# The Mean-Value Theorem for Elliptic Operators on Stratified Sets

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**Abstract**—In this paper, an analog of the mean-value theorem for harmonic functions is proved for an elliptic operator on the stratified set of “stratified” spheres whose radius is sufficiently small. In contrast to the classical case, the statement of the theorem has the form of a special differential relationship between the mean values over different parts of the sphere. The result is used to prove the strong maximum principle.

Key words: *Green’s formula, mean-value theorem, stratified sets, harmonic and subharmonic functions, strong maximum principle.*

The strong maximum principle for elliptic inequalities $\nabla(p\nabla u) - qu \geq 0$ on an arbitrary stratified set is not established in the most general form at present. Its proof in the case where the dimension of the strata is less than or equal to two is considered in [1], [2]. But the proof is rather tedious even in this case, although it is based on the well-known construction belonging to O. A. Oleinik and E. Hopf. The difficulties arise mainly due to the complicated geometry of stratified sets. This leads one to apply the barrier functions method in order to solve the problem.

In this paper, we consider a narrower (but meaningful) class of stratified sets. More precisely, we suppose that all strata are “flat”. Moreover, the coefficient $p$ is considered as a so-called stratified constant. Under these assumptions, the mean-value theorem for solutions of the equation $\nabla(p\nabla u) = 0$ can be established. Although this analog of the mean-value theorem is more complicated than the classical one, it provides, in some cases, simple proofs of the strong maximum principle without constraints on the dimension of the stratified set.

## 1. BASIC DEFINITIONS

The reader can find the definition of a stratified set adapted to applications to differential equations on this set as well as definitions of other related notions, for example, in [3]–[5]. Hence we restrict ourselves here to a short review of the basic definitions needed for a consistent exposition.

By a *stratified set* we mean a connected subset $\Omega$ in the Euclidean space $\mathbb{R}^n$ consisting of a finite number of smooth manifolds (strata) $\sigma_{ki}$. In the notation $\sigma_{ki}$, the first index, as usual, indicates the dimension of the strata $\sigma_{ki}$ and the second one comes from the autonomous numbering of $k$-dimensional strata. We impose the following two conditions:

(1) the boundary $\partial\sigma_{ki} = \overline{\sigma}_{ki} \setminus \sigma_{ki}$ of the stratum $\sigma_{ki}$ is the union of some strata $\sigma_{mi}$ with $m < k$;

(2) if $X \in \sigma_{ki}$ and $\sigma_{k+1,j_1}, \dots, \sigma_{k+1,j_m}$ are all the $(k+1)$-dimensional strata adjacent to $\sigma_{ki}$, then there exists a neighborhood $U_X \subset \mathbb{R}^n$ and a diffeomorphism $\varphi$ of this neighborhood such that the image of the intersection $U_X$ with the union of the strata $\sigma_{ki}$ with all adjacent strata $\sigma_{k+1,j}$ is the union of a part of the $k$-dimensional plane (which is the image of $\sigma_{ki} \cap U_X$) and parts of the $(k+1)$-dimensional subspaces (which are the images of $\sigma_{k+1,j_\nu} \cap U_X$). An example is given in Fig. 1.

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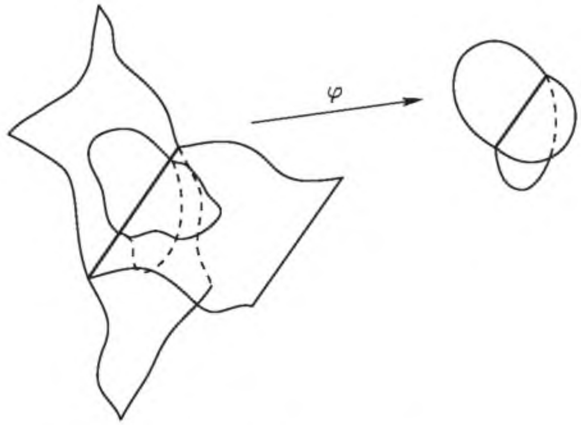


**Fig. 1.** Local straightening.

More precisely, a stratified set is a triple $(\Omega, \Sigma, \Phi)$, where $\Omega$ is the original connected subset of $\mathbb{R}^n$, $\Sigma$ is the collection of strata, and $\Phi$ is the "gluing" map which reconstructs the subset $\Omega$ from the system of strata $\Sigma$.

