EIKONAL APPROXIMATION IN THE TRANSITION RADIATION THEORY

N.F. Shul'ga , and V.V. Syshchenko

Simple variant of eikonal approximation in transition radiation theory is proposed. Spectral-angular density of radiation on fiber-like targets and nanotubes is calculated using Born and eikonal approximations. The conditions of validity of both approximations are considered.

1. INTRODUCTION

Transition radiation (TR) arises, in general case, when a charged particle moves in a medium with changing dielectric properties [1-3]. It was demonstrated in [4] that characteristics of TR (spectral-angular density and polarization) strongly depend on the details of the target geometry. However, precise calculations of TR characteristics could be carried out only in the cases of the simplest geometry, like the normal incidence of the particle onto the plane border between substances with different dielectric properties. To calculate spectral-angular density of TR in more complex situations we need some approximate methods.

We use the system of units in which the velocity of light $c = 1$.

2. BORN APPROXIMATION IN TRANSITION RADIATION THEORY

The spectral-angular density of TR could be written in a form [5]

$$\frac{dE}{d\omega\, do} = \frac{\omega^2}{(8\pi^2)^2}\left|\mathbf{k}\times\mathbf{I}\right|^2, \qquad (1)$$

where ω and $\mathbf{k}$ are the frequency and wave vector of radiated wave, and

$$\mathbf{I} = \int d^3r\, e^{-i\mathbf{kr}}(1-\varepsilon_\omega(\mathbf{r}))\mathbf{E}_\omega(\mathbf{r})\,. \qquad (2)$$

Here ε_ω is the dielectric permittivity of the target substance (we assume that it tends to unit on large distances from the target), and $\mathbf{E}_\omega$ is time Fourier component of the electric field produced in the target substance by the passing particle. For small radiation angles we can write (1) in a form

$$\frac{dE}{d\omega\, do} \approx \frac{\omega^4}{(8\pi^2)^2}\left|\mathbf{I}_\perp\right|^2, \qquad (3)$$

where $\mathbf{I}_\perp$ is the component of $\mathbf{I}$ perpendicular to the particle velocity $\mathbf{v}$.

For high frequencies the dielectric permittivity could be written in plasma form:

$$\varepsilon_\omega = 1-\omega_p^2/\omega^2, \qquad (4)$$

where ω_p is the plasma frequency. If the value $(1 - \varepsilon_\omega)$ in (2) could be treated as small perturbation, we can substitute into (2) non-disturbed Coulomb field of the moving particle,

$$\mathbf{E}_\omega^{(C)}(\mathbf{r})_\perp = \frac{2e\omega}{v^2\gamma} e^{i\frac{\omega}{v}z}\frac{\boldsymbol{\rho}}{\rho}K_1\left(\frac{\omega\rho}{v\gamma}\right) \qquad (5)$$

(where ρ and z are the longitudinal and transverse components of radius-vector $\mathbf{r}$, γ is Lorentz-factor of the particle, $K_1(x)$ is the modified Bessel function of the third kind), instead of complete electric field $\mathbf{E}_\omega$ in the target substance. This approach is analogous to Born approximation in quantum scattering theory (see, e.g., [6]).

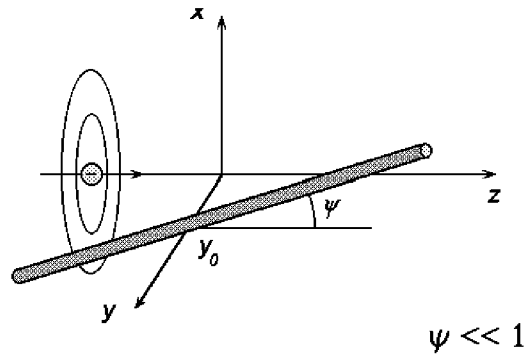

Fig. 1. *Incidence of a particle onto fiber-like target*

Born approximation gives the possibility to consider TR on targets with rather complex geometry, like dielectric fibers and nanotubes [5,7]. Let the ultrarelativistic electron incident onto the fiber-like target under small angle ψ to its axis (Fig. 1). The radiation efficiency (that is the spectral-angular density of radiation integrated over impact parameters) could be written in this case in the form

$$\frac{dK}{d\omega\, do} = L\psi\int dy_0 \frac{dE}{d\omega\, do} = \frac{Le^6 n_e^2\gamma}{m^2\omega\psi}F(\theta,\varphi)\,, \qquad (6)$$

where L is the total length of the fiber, n_e is the number of electrons per unit length of the fiber, and F is a function that determines the angular distribution of radiation. This function is plotted in Fig. 2 for different fibers with the same characteristic radius R. We can see that the intensity and angular distribution of TR substantially depend on the details of the fiber structure. This fact creates new possibilities in diagnostics of nanostructures.

