On the Index of the Dirichlet Problem for Elliptic Systems on the Plane

A. P. Soldatov
Belgorod State University, Belgorod, Russia

1. STATEMENT OF THE PROBLEM

For the elliptic system

$$\frac{\partial^2 u}{\partial y^2} - A_1 \frac{\partial^2 u}{\partial x\, \partial y} - A_0 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

with constant matrix coefficients $A_0, A_1 \in \mathbb{R}^{l\times l}$ in a bounded domain D, we consider the Dirichlet problem

$$u|_{\partial D} = f. \tag{2}$$

This problem was analyzed by Bitsadze [1] for domains D bounded by a smooth contour Γ. He also singled out the class of so-called weakly coupled systems (1); the Dirichlet problem for system (1) has the Fredholm property if and only if the system belongs to that class. In this case, the index of the problem proves to be zero. In the general case of second-order elliptic systems with continuous coefficients, these studies were continued in [2–5] and other papers.

To system (1), one naturally assigns the quadratic matrix pencil

$$P(z) = z^2 - A_1 z - A_0$$

and the characteristic polynomial $\det P(z)$ of degree $2l$. The ellipticity condition means that the roots of this polynomial do not lie on the real axis. Let σ be the set of roots of this polynomial in the upper half-plane. Denoting the multiplicity of a root $\nu \in \sigma$ by λ_ν, one can write

$$\det P(z) = \prod_{\nu\in\sigma} (z-\nu)^{l_\nu} (z-\bar{\nu})^{l_\nu}, \qquad \sum_{\nu\in\sigma} l_\nu = l.$$

By [6], the weak coupling condition is equivalent to the invertibility of the matrix $\int_{-\infty}^{\infty} P^{-1}(t)dt$. Another equivalent definition of weak coupling was also given there. For any elliptic system (1), there exist two matrices $B, J \in \mathbb{C}^{l\times l}$ such that

$$\det \begin{pmatrix} B & \bar{B} \\ BJ & \overline{BJ} \end{pmatrix} \neq 0, \qquad \det(z-J) = \prod_{\nu\in\sigma} (z-\nu)^{l_\nu}, \tag{3}$$

and the matrix relation

$$BJ^2 - A_1 BJ - A_0 B = 0 \tag{4}$$

is valid. Here the matrix J is defined modulo similarity and can also be chosen in Jordan form.

In the present paper, we assume that system (1) is weakly coupled and consider the Dirichlet problem in a domain D bounded by a piecewise smooth contour Γ. Numerous studies (e.g., see [7–10], where a detailed bibliography is given) deal with elliptic problems in piecewise smooth domains on the plane. The forthcoming considerations will be carried out in modified Hölder weighted spaces. Recall the definition [11] of these spaces.

On Γ, we choose a finite set F containing all corners of the contour and consider the family $\lambda = (\lambda_\tau,\ \tau \in F)$ of real numbers. By $C^\mu_\lambda\left(\bar{D}, F\right)$, $0 < \mu < 1$, we denote the space of all continuous functions $\varphi(z)$ on $\bar{D}\backslash F$ such that

$$\varphi_*(z) = \varphi(z) \prod_{\tau \in F} |z-\tau|^{\mu-\lambda_\tau} \in C^\mu\left(\bar{D}\right), \qquad \varphi_*(\tau) = 0, \qquad \tau \in F.$$

This is a Banach space with respect to the transferred norm; it is a weighted space in the following sense: the space C^μ_0 consists of bounded functions, and the multiplication by the weight function $\varrho(z) = \prod_F |z-\tau|^{\lambda_\tau}$ is an isomorphism of C^μ_0 onto C^μ_λ. By definition, if $\lambda = \mu$, then the space C^μ_μ consists of functions $\varphi \in C^\mu\left(\bar{D}\right)$ vanishing on F. The subspace C^μ_λ of continuously differentiable functions φ on $\bar{D}\backslash F$ whose partial derivatives φ_x and φ_y belong to $C^\mu_{\lambda-1}$ will be denoted by $C^{1,\mu}_\lambda$.

Further, we introduce the extension of $C^\mu_{(\lambda)}\left(\bar{D}, F\right) \supseteq C^\mu_\lambda\left(\bar{D}, F\right)$ by polynomials $p(z)$ in the complex variable z. Obviously, this extension is finite-dimensional; consequently, $C^\mu_{(\lambda)}\left(\bar{D}, F\right)$ is a Banach space with respect to the corresponding norm. The space $C^{1,\mu}_{(\lambda)}$ has a similar meaning relative to $C^{1,\mu}_\lambda$. The functions $\varphi \in C^\mu_{(\lambda)}\left(\bar{D}, F\right)$ are continuous on $\bar{D}\backslash\left\{\tau \in F,\ \lambda_\tau \le 0\right\}$ and behave as $O(1)|z-\tau|^{\lambda_\tau}$ as $z \to \tau$ if $\lambda_\tau \le 0$ and as $\varphi(\tau) + O(1)|z-\tau|^{\lambda_\tau}$ if $\lambda_\tau > 0$. Moreover, one can readily show that if $\lambda < 1$, then any function $\varphi \in C^1\left(\bar{D}\backslash F\right)$ whose derivatives are $O(1)|z-\tau|^{\lambda_\tau - 1}$ as $z \to \tau$ belongs to $C^\mu_{(\lambda)}\left(\bar{D}, F\right)$. Moreover, we have the inclusions

$$C^\mu\left(\bar{D}\right) \subseteq C^\mu_{(\lambda)}\left(\bar{D}, F\right), \qquad \lambda \le \mu, \qquad C^{1,\mu}\left(\bar{D}\right) \subseteq C^{1,\mu}_{(\lambda)}\left(\bar{D}, F\right), \qquad \lambda \le 1+\mu,$$

which become strict equalities for $\lambda = \mu$ and $\lambda = 1+\mu$, respectively.

In a similar way, one can define all these spaces on a piecewise smooth contour Γ. Obviously, the restriction operator $\varphi \to \varphi|_\Gamma$ is bounded: $C^\mu_{(\lambda)}\left(\bar{D}, F\right) \to C^\mu_{(\lambda)}(\Gamma, F)$, and similarly for the spaces $C^\mu_{(\lambda)}$. However, if we wish to have the same assertion for $C^{1,\mu}$, then the arcs forming the contour Γ should be subjected to an additional Lyapunov condition. More precisely, each arc $\Gamma_0 \subseteq \Gamma$ not containing points $\tau \in F$ as interior points should belong to the class $C^{1,\mu}$. As a result, for the pair (Γ, F), we obtain the class denoted by $C^{1,\mu}$. For curves Γ of this class, the restriction operator is bounded in the spaces $C^{1,\mu}_\lambda$ and $C^{1,\mu}_{(\lambda)}$.

Throughout the following, we consider the Dirichlet problem in the space $C^\mu_{(\lambda)}\left(\bar{D}, F\right)$ for $0 < |\lambda| < 1$ (i.e., for $0 < |\lambda_\tau| < 1$ for all $\tau \in F$) under the assumption that Γ does not contain cusps and the pair (Γ, F) belongs to the class $C^{1,\mu+0}$ (i.e., $C^{1,\mu+\varepsilon}$ for some $\varepsilon > 0$).

