}.$$

We can analogously consider discotic liquid crystals.

5. Conclusions

The proposed statistical approach for classifying equilibriums of condensed media with broken symmetries does not contain any model assumption about the free energy form and does not require the temperature to be close to the phase transition point. We used the method for classifying degenerate states of condensed media to describe quadrupole magnetics with vector and quadrupole order parameters [19], [20]. We obtained the analytic solution of the classification problem for equilibriums of superfluid systems: superfluid ^{3}He [5], a superfluid medium with d-pairing [21], and a solution of Fermi liquids with the vector order parameter [22].

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