Condition (1) is a standard assumption (see, for example, [6]) imposed on objects like cellular complexes. In the theory of stratified sets, two conditions due to Whitney are usually imposed instead of condition (2) (see, for example, [7]). These conditions admit a wide class of stratified sets, which is excessive for our purposes; just some of our results are applicable under the Whitney conditions.

Let us split the set $\Omega$ into two parts: $\Omega_0$ and $\partial\Omega_0$. For $\Omega_0$ one can take any open connected subset $\Omega$ (in the topology induced on $\Omega$ from $\mathbb{R}^n$) which is the union of its strata and satisfies $\overline{\Omega}_0 = \Omega$. In the topology induced on $\Omega$, the difference $\Omega \setminus \Omega_0$ is the topological boundary of $\Omega_0$; therefore, we denote it by $\partial\Omega_0$. In the general case, it can happen that, formally, $\Omega_0 = \Omega$, i.e. $\partial\Omega_0 = \varnothing$. In this case, the set of harmonic functions on $\Omega_0$, which will be defined in the next section, consists of constants (see [5]). In order to avoid this trivial case, we will further assume that $\partial\Omega_0 \neq \varnothing$.

Define a measure $\mu$ on $\Omega$:

$$\mu(\omega) = \sum_{\sigma_{ki}} \mu_k(\omega \cap \sigma_{ki}), \tag{1}$$

where $\mu_k$ is a $k$-dimensional Lebesgue measure on the strata $\sigma_{ki}$ and $\omega$ is a measurable set; it is obvious that *measurable* sets $\omega \subset \Omega$ are those whose intersections with all strata are Lebesgue-measurable.

The Lebesgue integral of a measurable function $f\colon \Omega \to \mathbb{R}$ with respect to such a measure reduces to the sum of the Lebesgue integrals over separate strata

$$\int_\Omega f\, d\mu = \sum_{\sigma_{ki}} \int_{\sigma_{ki}} f\, d\mu_k.$$

Further, we omit the measures $d\mu$ and $d\mu_k$ in the notation of all integrals.

The vector field $\vec{F}$ on $\Omega_0$ is said to be *tangent* to $\Omega_0$ if, for any stratum $\sigma_{ki} \subset \Omega_0$ and any point $X \in \sigma_{ki}$, the vector $\vec{F}(X)$ belongs to the space tangent to $\sigma_{ki}$ at the point $X$ of the space $T_X\sigma_{ki}$; for zero-dimensional strata, it is natural to put $\vec{F} = 0$.

Suppose that $X \in \sigma_{k-1i}$. The divergence of $\vec{F}(X)$ is defined by the relation

$$(\nabla\vec{F})(X) = (\nabla_{k-1}\vec{F})(X) + \sum_{\sigma_{kj} \succ \sigma_{k-1i}} \vec{F} \cdot \vec{\nu}\big|_{\overline{kj}}(X), \tag{2}$$

where $\nabla_{k-1}$ is the classical divergence operator (the first Laplace–Beltrami differential parameter) on $\sigma_{k-1,i}$, and the relationship $\sigma_{kj} \succ \sigma_{k-1,i}$ means that the strata $\sigma_{k-1,i}$ and $\sigma_{kj}$ are adjacent; $\vec{\nu}$ is the unit normal to $\sigma_{k-1,i}$ at the point $X$ oriented to the interior of the stratum $\sigma_{kj}$. Further, for example, for a function $u\colon \Omega \to \mathbb{R}$, a notation of the type $u\big|_{\overline{kj}}(X)$ $(X \in \sigma_{k-1,i} \succ \sigma_{kj})$ stands for the extension by continuity to the point $X$ of the restriction $u\big|_{\sigma kj}$ to $\sigma_{kj}$ of the function $u$. Of course, we assume that such an extension exists; this assumption holds, for example, when the above restriction is a function uniformly continuous on $\sigma_{kj}$. If $u$ is not continuous on $\Omega$ in the large, then, generally speaking, $u\big|_{\overline{kj}}(X) \neq u(X)$. One can prove that the divergence defined by formula (2) is the density of the flow of the vector field $\vec{F}$ with respect to the measure defined by formula (1).