2. REDUCTION TO THE RIEMANN–HILBERT PROBLEM

Consider the first-order elliptic system

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0 \tag{5}$$

with the matrix J in (3), (4), whose solutions are referred to as *J-analytic* or *hyperanalytic* functions [12]. The main results of the theory of analytic functions remain valid for the solutions of this system; in this case, the matrix

$$(\alpha + i\beta)_J = \alpha 1 + \beta J \tag{6}$$

plays the role of the complex variable $\alpha + i\beta$, where 1 stands for the identity matrix. For example, the Taylor series expansion has the form

$$\phi(z) = \sum_{k=0}^{\infty} (z-z_0)^k_J\, \phi^{(k)}(z_0), \qquad \phi^{(k)} = \frac{\partial^k \phi}{\partial x^k},$$

and the Cauchy integral is given by the relation

$$(I\varphi)(z) = \frac{1}{\pi i}\int_{\Gamma}(t-z)_J^{-1}dt_J\varphi(t), \tag{7}$$

where the contour Γ has a positive orientation with respect to D and, by analogy with (6), we set

$$(d\alpha + id\beta)_J = 1d\alpha + Jd\beta.$$

In this notation, any J-analytic function $\phi \in C\left(\bar{D}\right)$ in D can be represented by the Cauchy formula $2\phi = I\varphi^+$. This formula remains valid for functions $\phi \in C_\lambda^\mu\left(\bar{D}, F\right)$, $-1 < \lambda < 0$. In this case, the integral operator $I : C_\lambda^\mu\left(\bar{D}, F\right) \to C^\mu(\Gamma, F)$ is bounded, and we have the Sokhotskii–Plemelj formula

$$(I\varphi)^+(t_0) = \varphi(t_0) + (K\varphi)(t_0), \qquad t_0 \in \Gamma\backslash F. \tag{8}$$

Here $(K\varphi)(t_0)$ is a singular integral defined by analogy with (7) for $z = t_0$.

For the elliptic system (1), the solutions of Eq. (5), where J occurs in condition (3), play the same role as the analytic functions for the Laplace equation. This role is expressed by the representation

$$u = \operatorname{Re} B\phi \tag{9}$$

of the general solution of system (1). The fact that the function (9) is a solution of system (1) readily follows from (4). Here the derivative ϕ' is uniquely determined by u modulo a constant term $\eta \in \mathbb{C}^l$. This formula can be inverted by differentiation with the use of the relations $2u_x = B\phi' + \overline{B\phi'}$ and $2u_y = BJ\phi' + \overline{BJ\phi'}$, which follow from (7). Then, in view of (3),

$$\phi' = 2\left(C_1\frac{\partial u}{\partial x} + C_2\frac{\partial u}{\partial y}\right), \qquad \begin{pmatrix} C_1 & C_2 \\ \bar{C}_1 & \bar{C}_2 \end{pmatrix} = \begin{pmatrix} B & \bar{B} \\ BJ & \overline{BJ} \end{pmatrix}^{-1}. \tag{10}$$

In particular, ϕ' is a single-valued function. However, the function ϕ itself, given by the curvilinear integral

$$\phi(z) = \phi(z_0) + \int_{z_0}^{z} dt_J\left(C_1u_x + C_2u_y\right)(t),$$

is not necessarily single-valued in a multiply connected domain. In general, the nature of its multi-valuedness depends on the connectedness degree of the domain D, that is, on the number s of connected components of the contour $\Gamma = \partial D$. In an implicit form, the corresponding result [12] can be stated as follows.

To be definite, we assume that $z = 0 \in D$. Then there exist $l \times l$ matrices of functions $u_j \in C^\infty\left(\bar{D}\right)$, $1 \leq j \leq s$, whose columns satisfy Eq. (1) such that any solution of this equation can be uniquely represented in the form

$$u = \operatorname{Re} B\phi + \sum_{j=1}^{s} u_j\xi_j, \qquad \phi(0) = 0, \tag{11}$$

with some $\xi_j \in \mathbb{R}^l$. The solution of the Dirichlet problem (1), (2) is constructed in the class of functions of this form, where $\phi \in C_{(\lambda)}^\mu\left(\bar{D}, F\right)$, $0 < |\lambda| < 1$.

By m_λ we denote the number of points $\tau \in F$ such that $\lambda_\tau > 0$; the case in which $\lambda < 0$ is not excluded and corresponds to $m_\lambda = 0$. One can readily see that any J-analytic function $\phi \in C_{(\lambda)}^\mu\left(\bar{D}, F\right)$ can be uniquely represented in the form

$$\phi = \phi_0 + \sum_{j=1}^{m_\lambda} z_J^j\eta_j, \qquad \phi_0(0) = 0,$$

with some $\eta_j \in \mathbb{C}^l$. The substitution of this expression into (11) provides a similar representation in the class $\phi \in C^\mu_\lambda$ with the only difference that s should be replaced by $s+m_\lambda$. In turn, the substitution of the last representation into condition (2) reduces the Dirichlet problem to the equivalent boundary value problem

$$\operatorname{Re} B\phi^+ + \sum_{j=1}^{s+m_\lambda} b_j\xi_j = f, \qquad \phi(0)=0, \tag{12}$$

for a J-analytic function $\phi \in C^\mu_\lambda\left(\bar{D},F\right)$ and $\xi_j \in \mathbb{R}^l$, where we have set $b_j = u_j|_\Gamma$. Since $u_j \in C^1\left(\bar{D}\right)$ and the contour Γ contains no cusps, it follows that the matrix functions b_j satisfy the Lipschitz condition on Γ and hence belong to the class $C^\mu_{(\lambda)}(\Gamma,F)$, $\lambda<1$. It is also obvious that the multiplication operator $\varphi \to b_j\varphi$ is bounded in $C^\mu_{(\lambda)}$.

We choose a scalar function $\chi_\tau \in C^\infty$ that is identically equal to unity in a neighborhood of τ and vanishes in a neighborhood of $F\backslash\tau$. Then the operator

$$Pf = f - \sum_{\tau,\ \lambda_\tau>0} \chi_\tau f(\tau)$$

is the projection of the space $C^\mu_{(\lambda)}$ onto C^μ_λ, and problem (12) can be represented in the form of the system

$$\operatorname{Re} B\phi^+ + \sum_{j=1}^{s+m_\lambda} (Pb_j)\,\xi_j = Pf, \qquad \phi(0)=0, \qquad \sum_{j=1}^{s+m_\lambda} b_j(\tau)\xi_j = f(\tau), \qquad \lambda_\tau>0. \tag{13}$$

3. MAIN RESULTS

The operator of system (13) acts in the spaces $C^\mu_{(\lambda)}\left(\bar{D},F\right)\times\left(\mathbb{R}^l\right)^{s+m_\lambda} \to C^\mu_{(\lambda)}(\Gamma,F)\times\left(\mathbb{R}^l\right)^{m_\lambda}$; i.e., the system is a finite-dimensional extension of the Riemann–Hilbert problem

$$\operatorname{Re} B\phi^+ = f^0 \tag{13^0}$$

in the class of J-analytic functions $\phi \in C^\mu_\lambda$. The latter problem was studied in detail in [12], and the results can be restated for (13), i.e., for the original Dirichlet problem (1), (2).

For a sufficiently small $r>0$, the domain $D_\tau = D\cap\{|z-\tau|<r\}$ is a curvilinear sector with vertex τ, whose opening angle is denoted by θ_τ. It is assumed that the vertex τ is not a cusp of the contour Γ, which is expressed by the inequality $0<\theta_\tau<2\pi$. The curve $\Gamma_\tau = \Gamma\cap\{|z-\tau|<r\}$ consists of two smooth arcs $\Gamma_{\tau\pm0}$, which are the lateral sides of the sector D_τ. The notation is introduced so as to ensure that the passage from $\Gamma_{\tau-0}$ to $\Gamma_{\tau+0}$ through the vertex τ is performed in the positive sense of the contour Γ, i.e., in the sense for which the domain D lies to the left. Let $q_{\tau\pm0}\in\mathbb{C}$ be the unit tangent vector of the arc $\Gamma_{\tau\pm0}$ at the point τ directed from τ to the other endpoint of this arc, so that

$$q_\tau = q_{\tau-0}q_{\tau+0}^{-1} = e^{i\theta_\tau}. \tag{14}$$

In accordance with notation (6), consider the matrix

$$Q_\tau = q_{\tau-0,J}q_{\tau+0,J}^{-1}; \tag{15}$$

by (3), its eigenvalues are the complex numbers

$$q_{\tau(\nu)} = q_{\tau-0,\nu}q_{\tau+0,\nu}^{-1} = \varrho_{\tau(\nu)}e^{i\theta_{\tau(\nu)}}, \qquad \nu\in\sigma, \tag{16}$$

of multiplicity l_ν. Here, by analogy with (6), $(\alpha+i\beta)_\nu$ is understood as the complex number $\alpha+\nu\beta$.