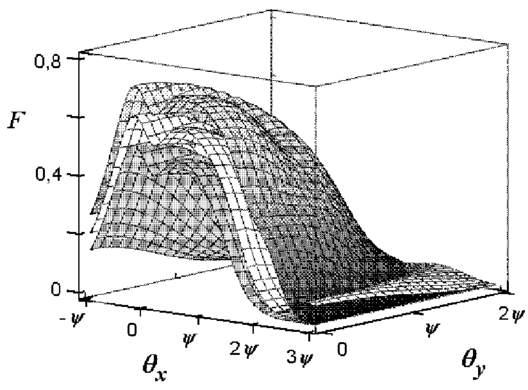

Fig. 2. *The function $F(\theta,\varphi)$ for uniform cylindrical fiber (upper plot), fiber with Gaussian distribution of the electron density in the plane perpendicular to the fiber axis (lower plot) and nanotube (middle plot) with the same characteristic radius, $R\omega/\gamma = 0.2\psi\gamma$. Here $\theta_x = \theta\cos\varphi$, $\theta_y = \theta\sin\varphi$, direction of the particle's velocity corresponds to $\theta_x = \theta_y = 0$, direction of the fiber's axis corresponds to $\theta_x = \psi$, $\theta_y = 0$; $\psi = 10^{-3}$, $\gamma = 2000$*

However, Born approximation in TR theory is valid only in the range of very high frequencies of radiated waves (that will be discussed below). So, we need to develop another approximate method with wider range of validity for radiation frequencies. One of possible ways to do this is to use the eikonal approximation.

3. EIKONAL APPROXIMATION IN TRANSITION RADIATION THEORY

Previously different authors had proposed their variants of eikonal approximation in TR theory. Particularly, M.L. Ter-Mikaelyan had done that in his well-known monograph [2]. However, his method substantially used one-dimensional character of the problem considered.

Another variant, based on the construction of Green function for Maxwell equations with charge and current of the moving particle, was developed by Alikhanyan and Chechin in [8]. However, their method leads to very awkward calculations in application to problems with complex geometry of the target.

In the present article we propose the variant of eikonal approximation based on the method of equivalent photons (see, e.g., [9]). In this approach we approximate the Coulomb field of the incident particle (5) by the packet of plane waves that permits us to describe the evolution of the field in target substance using free Maxwell equations without particle charge and current:

$$\left(\Delta + \omega^2\right)\mathbf{E}_\omega = \operatorname{grad}\operatorname{div}\mathbf{E}_\omega + \omega^2\left(1-\varepsilon_\omega\right)\mathbf{E}_\omega . \qquad (7)$$

According to the idea of eikonal approximation (see, e.g., [6]), we shall find the solution in the form of plane wave multiplied by the slowly changing function,

$$\mathbf{E}_\omega(\mathbf{r}) = e^{i\omega z}\,\Phi(\mathbf{r}), \qquad (8)$$

with the following boundary condition under $z \to -\infty$ (that is before interaction with the target):

$$\mathbf{E}^{(0)}_\omega(\mathbf{r})_\perp = \frac{2e\omega}{v^2\gamma} e^{i\omega z} \frac{\boldsymbol{\rho}}{\rho} K_1\left(\frac{\omega\rho}{v\gamma}\right).$$

As a result, we obtain the following expression for vector $\mathbf{I}_\perp$:

$$\mathbf{I}_\perp = i\frac{4e}{v^2\gamma}\int d^2\rho\; e^{-i\mathbf{k}_\perp \boldsymbol{\rho}} \frac{\boldsymbol{\rho}}{\rho} K_1\left(\frac{\omega\rho}{v\gamma}\right) \times\left\{\exp\left[-i\frac{\omega}{2}\int_{-\infty}^{\infty}\left(1-\varepsilon_\omega(\mathbf{r})\right)dz\right]-1\right\}. \qquad (9)$$

Let us compare the ranges of validity of both developed approximations in TR theory. All conditions [10] are summarized in Table 1. We see that eikonal results would be valid in more soft range of the spectrum, then Born ones (condition (I)), and have no restriction on the target thickness l along the particle trajectory (condition (II)). On the other hand, validity of the eikonal approximation needs the target with rather smoothly changing dielectric properties in transverse direction (condition (III), where ρ_{eff} is the effective transverse distance on which dielectric properties of the target are changing substantially).

Table 1. *Conditions of validity of approximations in TR theory*

Condition	Born approximation	Eikonal approximation
(I)	*Hard range of radiation spectrum:* $\frac{\omega_p^2}{\omega^2} \ll \gamma^{-2}$	*More soft range of the spectrum:* $\left(\theta^2,\gamma^{-2}\right) \ll \frac{\omega_p^2}{\omega^2} \ll 1$
(II)	*Thin target:* $\omega\frac{\omega_p^2}{\omega^2} l \ll 1$	*No restriction*
(III)	*No restriction*	$\rho_{eff} \gg \omega_p^{-1}$
(IV)	*No restriction*	*Small radiation angles* $\omega\frac{\theta^2}{2} l \ll 1$

For our problem of TR on the fiber-like target these conditions take the form presented in Table 2.