Formula (2) implies that, for the existence of the divergence of the vector filed $\vec{F}$, it suffices to suppose that the field $\vec{F}$ belongs to the class $C^1$ on each stratum $\sigma_{ki}$ (this yields the existence of the classical part of the divergence) and to require the existence of the previously described extensions (in order to define the "nonclassical" part of the divergence). The class of such fields will be denoted by $C^1(\Omega_0)$.

Suppose that the function $u\colon \Omega_0 \to \mathbb{R}$ is continuously differentiable in the interior of each stratum. Then one can consider the vector field $\nabla u$ whose restrictions to each strata $\sigma_{ki} \subset \Omega_0$ coincide with the field of the gradient $\nabla_k u$. Note that we do not suppose any relationships between the restrictions of the function $u$ to separate strata; hence the gradient $\nabla u$ is, in fact, a collection of independent vector fields. If $\nabla u$ belongs to the class $C^1(\Omega_0)$, then one can consider the operator $\Delta u = \nabla(\nabla u)$; one can naturally call it the *Laplace operator*. As usual, we use the same symbol $\nabla$ for the gradient and for the divergence. We will keep in mind that the divergence acts on the vector field and the gradient acts on scalar functions. Further, we suppose that $u$ is continuous on $\Omega_0$. If, moreover, $\nabla u \in C^1(\Omega_0)$, then we write $u \in C^2(\Omega_0)$.

## 2. THE MEAN-VALUE THEOREM

In this section, we suppose that each stratum $\sigma_{ki}$ is a $k$-dimensional polyhedron, although some results from this section remain valid in the general case. Consider the operator $\Delta_p$, acting by the formula

$$\Delta_p u = \nabla(p\nabla u),$$

where the function $p\colon \Omega_0 \to \mathbb{R}$ can possess discontinuities on $\Omega_0$, but it its restrictions to each stratum are constants; we call such a function a *stratified constant*. It can be easily seen that if $\vec{F} \in C^1(\Omega_0)$, then $p\vec{F} \in C^1(\Omega_0)$, and hence the expression $\Delta_p u$ is meaningful for any function $u \in C^2(\Omega_0)$.

The explicit expression for $\Delta_p u$ at the point $X \in \sigma_{k-1,i}$ by (2) has the following form:

$$\Delta_p u(X) = \nabla_{k-1}(p\nabla_{k-1}u)(X) + \sum_{\sigma_{kj}\succ\sigma_{k-1,i}} p\nabla u \cdot \vec{\nu}\big|_{\overline{kj}}(X).$$

Those strata which are not adjacent to strata of higher dimension (such as strata of maximal dimension) are called *free*. Further, we assume that the coefficient $p$ is positive on such strata (generally, $p \geq 0$). It is obvious that, on free strata, the operator $\Delta_p u$ reduces to the standard $(k-1)$-dimensional Laplace operator: $\Delta_p u = p\Delta_{k-1}u$. It is also obvious that the union of the closures of free strata coincides with $\Omega$.

Suppose that $X_0 \in \sigma_{ki} \subset \Omega_0$, and $r > 0$ is less than the distance to all other strata (with the exception of $\sigma_{ki}$) whose dimension is less than or equal to $k$, and $B_r(X_0)$ is the ball in the space $\mathbb{R}^n$. The intersection $\Omega_0 \cap \partial B_r(X_0) = S_r(X_0)$ is called a *stratified sphere* in $\Omega_0$, or simply a *sphere*. An example of such a sphere is shown in Fig. 2.