Since the points $\nu\in\sigma$ lie in the upper half-plane, it follows that the affine transformation $z\to z_\nu$ is invariant in this half-plane and leaves the points of the real axis fixed. In particular, it does not change the orientation of the plane. Therefore, by analogy with (14), the quantity $\theta_{\tau(\nu)}$ is the opening angle of the curvilinear sector $D_{\tau(\nu)}$ onto which D_τ is mapped under this transformation. Note that the relations $0 < \theta_{\tau(\nu)} < \pi$, $\theta_{\tau(\nu)} = \pi$, and $\pi < \theta_{\tau(\nu)} < 2\pi$ are equivalent to $0 < \theta_\tau < \pi$, $\theta_\tau = \pi$, and $\pi < \theta_\tau < 2\pi$, respectively. One can readily justify this fact by varying ν along the segment joining ν with the imaginary unit i.

To the matrix Q_τ, we assign the complex powers $Q_\tau^\zeta = e^{\zeta \ln Q_\tau}$ and $\bar{Q}_\tau^\zeta = e^{\overline{\zeta \ln Q_\tau}}$, which are treated in the sense of functions of matrices. The bar on the right-hand side of the second relation stands for complex conjugation. By using these powers, we introduce the family of entire matrix functions

$$X_\tau(\zeta) = \zeta^{-1}\left[Q_\tau^\zeta\left(B^{-1}\bar{B}\right) - \left(B^{-1}\bar{B}\right)\bar{Q}_\tau^\zeta\right], \qquad X_\tau^0(\zeta) = \zeta^{-1}\left(Q_\tau^\zeta - \bar{Q}_\tau^\zeta\right). \tag{17}$$

By (14), the eigenvalues of the matrix $X_\tau^0(\zeta)$ are $\zeta^{-1}\left(q_{\tau(\nu)}^\zeta - \bar{q}_{\tau(\nu)}^\zeta\right)$; they have the multiplicities l_ν, $\nu \in \sigma$. Therefore,

$$\det X_\tau^0(\zeta) = \zeta^{-l}\prod_{\nu\in\sigma}\left(q_{\tau(\nu)}^\zeta - \bar{q}_{\tau(\nu)}^\zeta\right)^{l_\nu} = (2i)^l\prod_{\nu\in\sigma}\left[\frac{\sin\theta_{\tau(\nu)}\zeta}{\zeta}\right]^{l_\nu}. \tag{18}$$

In particular, $\det X_\tau^0(\zeta) \neq 0$ in the strip $|\operatorname{Re}\zeta| \leq \delta$ for sufficiently small $\delta > 0$.

In each strip $\alpha_1 \leq \operatorname{Re}\zeta \leq \alpha_2$, we have $X_\tau(\zeta)\left(X_\tau^0\right)^{-1}(\zeta) \to B^{-1}\bar{B}$ uniformly with respect to $\operatorname{Re}\zeta$ as $\operatorname{Im}\zeta \to \infty$. Therefore, in this strip, the determinant $\det X_\tau(\zeta)$ has finitely many zeros. The number of these zeros counting multiplicities in the open strip lying between the lines $\operatorname{Re}\zeta = 0$ and $\operatorname{Re}\zeta = \alpha$, $\alpha \neq 0$, is denoted by $n_\tau(\alpha)$. The number n_τ^0 has a similar meaning with respect to the line $\operatorname{Re}\zeta = 0$. Next, we take a number $\delta > 0$ small enough to ensure that $n_\tau(\delta) = n_\tau(-\delta) = 0$ for all τ and introduce the integers

$$\Delta_\tau^\pm = \frac{1}{2\pi}\arg\left(\frac{\det X_\tau}{\det X_\tau^0}\right)\Bigg|_{\pm\delta - i\infty}^{\pm\delta + i\infty}. \tag{19}$$

Note that

$$\Delta_\tau^+ - \Delta_\tau^- = n_\tau^0 \tag{20}$$

by the Rouché theorem.

Theorem 1. *The Dirichlet problem* (1), (2) *has the Fredholm property in the class* $C_{(\lambda)}^\mu$, $0 < |\lambda| < 1$, *if and only if*

$$\det X_\tau(\zeta) \neq 0, \qquad \operatorname{Re}\zeta = \lambda_\tau, \qquad \tau \in F, \tag{21}$$

and its index is given by the formula

$$\varkappa = -\sum_{\tau\in F}\left[\Delta_\tau^\pm \pm n_\tau\left(\lambda_\tau\right)\right], \tag{22}$$

where the sign $\pm$ *coincides with the sign* $\operatorname{sgn}\lambda_\tau$.

Proof. Obviously, the operators of problems (13) and (13^0) are Fredholm equivalent, and their indices $\varkappa$ and $\varkappa^0$ are related by the formula

$$\varkappa = \varkappa^0 + (s-2)l, \tag{23}$$

where we have used the condition $\phi(0) = 0$ in problem (13).

On the other hand, a criterion for the Fredholm property of problem (13^0) and a formula for its index were given in [12]. Let m be the number of elements of the set F. By [12], the end symbol of problem (13^0) is the block diagonal $m \times m$ matrix with diagonal blocks

$$X(\zeta;\tau) = \begin{pmatrix} B & \bar{B} \\ BQ_\tau^\zeta & \bar{B}\,\bar{Q}_\tau^\zeta \end{pmatrix} \tag{24}$$

considered on the lines $\operatorname{Re}\zeta = \lambda_\tau$. The matrix $X^0(\zeta;\tau)$ has the same meaning with B replaced by the identity matrix. In this notation, the criterion for the Fredholm property of problem (13^0) in the space C^μ_λ is given by the inequality

$$\det X(\zeta;\tau) \neq 0, \qquad \operatorname{Re}\zeta = \lambda_\tau, \qquad \tau \in F, \tag{25}$$

and its index $\varkappa^0(\lambda)$ for $1/2 \le \lambda < 0$ is given by the formula

$$\varkappa^0(\lambda) = -\sum_\tau \frac{1}{2\pi i} \ln \frac{\det X(\zeta;\tau)}{\det X^0(\zeta;\tau)}\bigg|_{\lambda_\tau - i\infty}^{\lambda_\tau + i\infty} + l(2-s). \tag{26}$$

In the general case of an arbitrary weight order λ, this formula should be supplemented by the relation

$$\varkappa^0(\lambda') - \varkappa^0(\lambda'') = \sum_\tau k_\tau, \tag{27}$$

where k_τ is the number of zeros of the function $\det X(\zeta;\tau)$ in the strip between the lines $\operatorname{Re}\zeta = \lambda'_\tau$ and $\operatorname{Re}\zeta = \lambda''_\tau$ with the plus sign if $\lambda'_\tau \le \lambda''_\tau$ and with the minus sign otherwise.