Consider now two limiting cases. At first, let us find the range of parameters in which the results of Born and eikonal approximations coincide. It could be demonstrated that for this goal we must expand the condition (II) to the eikonal formulae, the condition (IV) to the Born ones and also apply the additional condition (V) (see Table 3).

Table 2. *Conditions of validity of approximations for radiation on fiber-like target*

Condition	Born approximation	Eikonal approximation
(I)	$\frac{\omega_p^2}{\omega^2} \ll \gamma^{-2}$	$\left(\theta^2,\gamma^{-2}\right) \ll \frac{\omega_p^2}{\omega^2} \ll 1$
(II)	$\omega\frac{\omega_p^2}{\omega^2}\frac{2R}{\psi} \ll 1$	*No restriction*
(III)	*No restriction*	$R\omega_p \gg 1$
(IV)	*No restriction*	$\omega\theta^2\frac{R}{\psi} \ll 1$

Table 3. *Ranges of parameters in which both approximations give the same results*

Condition	Born approximation	Eikonal approximation
(I)	$\frac{\omega_p^2}{\omega^2} << \gamma^{-2}$	$\left(\theta^2, \gamma^{-2}\right) << \frac{\omega_p^2}{\omega^2} << 1$
(II)	$\omega \frac{\omega_p^2}{\omega^2} \frac{2R}{\psi} << 1$	~~No restriction~~
(III)	*No restriction*	$R\omega_p >> 1$
(IV)	~~No restriction~~	$\omega\theta^2 \frac{R}{\psi} << 1$ and $\omega\gamma^{-2}\frac{R}{\psi} << 1$
(V)	$\psi\gamma >> 1$	

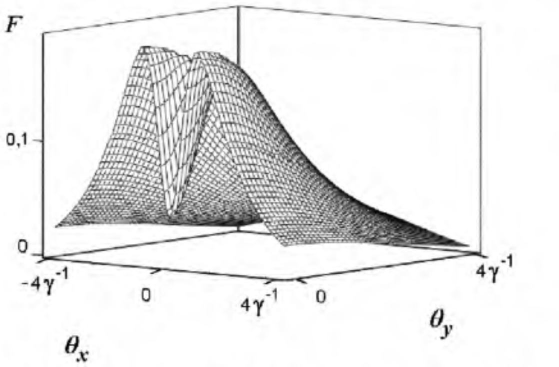

Fig. 3. *The function F for the uniform cylindrical fiber in the case when Born and eikonal approximations give the same results: $R\omega/\gamma = 0.015\psi\gamma$, $\psi = 0.1$, $\gamma = 2000$, $\hbar\omega_p = 20\ eV$ (for example, $\hbar\omega = 13\ keV$, $R = 10^{-5}\ cm$)*

In this range of parameters the angular distribution of TR on the uniform cylindrical fiber will look like it is pictured on Fig. 3. We see that the angular distribution has the form of empty cone that is characteristic to the problem on TR under normal incidence of the particle onto flat dielectric plate. However, the angular distribution in our case do not possess axial symmetry around the particle velocity. This fact means that in the given range of parameters TR just carries some information about the target geometry.

Another interesting limiting case is the case of very thick fiber, when its radius is much larger than the characteristic transverse dimension of the relativistically compressed Coulomb field of the incident particle, $R >> \gamma / \omega$ (see Fig. 4).

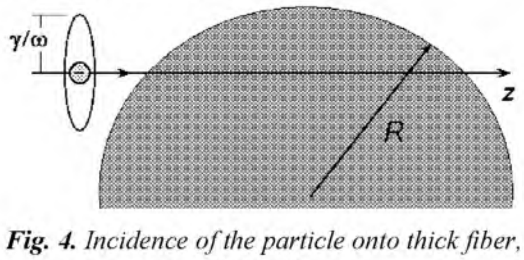

Fig. 4. *Incidence of the particle onto thick fiber, $R >> \gamma / \omega$*

In this case we obtain from (9)

$$\mathbf{I}_\perp = \frac{8\pi e}{v\omega} \frac{\mathbf{k}_\perp}{k_\perp^2 + (\omega/\gamma)^2} \times \left\{ \exp\left[-i \frac{\omega}{2} \frac{\omega_p^2}{\omega^2} \frac{2}{\psi} \sqrt{R^2 - y_0^2} \right] - 1 \right\}.$$

This result could be simply interpreted when we compare it to the corresponding formula for the case of normal incidence of the particle onto flat dielectric plate of thickness a:

$$\mathbf{I}_\perp = \frac{8\pi e}{v\omega} \frac{\mathbf{k}_\perp}{k_\perp^2 + (\omega/\gamma)^2} \left\{ \exp\left[-i \frac{\omega}{2} \frac{\omega_p^2}{\omega^2} a \right] - 1 \right\}.$$

We can see that very thick fiber looks for the incident particle like a plate with local thickness $\frac{2}{\psi}\sqrt{R^2 - y_0^2}$.