The sphere $S_r$ (when we consider an arbitrary sphere, we do not specify its center) can be considered as a stratified set; more precisely, we declare that intersections of the form $S_r \cap \sigma_{ki}$ are the $(k-1)$-dimensional strata of $S_r$. The union of all $k$-dimensional strata of the sphere $S_r$ will be denoted by $S_r^k$. The measure on a stratified sphere is defined (as above) in the same way as on an arbitrary stratified set. All integrations below over spheres involve this measure.

The following analog of the second Green formula is proved in [5]: for arbitrary sufficiently smooth functions $u$ and $v$ (for example, $C^2(\Omega_0)$-smoothness in the interior of $\Omega_0$ is sufficient), the following formula holds:

$$\int_{\Omega_0} (v\Delta_p u - u\Delta_p v) = \int_{\partial\Omega_0} \big(u(p\nabla v)_\nu - v(p\nabla u)_\nu\big), \tag{3}$$

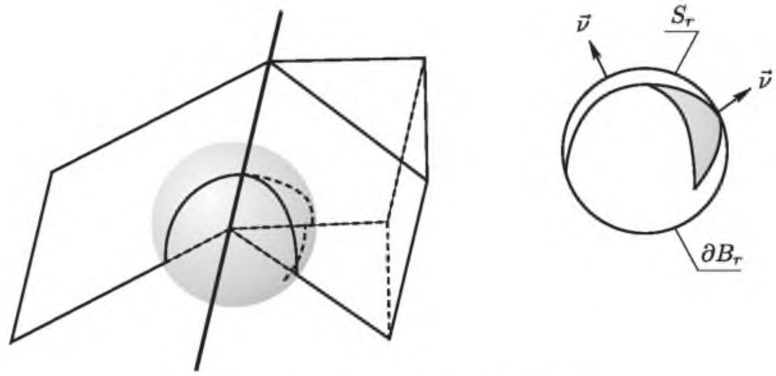


**Fig. 2.** Example of a stratified sphere.

where $\vec{\nu}$ is defined as above on each stratum from $\partial\Omega_0$. By setting $v \equiv 1$ in this formula for a function $u \in C^2(\Omega_0)$ and an arbitrary sphere $S_r$, we obtain the identity

$$\int_{\partial S_r} (p\nabla u)_\nu = 0, \tag{4}$$

where $\vec{\nu}$ is the normal to $S_r$, and $(p\nabla u)_\nu$ in this case (unlike the general case considered in [5]) is equal to

$$(p\nabla u)_\nu = p\nabla u \cdot \vec{\nu} = p\frac{\partial u}{\partial \nu}.$$

Equation (4) implies an analog of the mean-value theorem for harmonic functions. To obtain it, we need the following assertion. Further, it will be convenient to assume that $\vec{\nu}$ is an outer normal.

**Lemma 1.** *Suppose that $u \in C^1(\Omega_0)$. Then, for the outer normal $\vec{\nu}$ to $S_r(X)$, we have*

$$\frac{d}{dr} M_{S_r^k}(u) = \frac{1}{|S_r^k|_p} \int_{S_r^k(X)} (p\nabla u)_\nu \, d\mu_r, \tag{5}$$

*where*

$$|S_r^k|_p = \int_{S_r^k} p \, d\mu_r,$$

*$M_{S_r^k}(u)$ is the weighted mean value of the function $u$ and*

$$M_{S_r^k}(u) = \frac{1}{|S_r^k|_p} \int_{S_r^k} pu \, d\mu_r.$$

**Proof.** By definition, we have

$$M_{S_{r+\Delta r}^k(X)}(u) = \frac{1}{|S_{r+\Delta r}^k(X)|_p} \int_{S_{r+\Delta r}^k} pu \, d\mu_{r+\Delta r}.$$