It follows from the comparison of the matrices (17) and (24) that

$$\begin{pmatrix} B^{-1} & 0 \\ Q^\zeta_\tau B^{-1} & -B^{-1} \end{pmatrix} X(\zeta;\tau) = \begin{pmatrix} 1 & B^{-1}\bar{B} \\ 0 & \zeta X_\tau(\zeta) \end{pmatrix}. \tag{28}$$

By setting $B = 1$ here, one can write out a similar relation for the matrices $X^0(\zeta;\tau_0)$ and $X^0_\tau(\zeta)$. Therefore, conditions (21) and (25) are equivalent, and formula (26) with $\lambda = -\delta$ in notation (19) can be represented in the form

$$\varkappa^0(-\delta) = -\sum_\tau \Delta^-_\tau + l(2-s).$$

By the same argument, relation (27) applied to the weight orders $\lambda' = \lambda$ and $\lambda'' = -\delta$ gives

$$\varkappa^0(\lambda) - \varkappa^0(-\delta) = \sum_\tau k_\tau, \qquad k_\tau = \begin{cases} n_\tau(\lambda_\tau) & \text{if } \lambda_\tau < 0 \\ -n_\tau(\lambda_\tau) - n^0_\tau & \text{if } \lambda_\tau > 0. \end{cases}$$

By combining the last two relations with (20) and (23), we obtain (22), which completes the proof of the theorem.

Note that the integers (19) and hence the index (22) of the Dirichlet problem can be evaluated in a more explicit form if the function $\zeta^l \det X_\tau(\zeta)$ is even. Then, by (18), the ratio $\det X/\det X^0$ is also even; therefore, the quantities (19) have opposite signs. This, together with (20), implies that

$$\Delta^\pm_\tau = \pm\frac{1}{2} n^0_\tau. \tag{29}$$

It follows from the same considerations that $n_\tau(\alpha) = -n_\tau(-\alpha)$; therefore, by (22), the indices of the Dirichlet problem in the classes $C^\mu_{(\pm\lambda)}$ have opposite signs.

Just as in [12], we supplement Theorem 1 with the corresponding result on the smoothness and asymptotics of the solution of the Dirichlet problem.

Theorem 2. *If, under the assumptions of Theorem* 1, *the function* $u \in C^\mu_{(\lambda)}(\bar{D}, F)$ *is a solution of the Dirichlet problem with right-hand side* $f \in C^{1,\mu}_{(\lambda)}(\Gamma, F)$, *then* $u \in C^{1,\mu}_{(\lambda)}(\bar{D}, F)$.

Proof. As was mentioned above, the matrix functions b_j belong to the class $C^{1,\mu}_{(\lambda)}(\Gamma, F)$. Consequently, if $\phi \in C^\mu_{(\lambda)}(\bar{D}, F)$ in problem (13), then the right-hand side $f^0 = Pf - \sum (Pb_j)\,\xi_j$ of problem (13^0) belongs to the class $C^{1,\mu}_{(\lambda)}(\Gamma, F)$. Therefore, it remains to apply the corresponding result [12] on the smoothness of the solution to this problem.

The family of spaces $C^{\mu}_{(\lambda)}$ is monotone decreasing (by inclusion) with respect to each of the parameters μ and λ_τ, $\tau \in F$. Thus one can introduce the classes

$$C^{\mu}_{(\lambda+0)} = \bigcup_{\varepsilon>0} C^{\mu}_{(\lambda+\varepsilon)}, \qquad C^{\mu}_{(\lambda-0)} = \bigcap_{\varepsilon>0} C^{\mu}_{(\lambda-\varepsilon)}.$$

If $\lambda = 0$, then we denote them by $C^{\mu}_{(+0)}$ and $C^{\mu}_{(-0)}$ for brevity. As an example, by analogy with the complex powers Q^{ζ}_{τ} contained in (17), in the curvilinear sector D_τ, we consider the matrix functions $\ln(z-\tau)_J$ and $(z-\tau)^{\zeta_0}_J$. One can readily see that $(z-\tau)^{\zeta_0}_J \left[\ln(z-\tau)_J\right]^j \in C^{\mu}_{(\alpha-0)}$, $\operatorname{Re}\zeta_0 = \alpha$, $j = 0, 1, \ldots$

Let the point ζ_0 be a zero of the function $\det X_\tau(\zeta)$; thus the matrix function $X^{-1}_\tau(\zeta)$ has a pole of some order $r(\zeta_0) \geq 1$ at this point. Note that $r(\zeta_0)$ does not exceed the multiplicity of the zero ζ_0 of the function $\det X_\tau(\zeta)$. To describe the asymptotic behavior of the solution $u(z)$ in the sector D_τ as $z \to \tau$, to each number $\alpha \in \mathbb{R}$, we assign the J-analytic function $\phi_\tau(z;\alpha) \in C^{\mu}_{(\alpha-0)}$ of the form

$$\phi_\tau(z;\alpha) = \sum_{\operatorname{Re}\zeta_0=\alpha} \sum_{j=0}^{r(\zeta_0)-1} (z-\tau)^{\zeta_0}_J \left[\ln(z-\tau)_J\right]^j c_j(\zeta_0) \tag{30}$$

with some coefficients $c_j(\zeta_0) \in \mathbb{C}^l$, where the summation is performed over all zeros ζ_0 of the function $\det X_\tau(\zeta)$ on the line $\operatorname{Re}\zeta_0 = \alpha$. Strictly speaking, we use this expression only for $\alpha \neq 0$. If $\alpha = 0$, then to the points ζ_0, we add the point $\zeta_0 = 0$ and suppose that $r(0) = 0$ for $\det X_\tau(0) \neq 0$. In this case, we set

$$\phi_\tau(z;0) = \sum_{\operatorname{Re}\zeta_0=0,\,\zeta_0\neq 0} \sum_{j=0}^{r(\zeta_0)-1} (z-\tau)^{\zeta_0}_J \left[\ln(z-\tau)_J\right]^j c_j(\zeta_0) + \sum_{j=0}^{r(0)} \left[\ln(z-\tau)_J\right]^j c_j(0). \tag{30^0}$$

Note that $\phi'_\tau(z) \in C^{\mu}_{\alpha-1}(D_\tau, \tau)$ in both cases.

In the following assertion, we no longer assume that condition (21) is satisfied.

Theorem 3. *Let u be a solution of the Dirichlet problem in the class $C^{\mu}_{(\lambda-0)}\left(\bar{D}, F\right)$ [more precisely, the function ϕ occurring in the representation* (12) *belongs to this class*], *and let the right-hand side f of this problem belong to the class $C^{\mu}_{(\lambda_\tau+0)}(\Gamma_\tau, \tau)$ on the lateral boundary Γ_τ of the sector D_τ for some $\tau \in F$. Then there exists a function $\phi_\tau(z;\lambda_\tau)$ of the form* (30) *or* (30^0) *depending on the value of λ_τ such that*

$$u_\tau = u - \operatorname{Re} B\phi_\tau \in C^{\mu}_{\lambda_\tau+0}\left(\bar{D}_\tau, \tau\right). \tag{31}$$

If, in addition, $f \in C^{1,\mu}_{(\lambda_\tau+0)}(\Gamma_\tau, \tau)$, then $u_\tau \in C^{1,\mu}_{(\lambda_\tau+0)}\left(\bar{D}_\tau, \tau\right)$.

Proof. Just as in Theorem 2, we consider problem (13^0) with right-hand side $f^0 \in C^{\mu}_{\lambda_\tau+0}(\Gamma_\tau, \tau)$. For its solution ϕ, one can use the corresponding result [12] on the asymptotics. It is stated similarly to the assertion to be justified with the only difference that $r(\zeta_0)$ in (30) should be replaced by the order of the pole $\tilde{r}(\zeta_0)$ of the function $X^{-1}(\zeta;\tau)$ at the point ζ_0. Here there is no difference between the cases $\alpha \neq 0$ and $\alpha = 0$. It follows from (28) that the quantities $r(\zeta_0)$ and $\tilde{r}(\zeta_0)$ are related by the formula

$$\tilde{r}(\zeta_0) = r(\zeta_0) + (1 - \operatorname{sgn}|\zeta_0|).$$

Thus we obtain all assertion of the theorem.