Hence, in the limit of thick fiber the angular distribution of TR has a shape like in the case of flat plate (Fig. 5). The details of the fiber structure could make influence only to the absolute value of radiation intensity, not to the shape of angular distribution.

4. CONCLUSION

The simple variant of eikonal approximation in TR theory based on the method of equivalent photons is developed.

The results of eikonal and Born approximations coincide in a rather wide range of parameters. This fact permits to expand result obtained using Born approximation to more soft range of the radiation spectrum.

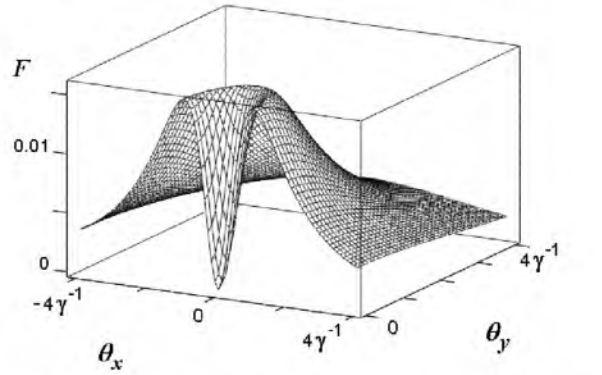

Fig. 5. *The function F for the uniform cylindrical fiber in the case $R\omega/\gamma = 0.15\psi\gamma$, $\psi = 0.1$, $\gamma = 2000$*

Spectral-angular distribution of TR is sensitive to the details of the target geometry, that could be used in diagnostics of nanostructures.

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REFERENCES

1. V.L. Ginzburg, V.N. Tsytovich. *Transition Radiation and Transition Scattering.* Bristol, New York: Adam Hilger Publ., 1990, 360 p.

2. M.L. Ter-Mikaelyan. *High-Energy Electromagnetic Processes in Condensed Media.* New York: “Wiley”, 1972, 457 p.
3. G.M. Garibian, C. Yang. *X-ray Transition Radiation.* Yerevan: Publ. Acad. of Science of Armenia, 1983, 320 p. (in Russian).
4. N.F. Shul’ga, S.N. Dobrovol’skii. Theory of transition radiation from relativistic electron in thin metal target //*JETP*. 2000, v. 90, p. 579-583.
5. N.F. Shul’ga, V.V. Syshchenko. Polarization of transition radiation on some sorts of targets //*Problems of Atomic Science and Technology.* 2001, N6(1), p. 137-137.
6. R.G. Newton. *Scattering theory of waves and particles.* New York: “McGraw-Hill Book Company”, 1968, 608 p.
7. N.F. Shul’ga, V.V. Syshchenko. Transition radiation of high energy particles on fiber-like targets //*Phys. Lett. A*. 2003, v. 313, p. 307-311.
8. A.I. Alikhanyan, V.A. Chechin. Application of eikonal approximation in X-ray transition radiation theory //*Proc. P.N. Lebedev Phys. Inst.* 1982, v. 140, p. 146-158 (in Russian).
9. J.D. Jackson. *Classical Electrodynamics.* New York: “John Wiley & Sons”, 1998.
10. N.F. Shul’ga, V.V. Syshchenko. Transition radiation by relativistic electrons in inhomogeneous substance //H. Wiedemann (ed.), *Advanced Radiation Sources and Applications.* Dordrecht: “Springer”, 2006, p. 129-148.

ЭЙКОНАЛЬНОЕ ПРИБЛИЖЕНИЕ В ТЕОРИИ ПЕРЕХОДНОГО ИЗЛУЧЕНИЯ

Н.Ф. Шульга, В.В. Сыщенко

Предложен простой вариант эйконального приближения в теории переходного излучения. Рассчитана спектрально-угловая плотность излучения на нитевидных мишенях и нанотрубках с использованием борновского и эйконального приближений. Рассмотрены условия применимости обоих приближений.

ЕЙКОНАЛЬНЕ НАБЛИЖЕННЯ В ТЕОРІЇ ПЕРЕХІДНОГО ВИПРОМІНЮВАННЯ

М.Ф. Шульга, В.В. Сищенко

Запропоновано простий варіант ейконального наближення в теорії перехідного випромінювання. Розраховано спектрально-кутову густину випромінювання на нитковидних мішенях та нанотрубках з використанням борнівського та ейконального наближень. Розглянуто умови застосовності обох наближень.