The transfer of integration from the sphere of radius $r + \Delta r$ to the sphere of radius $r$ and the identity $p \, d\mu_{r+\Delta r} = p(1 + \Delta r/r)^{(k-1)} \, d\mu_r$, yield the following identity:

$$M_{S_{r+\Delta r}^k(X)}(u) = \frac{1}{|S_r^k|_p} \int_{S_r^k} pu\left(Z + Z \cdot \frac{\Delta r}{r}\right) d\mu_r,$$

where $Z \in S_r^k$. Hence, for the increment $\Delta M_{S_r^k(X)}(u)$, we have

$$\Delta M_{S_r^k}(u) = \frac{1}{|S_r^k|_p} \int_{S_r^k} p\left(u\left(Z + Z \cdot \frac{\Delta r}{r}\right) - u(Z)\right) d\mu_r.$$

By using the finite-difference formula in the integral, we obtain the expansion

$$\Delta M_{S_r^k}(u) = \frac{\Delta r}{|S_r^k|_p} \int_{S_r^k} p\nabla u(Z) \cdot \vec{\nu}\, d\mu_r + o(\Delta r)$$

which proves the statement of the lemma. □

It worth noting that formula (5) can be rewritten in the following form:

$$\alpha_k r^k \frac{d}{dr} M_{S_r^k}(u) = \int_{S_r^k} p\nabla u(Z) \cdot \vec{\nu}\, d\mu_r,$$

where $\alpha_k$ is independent of the radius. An explicit expression for $\alpha_k$ is now obvious. Note that if $p = 0$ on all $k$-dimensional strata intersecting the considered sphere, then $\alpha_k = 0$. The summation of all these formulas over all $k$ from $k = 0$ to $k = d - 1$ yields

$$\sum_{k=0}^{d-1} \alpha_k r^k \frac{d}{dr} M_{S_r^k}(u) = \int_{S_r} p\nabla u(Z) \cdot \vec{\nu}\, d\mu_r = \int_{S_r} (p\nabla u)_\nu\, d\mu_r. \tag{6}$$

Using formula (4) for the solution of the equation

$$\Delta_p u = 0, \tag{7}$$

we obtain an analog (in differential form) of the classical mean-value theorem (a special case of such a statement was also considered in [8]).

**Theorem 1.** *Suppose that $u$ is a solution of Eq. (7). Then, for any $X \in \Omega_0$ and any stratified sphere, the following relation holds:*

$$\sum_{k=0}^{d-1} \alpha_k r^k \frac{d}{dr} M_{S_r^k(X)}(u) = 0. \tag{8}$$

Note that this formula holds for any stratified constant, i.e., $p$ can vanish on strata of any dimension (but not on free strata). In particular, Theorem 1 allows one to deduce the mean-value theorem in the standard integral form (not as a statement concerning the local differential form) for the case in which $p$ differs from zero only on free strata of the same dimension $d$.

Indeed, in this case, all the summands in Eq. (8), with the exception of the summand corresponding to $k = d - 1$, vanish and we have (for brevity, we again skip the notation of the center of a sphere)

$$\alpha_k r^{d-1} \frac{d}{dr} M_{S_r^{d-1}}(u) = 0.$$

This proves that $M_{S_r^{d-1}}(u) = \text{const}$. But, in this case, $M_{S_r^{d-1}}(u) = M_{S_r}(u)$. It now remains to pass to the limit as $r \to 0$ to obtain the desired result. In particular, the classical mean-value theorem for harmonic functions follows from our proof.

## 3. APPLICATIONS OF MEAN-VALUE THEOREM

In this section, we consider applications of the mean-value theorem related to the maximum principle. In the case considered below, the solution of the inequality $\Delta_p u \geq 0$ can possess points of local maximum on $\Omega_0$ without being a constant on $\Omega_0$. However, such a solution cannot possess any so-called points of nontrivial maximum; a point $X_0 \in \Omega_0$ is called a *point of nontrivial maximum* if the inequality $u(X) \leq u(X_0)$ holds in the neighborhood of the point $X_0$, but $u$ is not a constant function in any neighborhood of this point.