Corollary. *Let $n^0_\tau = 0$, i.e., $\det X_\tau(\zeta) \neq 0$ for $\operatorname{Re}\zeta = 0$. Then each solution $u \in C^{\mu}_{-0}$ of the Dirichlet problem whose right-hand side satisfies, the condition $f \in C^{\mu}_{(+0)}(\Gamma_\tau, \tau)$ also belongs to the class $C^{\mu}_{(+0)}\left(\bar{D}_\tau, \tau\right)$ in the sector D_τ.*

4. SELF-ADJOINT ELLIPTIC SYSTEMS

We apply the obtained results to the Lagrange self-adjoint elliptic system

$$\sum_{i,j=1}^{2} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \qquad A_{ij}^{\mathrm{T}} = A_{ji}, \tag{32}$$

where $x_1 = x$, $x_2 = y$, and the symbol T stands for matrix transposition. The Dirichlet problem (2) for this system is considered in the classes $C^{\mu}_{-0} = C^{\mu}_{(-0)}$ and $C^{\mu}_{(+0)}$. By Theorems 1 and 2, the class $V^{\pm} \subseteq C^{\mu}_{(\pm 0)}$ of solutions of the homogeneous problem is finite-dimensional and lies in $C^{1,\mu}_{(\pm 0)}$. In particular, if $v \in V^{\pm}$, then the conormal derivative

$$\frac{\partial v}{\partial \nu} = \sum_{i,j=1}^{2} n_i A_{ij}\frac{\partial u}{\partial x_j}, \tag{33}$$

where n_1 and n_2 are the components of the outward normal, belongs to the class $C^{\mu}_{-1\pm 0}(\Gamma, F)$. Therefore, its product by a function in $C^{\mu}_{(\mp 0)}$ belongs to the class $C^{\mu}_{-1+0}(\Gamma, F)$ and hence is integrable. Here and throughout the following, the product uv of two l-vectors is understood as the inner product $u_1v_1 + \cdots + u_lv_l$.

Theorem 4. *Let system* (32) *be weakly coupled, and let relation* (29) *be valid for all* $\tau \in F$. *Then the conditions*

$$\int_{\Gamma} f\frac{\partial v}{\partial \nu} ds = 0, \qquad v \in V^{\mp}, \tag{$34^{\pm}$}$$

are necessary and sufficient for the solvability of the problem in the class $C^{\mu}_{(\pm 0)}$. *Accordingly, the index* $\varkappa^{\pm}$ *of the problem in this class is given by the relation*

$$\varkappa^{\pm} = \dim V^{\pm} - \dim V^{\mp} = \mp\frac{1}{2}\sum\nolimits_{\tau} n^0_{\tau}. \tag{35}$$

Proof. Since system (32) is self-adjoint, we have the identity

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[\left(A_{ij}\frac{\partial u}{\partial x_j}\right)v - u\left(A_{ij}\frac{\partial v}{\partial x_j}\right)\right] = 0$$

for arbitrary solutions u and v. Consequently, if $u, v \in C^{1,\mu}_{(\mp 0)}$, then the bracketed expression in this identity belongs to the class C^{μ}_{-1+0}. Therefore, one can use the Green formula, which implies that

$$\int_{\Gamma}\left[\left(\frac{\partial u}{\partial \nu}\right)v - u\frac{\partial u}{\partial \nu}\right] ds = 0$$

in notation (33). In particular, condition ($34^{\pm}$) is necessary for the solvability of problem (2), (32) in the class $C^{1,\mu}_{(\pm 0)}$. By the density argument, this fact remains valid for the class $C^{\mu}_{(\pm 0)}$.

Further, let us show that the number of linearly independent conditions ($34^{\pm}$) coincides with the dimension $\dim V^{\mp}$. In other words, $v \to \partial v/\partial \nu$ is a linear one-to-one correspondence in the class $V = V^{-}$.

Indeed, let $\partial v/\partial \nu = 0$ on Γ for some function $v \in V$. Then, by (34) and the relation $v|_{\Gamma} = 0$, for the partial derivatives v_x and v_y on Γ, we have the homogeneous system of linear equations

$$(n_1A_{11} + n_2A_{21})\,v_x + (n_1A_{12} + n_2A_{22})\,v_y = 0, \qquad -n_2v_x + n_1v_y = 0. \tag{36}$$

For the coefficient matrix of this system, we have

$$\begin{pmatrix} \sum_i n_i A_{i1} & \sum_i n_i A_{i2} \\ -n_2 & n_1 \end{pmatrix} \begin{pmatrix} n_1 & -n_2 \\ n_2 & n_1 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ 0 & 1 \end{pmatrix},$$

where for brevity, we have set $Q_1 = \sum_{i,j} n_i n_j A_{ij}$ and $Q_2 = \sum_i n_i \left(n_1 A_{i2} - n_2 A_{i1}\right)$. This relation, together with the ellipticity condition $\det Q_1 \neq 0$ for Eq. (32), implies that system (36) has only the zero solution, i.e., $v_x = v_y = 0$ on Γ. For v, we write out the representation (9), where, by (10), the derivative ϕ' vanishes on Γ. It follows from the Cauchy theorem that $\phi' \equiv 0$ in the domain D, and hence $v \equiv 0$.

Thus relations ($34^{\pm}$) define $\dim V^{\mp}$ linearly independent orthogonality conditions. Theorem 1, together with (29), implies that the index $\varkappa^{\pm} = \dim V^{\pm} - k^{\pm}$ of the problem in the class $C^{\mu}_{(\pm)}$ is determined by the right-hand side of (35). In addition, by virtue of the preceding considerations, for the number $k^{\pm}$ of linearly independent solvability conditions for this problem, we have the lower bound $k^{\pm} \geq \dim V^{\mp}$. Therefore, $0 = \varkappa^+ - \varkappa^- = (\dim V^+ - k^-) + (\dim V^- - k^+)$, which is possible only if $k^{\pm} = \dim V^{\mp}$. This readily implies all assertions of the theorem.

The absolute value of the indices (35) coincides with the dimension of the space V^0 in the expansion $V^- = V^0 \oplus V^+$. By Theorem 3, each of the functions in a basis of the space V^0 admits the asymptotics (31) at the points $\tau \in F$ with functions ϕ_τ of the form (30^0) nonzero at least for some τ. Obviously, the doubled dimension of the space V^0 is equal to the sum $\sum_\tau n^0_\tau$.

As was mentioned above, relation (29) is necessarily valid in the case of an even function $\zeta^l \det X_\tau(\zeta)$. In this case, under assumption (25), the indices $\varkappa^{\pm}$ of the problem in the spaces $C^{\mu}_{(\pm\lambda)}$ are also opposite. If, as before, $V^{\pm} \subseteq C^{\mu}_{(\pm\lambda)}$ stands for the classes of solutions of the homogeneous problem, then, by Theorems 2 and 3, $V^+ \subseteq C^{1,\mu}_{(\lambda+0)}$ and $V^- \subseteq C^{1,\mu}_{(\lambda-0)}$, which implies that the solvability conditions ($34^{\pm}$) are meaningful in this case as well. By using the same reasoning, one can show that Theorem 4 is valid also in the spaces $C^{\mu}_{(\pm\lambda)}$.