**Lemma 2.** *Suppose that $u \in C^2(\Omega_0)$, $X$ is a point of nontrivial local maximum of the function $u$, and $u(X) = 0$. Further, suppose that*

$$\lim_{r\to 0} \frac{1}{r^{d-1}} \int_{S_r(X)} pu\, d\mu = 0. \tag{9}$$

*Then there exists an arbitrarily small positive $r$ such that*

$$\sum_{k=0}^{d-1} \alpha_k r^k \frac{d}{dr} M_{S_r^k(X)}(u) < 0. \tag{10}$$

**Proof.** In the obvious formula

$$\sum_{k=0}^{d-1} \alpha_k r^k \frac{d}{dr} M_{S_r^k}(u) = \frac{d}{dr}\Bigl(\sum_{k=0}^{d-1} \alpha_k r^k M_{S_r^k}(u)\Bigr) - \sum_{k=1}^{d-1} k\alpha_k r^{k-1} M_{S_r^k}(u), \tag{11}$$

the expression in parentheses is equal to

$$\int_{S_r} pu\, d\mu = \phi(r).$$

The assumptions of the lemma imply that $u(Y) \le 0$ in a sufficiently small neighborhood of the point $X$; therefore, $M_{S_r^k} \le 0$. Hence Eq. (11) justifies the following estimate:

$$\begin{aligned}\sum_{k=0}^{d-1} \alpha_k r^k \frac{d}{dr} M_{S_r^k}(u) &\le \frac{d}{dr}\Bigl(\sum_{k=0}^{d-1} \alpha_k r^k M_{S_r^k}(u)\Bigr) - \frac{d-1}{r} \sum_{k=0}^{d-1} \alpha_k r^k M_{S_r^k}(u) \\ &= \phi'(r) - \frac{d-1}{r}\phi(r) = r^{d-1}\Bigl(\frac{\phi(r)}{r^{d-1}}\Bigr)'. \end{aligned} \tag{12}$$

Suppose that, in spite of the statement of the lemma, the sum on the left-hand side of this inequality is nonnegative for all sufficiently small $r$. Then the function

$$\psi(r) = \frac{\phi(r)}{r^{d-1}}$$

is nondecreasing. By assumption,

$$\lim_{r\to 0} \psi(r) = 0.$$

Hence $\psi(r) \ge 0$ for all sufficiently small $r$. But then

$$\phi(r) = \int_{S_r} pu\, d\mu \ge 0.$$

On the other hand, it is obvious that this integral is nonpositive. Therefore,

$$\int_{S_r} pu\, d\mu = 0$$

for all sufficiently small $r$. This yields $u = 0$, at least on those parts $S_r^k$ of the sphere $S_r$ which are the intersections of this sphere with free strata (recall that $p > 0$ in these parts). But then $u = 0$ on the entire sphere $S_r$, because the function $u$ is continuous and the union of the closures of the above parts $S_r^k$ coincides with $S_r$. Since this is valid for all sufficiently small $r$, then $u = 0$ in some neighborhood of the point $X$; this leads to a contradiction with the existence of a nontrivial maximum at the point $X$. □

As an illustration to this assertion, consider a simple proof of the strong maximum principle for a solution of the inequality $\Delta_p u \ge 0$ in some special cases. The first of these cases will be described in the next theorem; it clearly contains the classical strong maximum principle for subharmonic functions.

**Theorem 2.** *Suppose that $u \in C^2(\Omega_0)$ is a solution of the inequality $\Delta_p u \geq 0$. Suppose that $p > 0$ only on free strata of the same dimension. Then the function $u$ has no points of nontrivial local maximum in $\Omega$.*

**Proof.** Assume the contrary: $X_0$ is a point of nontrivial local maximum. Without loss of generality, one can assume that $u(X_0) = 0$ (in the opposite case, instead of $u$, one can take the function $u - u(X_0)$). Green's formula (3) and the inequality $\Delta_p u \geq 0$ readily imply that

$$\int_{S_r} (p\nabla u)_\nu = \int_{S_r} p\,\frac{\partial u}{\partial \nu} \geq 0 \tag{13}$$

for any admissible $r > 0$.