5. SYSTEMS OF TWO EQUATIONS

If $l = 2$, then the determinant of the matrices X_τ can be evaluated in closed form. We set

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^* = \begin{pmatrix} B_{22} & -B_{12} \\ -B_{21} & B_{11} \end{pmatrix} = (\det B) B^{-1}. \tag{37}$$

In this notation, we have

$$\begin{gathered} B^* \bar{B} = \begin{pmatrix} r & -2ip_2 \\ 2ip_1 & \bar{r} \end{pmatrix}, \\ r = B_{22}\bar{B}_{11} - B_{12}\bar{B}_{21}, \qquad p_j = \operatorname{Im}\left(B_{1j}\bar{B}_{2j}\right), \qquad j = 1, 2, \\ |\det B|^2 = \det B^* \det \bar{B} = |r|^2 - 4p_1 p_2. \end{gathered} \tag{38}$$

By (37), in definition (15) of the matrix X_τ, one can replace B^{-1} by B^*, which is assumed below. Therefore,

$$\zeta X_\tau(\zeta) = Q^\zeta_\tau \left(B^* \bar{B}\right) - \left(B^* \bar{B}\right) \bar{Q}^\zeta_\tau. \tag{39}$$

As was mentioned in Section 1, the matrix J occurring in relations (3) and (4) can be chosen in Jordan form. We have only the following three cases for it:

$$J_I = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \qquad \nu_1 \neq \nu_2; \qquad J_{II} = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}; \qquad J_{III} = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}. \tag{40}$$

In the last case $J = J_{III}$, the matrices Q^ζ and $\bar{Q}^\zeta$ are scalar, and relation (39) can be reduced to the form $X_\tau(\zeta) = \zeta^{-1}\left(Q^\zeta_\tau - \bar{Q}^\zeta_\tau\right)\left(B^* \bar{B}\right) = X^0_\tau(\zeta)\left(B^* \bar{B}\right)$. In particular, the determinants of the matrices X and X^0 differ by a constant factor, whence it follows that $\Delta_\tau^\pm = 0$. Note that, in this case, system (1) can be reduced to a single scalar equation. Therefore, below we restrict our considerations to the first two cases in (40).

First, let $J = J_I$. In this case, the matrix Q_τ occurring in (15) has the diagonal form with diagonal entries (16). For brevity, we set $q_j = q_{\tau(\nu_j)}$, $j = 1, 2$, and in a similar way, we write $q_j = \varrho_j e^{i\theta_j}$. Then $q_j^\zeta = \varrho_j^\zeta e^{i\theta_j\zeta}$, $\bar q_j^\zeta = \varrho_j^\zeta e^{-i\theta_j\zeta}$, $Q_\tau^\zeta = \operatorname{diag}\left(q_1^\zeta, q_2^\zeta\right)$, and $\bar Q_\tau^\zeta = \operatorname{diag}\left(\bar q_1^\zeta, \bar q_2^\zeta\right)$. This, together with (38) and (39), implies that

$$\zeta X_\tau(\zeta) = \begin{pmatrix} r\left(q_1^\zeta - \bar q_1^\zeta\right) & -2ip_2\left(q_1^\zeta - \bar q_2^\zeta\right) \\ 2ip_1\left(q_2^\zeta - \bar q_1^\zeta\right) & \bar r\left(q_2^\zeta - \bar q_2^\zeta\right) \end{pmatrix},$$

$$\begin{aligned} \zeta^2 \det X_\tau(\zeta) &= |r|^2\left(q_1^\zeta - \bar q_1^\zeta\right)\left(q_2^\zeta - \bar q_2^\zeta\right) - 4p_1p_2\left(q_1^\zeta - \bar q_2^\zeta\right)\left(q_2^\zeta - \bar q_1^\zeta\right) \\ &= |\det B|^2\left(q_1^\zeta - \bar q_1^\zeta\right)\left(q_2^\zeta - \bar q_2^\zeta\right) - 4p_1p_2\left(q_1^\zeta \bar q_2^\zeta + q_2^\zeta \bar q_1^\zeta - q_1^\zeta \bar q_1^\zeta - q_2^\zeta \bar q_2^\zeta\right). \end{aligned}$$

A similar expression can be written out for X^0 with $r = \det B = 1$ and $p_j = 0$. After obvious transformations, we have

$$\begin{gathered} -\frac{\zeta^2 \det X_\tau(\zeta)}{4|\det B|^2 \varrho_1^\zeta \varrho_2^\zeta} = \sin\theta_1\zeta \sin\theta_2\zeta - \beta\left[\left(\frac{\varrho_1^\zeta}{\varrho_2^\zeta} - \frac{\varrho_2^\zeta}{\varrho_1^\zeta}\right)^2 + 4\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)\zeta\right], \\ \beta = \frac{p_1p_2}{|\det B|^2} = \frac{\operatorname{Im}\left(B_{11}\bar B_{21}\right)\operatorname{Im}\left(B_{12}\bar B_{22}\right)}{|\det B|^2}. \end{gathered} \tag{41_I}$$

Note that here the quantity $\beta = \beta(B)$ is an invariant of system (1), i.e., is independent of the choice of the matrix B in relations (3) and (4) for $J = J_I$. Indeed, by [13], any other matrix $\tilde B$ of this kind is related to B by the formula $\tilde B_{ij} = \lambda_j B_{ij}$, $1 \le i, j \le 2$, with some $\lambda_j \ne 0$. This implies that $\tilde p_1 \tilde p_2 = |\lambda_1|^2 |\lambda_2|^2 p_1p_2$ and $|\det \tilde B|^2 = |\lambda_1|^2 |\lambda_2|^2 |\det B|^2$, whence it follows that $\beta\left(\tilde B\right) = \beta(B)$. The expression for the determinant of the block matrix occurring in (3) was given in [13]. In the above-introduced notation, we have

$$\frac{1}{2}\det\begin{pmatrix} B & \bar B \\ BJ & \bar B \bar J \end{pmatrix} = |\nu_1 - \nu_2|^2 |\det B|^2 \left(\beta - \frac{\operatorname{Im}\nu_1 \operatorname{Im}\nu_2}{|\nu_1 - \nu_2|^2}\right).$$

Therefore, condition (3) is equivalent to the condition

$$\beta \ne \frac{\operatorname{Im}\nu_1 \operatorname{Im}\nu_2}{|\nu_1 - \nu_2|^2}. \tag{42_I}$$

For a given number $\beta \in \mathbb{R}$ satisfying this condition, one can choose an invertible matrix B related to β by the corresponding formula (41_I). In turn, by [13], this matrix defines a unique elliptic system (1) for which relations (3) and (4) are valid. In closed form, the coefficients A_0 and A_1 of this system can be found from the matrix relation

$$\begin{pmatrix} 0 & 1 \\ A_0 & A_1 \end{pmatrix} = \begin{pmatrix} BJ & \bar B \bar J \\ BJ^2 & \bar B \bar J^2 \end{pmatrix}\begin{pmatrix} B & \bar B \\ BJ & \bar B \bar J \end{pmatrix}^{-1}, \qquad J = J_I.$$

Now consider the case $J = J_{II}$ in (40). In this case, for the matrices $q_{\tau\pm0,J}$ occurring in (15), we have the expression

$$q_J = q_\nu \begin{pmatrix} 1 & q_\nu^{-1}\operatorname{Im} q \\ 0 & 1 \end{pmatrix}.$$

Consequently,

$$Q_\tau = q_{\tau(\nu)} \begin{pmatrix} 1 & \delta_\tau \\ 0 & 1 \end{pmatrix}, \qquad \delta_\tau = \frac{\operatorname{Im} q_{\tau-0}}{q_{\tau-0}} - \frac{\operatorname{Im} q_{\tau+0}}{q_{\tau+0}}.$$

In notation (14), the complex number δ_τ can be represented in the form

$$\delta_\tau = (\sin\theta_\tau)/(q_{\tau-0,\nu})\,(q_{\tau-0,\nu})\,. \tag{43}$$