Suppose that $d$ is the dimension of the free strata in $\Omega_0$. Since $p = 0$ on all strata from $\Omega_0$ whose dimension is less than $d$, we have

$$\frac{\phi}{r^{d-1}} = \frac{\int_{S_r(X)} pu}{r^{d-1}} = \frac{\int_{S_r^{d-1}(X)} pu}{r^{d-1}},$$

and since $u$ is continuous and $u(X) = 0$, the integral in the numerator of the last fraction is of order $o(r^{d-1})$. According to Lemma 2 and equality (6), there exists an arbitrarily small $r > 0$ such that

$$\int_{S_r} p\,\frac{\partial u}{\partial \nu} < 0.$$

This contradicts Eq. (13). □

In the general case, it is difficult to verify condition (9). However, for $d = 2$ (where $d$ is the maximal dimension of strata from $\Omega_0$), Lemma 2 allows one to give a simple proof of the strong maximum principle. More precisely, the following assertion holds.

**Theorem 3.** *Suppose that the maximal dimension of the strata from in $\Omega_0$ is equal to two. Suppose that the coefficient $p$ is positive at least on all free strata. Then the solution of the inequality $\Delta_p u \geq 0$ can have no nontrivial local maximum in $\Omega_0$.*

**Proof.** Suppose that the converse statement holds. Let $X_0$ be a point of local nontrivial maximum of the function $u$. Without loss of generality, we will suppose that $u(X_0) = 0$. First, we consider the case in which $X_0$ coincides with one of the zero-dimensional strata; let it be $\sigma_{0m}$. Further, we suppose that all $\sigma_{1i_1}, \ldots, \sigma_{1i_l}$ are one-dimensional strata adjacent to $\sigma_{0m}$. Denote by $X_1, \ldots, X_l$ the intersection points of these strata with $\partial B_r(X_0)$. The set of these points forms exactly the part $S_r^0$ of the sphere $S_r = \partial B_r(X_0) \cap \Omega_0$. Let us prove that

$$\frac{1}{r}\int_{S_r} pu \to 0 \tag{14}$$

as $r \to 0$.

By the definition of the integral, we have

$$\frac{1}{r}\int_{S_r} pu = \frac{1}{r}\int_{S_r^0} pu + \frac{1}{r}\int_{S_r^1} pu = \frac{1}{r}\sum_{i=1}^{l}(pu)(X_i) + \frac{1}{r}\int_{S_r^1} pu.$$

Using the same arguments as in the proof of the previous lemma, one can easily show that the last integral in the above equality converges to zero as $r \to 0$. Moreover,

$$\frac{1}{r}\sum_{i=1}^{l}(pu)(X_i) = \frac{1}{r}\sum_{i=1}^{l} p\big(u(X_i) - u(X_0)\big).$$

Now it is obvious that

$$\frac{1}{r}\int_{S_r^0} pu \to \sum_{\sigma_{1i_k}\succ\sigma_{0m}} p\nabla u\cdot\vec{\nu}\big|_{\overline{1i_k}}(X_0) = \Delta_p u \geq 0 \tag{15}$$

as $r \to 0$. On the other hand,

$$\frac{1}{r}\int_{S_r^0} pu \leq 0. \tag{16}$$

Comparing Eqs. (15) and (16), we see that

$$\frac{1}{r}\lim_{r\to 0} pu = 0.$$

Hence (14) is proved. Now, Lemma 2 implies that there exists an arbitrarily small $r > 0$ such that

$$\int_{S_r} p\,\frac{\partial u}{\partial \nu} < 0.$$

However, according to what has been said, the inequality $\Delta_p u \geq 0$ yields Eq. (13), and hence we obtain a contradiction.

The case in which $X_0$ belongs to one or two-dimensional strata can be treated in a simpler way. Moreover, when $X_0$ belongs to two-dimensional strata, the situation is classical ($u$ is harmonic in these strata) and no proof is needed. □

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