Therefore,

$$Q_\tau^\zeta = q_{\tau(\nu)}^\zeta \begin{pmatrix} 1 & \delta_\tau\zeta \\ 0 & 1 \end{pmatrix}, \qquad \bar{Q}_\tau^\zeta = \bar{q}_{\tau(\nu)}^\zeta \begin{pmatrix} 1 & \bar{\delta}_\tau\zeta \\ 0 & 1 \end{pmatrix}.$$

This, together with (38) and (39), implies that

$$\zeta X(\zeta) = \begin{pmatrix} r\left(q^\zeta - \bar{q}^\zeta\right) + 2ip_1\delta\zeta q^\zeta & -2ip_2\left(q^\zeta - \bar{q}^\zeta\right) + \zeta\left(\bar{r}\delta q^\zeta - r\bar{\delta}\bar{q}^\zeta\right) \\ 2ip_1\left(q^\zeta - \bar{q}^\zeta\right) & \bar{r}\left(q^\zeta - \bar{q}^\zeta\right) - 2ip_1\bar{\delta}\zeta\bar{q}^\zeta \end{pmatrix},$$

where we have denoted $q = q_{\tau(\nu)}$ and $\delta = \delta_\tau$ for brevity. A similar expression is valid for the matrix X^0 with $r = 1$ and $p_j = 0$ as well. After simple manipulations, we obtain

$$-\frac{\zeta^2 \det X_\tau(\zeta)}{4|\det B|^2\varrho^{2\zeta}} = \sin^2\theta\zeta - \beta|\delta|^2\zeta^2, \qquad \beta = \frac{p_1^2}{|\det B|^2}. \tag{41_{II}}$$

By [13], the two matrices $\tilde{B}$ and B occurring in (3) and (4) are related by the formula $\tilde{B}_{i1} = \lambda_1 B_{i1}$, $\tilde{B}_{i2} = \lambda_2 B_{i1} + \lambda_1 B_{i2}$ with some scalars $\lambda_1 \neq 0$ and λ_2. Therefore, just as in the case of J_I, one can readily show that the quantity β is an invariant of system (1). It also follows from [13] that

$$\frac{1}{2}\det\begin{pmatrix} B & \bar{B} \\ BJ & \bar{B}\bar{J} \end{pmatrix} = |\det B|^2\left(\beta - \operatorname{Im}^2\nu\right)$$

in the case under consideration; therefore, condition (3) can be reduced to the form

$$\beta \neq \operatorname{Im}^2\nu. \tag{42_{II}}$$

If a given number β satisfies condition (42_{II}) and the invertible matrix B is related to it by the corresponding formula (41_{II}), then, just as above, one can show that this matrix defines a unique elliptic system with properties (3) and (4), where $J = J_{II}$.

The determinant of the matrix X^0 satisfies the similar relations (41) with $\det B = 1$ and $\beta = 0$. These expressions imply that $\det X(\zeta)$ is an even function, whence we obtain (29). In this connection, the problem of evaluating the number n_τ^0 of zeros of the function $\det X(\zeta)$ on the line $\operatorname{Re}\zeta = 0$ becomes important. By (41), these zeros coincide with the zeros of the function

$$|\det B|^{-2}\frac{\det X_\tau}{\det X_\tau^0} = 1 - \beta h, \tag{44}$$

where we have set

$$h_I(\zeta) = \frac{4}{(\sin\theta_1\zeta)(\sin\theta_2\zeta)}\left[\sinh^2\left(\ln\frac{\varrho_1}{\varrho_2}\right)\zeta + \sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\zeta\right], \qquad h_{II}(\zeta) = \frac{|\delta_\tau|^2\zeta^2}{\sin^2\theta_{\tau(\nu)}\zeta},$$

respectively, for the two cases. [Recall that $\varrho_j = \varrho_{\tau(\nu_j)}$ and $\theta_j = \theta_{\tau(\nu_j)}$.] On the imaginary axis, we have

$$h_I(it) = \frac{4}{(\sinh t\theta_1)(\sinh t\theta_2)}\left[\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right)t - \sin^2\left(\ln\frac{\varrho_1}{\varrho_2}\right)t\right], \qquad h_{II}(it) = \frac{|\delta_\tau|^2 t^2}{\sinh^2 t\theta_{\tau(\nu)}},$$

and, in particular,

$$h_I(0)=\frac{4}{\theta_1\theta_2}\left[\left(\frac{\theta_1-\theta_2}{2}\right)^2-\left(\ln\frac{\varrho_1}{\varrho_2}\right)^2\right],\qquad h_{II}(0)=\frac{|\delta_\tau|^2}{\theta^2_{\tau(\nu)}}. \tag{45}$$

Since $h(it)\to 0$ as $t\to\infty$, we find that the equation $1-\beta h(it)=0$ with $\beta h(0)\geq 1$ has at least one root on the half-line $t\geq 0$. Therefore, $n^0_\tau\geq 2$ for $\beta h(0)\geq 1$. This fact can be refined.

Lemma 1. *If $J=J_{II}$, then*

$$n^0_\tau=\begin{cases}2 & \text{if}\quad \beta h(0)\geq 1\\ 0 & \text{if}\quad \beta h(0)<1.\end{cases} \tag{46}$$

A similar assertion is valid for $J=J_I$ under the additional assumption

$$\varrho_{\tau(\nu_1)}=\varrho_{\tau(\nu_2)},\qquad \theta_{\tau(\nu_1)}=3\theta_{\tau(\nu_2)}. \tag{47}$$

Proof. The function $g(x)=x^2/\sinh^2 x$ is strictly monotone decreasing from 1 to 0 on the half-line $[0,\infty)$. Since $1-\beta h(it)=1-\beta h(0)g(t)$, we have the first assertion of the lemma.

Let $J=J_I$, and let relation (46) be valid; i.e., $\varrho_1=\varrho_2$ and $\theta_1=3\theta_2$. Then

$$h(it)=\frac{4\sinh t\theta_1}{\sinh 3t\theta_1}=\frac{4}{3+4\sinh^2 t\theta_1}.$$

The function $g(x)=1/\left(1+3\sinh^2 x\right)$ is strictly monotone decreasing from 1 to 0 on the half-line $[0,\infty)$. Since $1-\beta h(it)=1-\beta h(0)g\left(t\theta_1\right)$, we arrive at the second assertion of the lemma.

Note that under assumption (46), one can also readily write out the distribution of zeros of the function $\det X_\tau(\zeta)$ on the entire plane and, in particular, obtain a closed-form expression for the integer-valued characteristic $n_\tau(\alpha)$. Indeed, in this case, the expression (44_I) becomes much simpler:

$$\zeta^2 X_\tau(\zeta)=4|\det B|^2\varrho_1^{2\zeta}\varrho_2^{2\zeta}\left(\sin^2\theta_1\zeta\right)\left(4\sin^2\theta_1\zeta+4\beta\sin\theta_1\zeta-3\right).$$

Then we encounter the problem as to when relations (47) are valid, i.e., when there exist unit vectors $q_{\tau\pm 0}$ satisfying these relations. This question can be answered only partly.

Lemma 2. *For given $\nu_1\neq\nu_2$, there exist unit vectors $q_{\tau\pm 0}\in\mathbb{C}$ such that the first relation in* (47) *is valid.*

Proof. We have $q_{\tau\pm 0}=q^1_{\tau\pm 0}+iq^2_{\tau\pm 0}$, $q^j_{\tau\pm 0}\in\mathbb{R}$. Then

$$q_k=\frac{q^1_{\tau-0}+\nu_k q^2_{\tau-0}}{q^1_{\tau+0}+\nu_k q^1_{\tau+0}},\qquad k=1,2.$$

The problem is to choose $q_{\tau\pm 0}$ such that $|q_1|=|q_2|$. This condition is equivalent to the relation

$$\frac{q^1_{\tau-0}+\nu_1 q^2_{\tau-0}}{q^1_{\tau-0}+\nu_2 q^1_{\tau-0}}=\frac{q^1_{\tau+0}+\nu_1 q^2_{\tau+0}}{q^1_{\tau+0}+\nu_2 q^1_{\tau+0}},$$

or

$$\frac{1+\nu_1 t_-}{1+\nu_2 t_-}=\frac{1+\nu_1 t_+}{1+\nu_2 t_+},\qquad t_\pm=\frac{q^2_\tau\pm 0}{q^1_\tau\pm 0}. \tag{48}$$

On the plane w, we consider the image L of the real axis $\mathbb{R}$ under the linear-fractional transformation

$$w=\frac{1+\nu_1 t}{1+\nu_2 t}. \tag{49}$$

Obviously, L is a circle passing through the points $w = 1$ and $w = \nu_1/\nu_2$. The point $w = 0$ lies outside this circle, since the points $t = -1/\nu_j$, $j = 1, 2$, lie in the upper half-plane, i.e., on one side of the axis $\mathbb{R}$. Without loss of generality, the numbering of ν_j can be chosen so as to ensure that $\operatorname{Im}(\nu_1/\nu_2) > 0$. If

$$R_0 < R < R_1; \qquad R_0 = \min_L |w|, \qquad R_1 = \max_L |w|,$$

then the circle $|w| = R$ meets L at two points $w_\pm$. If $t_\pm$ corresponds to $w_\pm$ under the transformation (49), then formula (48) defines unit vectors $q_{\tau\pm 0}$ with the desired property.

The Jordan matrix J occurring in relations (3) and (4) is referred to as the Jordan form of system (1).

Theorem 5. *For a given matrix $J = J_I$ or $J = J_{II}$ occurring in* (40), *there exists a contour Γ with the unique corner τ and an elliptic system* (1) *with Jordan form J such that the index of the Dirichlet problem* (1), (2) *in the class $C^\mu_{(+0)}$ is negative.*

Proof. Since the function $\det X_\tau(\zeta)$ is even, it follows from Theorem 1 that the index $\varkappa^+$ of the Dirichlet problem in the class $C^\mu_{(+0)}$ is equal to $-n^0_\tau/2$.

First, let $J = J_{II}$. In this case, the quantity $h(0)$ occurring in (45) is positive, and one can choose a number $\beta > 0$ from condition (42_{II}) and the condition $\beta h(0) \geq 1$. Let B be an invertible matrix related to β by formula (41_{II}). Then the corresponding elliptic system is the desired one.

If $J = J_I$, then, following Lemma 2, we choose $q_{\tau\pm 0}$ and consider a contour Γ for which these vectors are tangent at the point τ to the lateral sides of the sector D_τ. Then, by (45), $h(0)$ is positive, and it remains to repeat the above-performed considerations.

Note that if $J = J_{II}$, then the contour Γ can be chosen arbitrarily. The negativeness of the index $\varkappa^+$ in the theorem implies that, in the case of the uniqueness of the solution of the Dirichlet problem, its existence is possible only under additional orthogonality conditions imposed on the right-hand side f of Eq. (2). This phenomenon takes place only in the case of a corner. If τ is a regular point, i.e., $q_{\tau-0} = -q_{\tau+0}$, then, in notation (41), we have $\varrho_1 = \varrho_2$, $\theta_1 - \theta_2 = 2\pi$ for $J = J_I$, and $\delta_\tau = 0$ for $J = J_{II}$. In both cases, the second term on the right-hand side in relation (41) identically vanishes; consequently, $\varkappa^+ = 0$.

6. THE LAMÉ SYSTEM WITH A NEGATIVE PARAMETER

Consider system (32) with coefficients

$$\begin{aligned} A_{11} &= \begin{pmatrix} \lambda+2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}, \\ A_{21} &= \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \qquad A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda+2 \end{pmatrix}. \end{aligned} \tag{50}$$

For a positive value of λ, it is the Lamé system of plane isotropic elasticity. However, it can also be considered for negative values of the parameter λ. The characteristic polynomial of this system is equal to $(\lambda+2)(1+z^2)^2$, and hence the ellipticity condition can be reduced to the inequality $\lambda \neq -2$. If $\lambda = -1$, then the system becomes the Laplace equation, and this case can be excluded from considerations. If $\lambda \neq -1$, then relations (3) and (4) are valid for the matrices

$$B = \begin{pmatrix} 1 & 0 \\ i & -\varkappa \end{pmatrix}, \qquad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}, \qquad \varkappa = \frac{\lambda+3}{\lambda+1}.$$

In particular, if $\lambda = -3$, then the system in question is strongly coupled. Therefore, the conditions imposed on the parameter λ can be reduced to the inequalities

$$\lambda \neq -1, -2, -3. \tag{51}$$

In this case, conditions (42_{II}) and (43_{II}) can be combined into a single condition of the form $\beta = 1/\varkappa^2 \neq 1$. Here the validity of the inequality is provided by the condition $\lambda \neq -2$. In the case under consideration, relations (43) and (45_{II}) acquire the form

$$h(0) = \frac{|\delta_\tau|^2}{\theta_\tau^2}, \qquad \delta_\tau = \frac{\sin\theta_\tau}{q_{\tau-0}q_{\tau+0}},$$

where we have used the fact that if $\nu = i$, then $z \to z_\nu$ is the identity mapping. Therefore, $h(0) = (\sin\theta_\tau)^2/\theta_\tau^2$, and the expression (46) can be reduced to the form

$$n_\tau^0 = \begin{cases} 2 & \text{if} \quad |\sin\theta_\tau| \geq |\varkappa|\theta_\tau \\ 0 & \text{if} \quad |\sin\theta_\tau| < |\varkappa|\theta_\tau. \end{cases}$$

The inequality $|\sin\theta_\tau| \geq |\varkappa|\theta_\tau$, that is, the inequality

$$\left|\frac{\lambda+3}{\lambda+1}\right| \leq \frac{|\sin\theta_\tau|}{\theta_\tau}, \qquad 0 < \theta_\tau < 2\pi, \qquad \theta_\tau \neq \pi, \tag{52}$$

together with (51), singles out two disjoint intervals

$$-3 \leq \lambda < \lambda_\tau^0, \qquad \lambda_\tau^1 \leq \lambda < -2 \tag{53}$$

of the parameter λ in which $n_\tau^0 = 2$. For the remaining values of λ, we have $n_\tau^0 = 0$. By applying Theorem 4 to system (32), (50), we obtain the following assertion.

Theorem 6. *Let $V^\pm$ be the class of all solutions $v \in C^\mu_{(\pm 0)}$ of the homogeneous Dirichlet problem for system* (32), (50). *Then conditions* ($34^\pm$) *are necessary and sufficient for the solvability of the problem in the class $C^\mu_{(\pm 0)}$, and its index $\varkappa^\pm$ is given by the relation*

$$\varkappa^\pm = \mp\sum_\tau \chi\left(\frac{|\sin\theta_\tau|}{\theta_\tau} - \left|\frac{\lambda+3}{\lambda+1}\right|\right), \qquad \chi(t) = \begin{cases} 1 & \textit{if} \quad t \geq 0 \\ 0 & \textit{if} \quad t < 0. \end{cases}$$

Note that system (32), (50) is strongly elliptic for $\lambda > -2$. In this case, by (53), inequality (52) fails, and therefore, $\varkappa^+ = \varkappa^- = 0$. In other words, the negative index $\varkappa^+$ can appear only if $\lambda < -2$. More precisely, if the contour Γ has a unique corner τ and $\lambda_\tau^0 \leq \lambda < -2$, then $\varkappa^- = \dim V^- - \dim V^+ = 1$. Consequently, there exists exactly one (to within proportionality) solution of the homogeneous Dirichlet problem, which admits the asymptotics (31) with a nonzero function ϕ_τ of the form (30^0) at the point τ.

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