The Bitsadze–Samarskii Problem for Douglis Analytic Functions

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1. STATEMENT OF THE PROBLEM

Let $D \subseteq \mathbb{R}^2$ be a domain bounded by a piecewise smooth contour Γ without return points. We choose a finite subset $F \subseteq \Gamma$ that contains all corner points of the contour and consider a continuously differentiable mapping $\alpha : \Gamma\backslash F \to D$. Suppose that the function α and its derivative α' (with respect to the arc length) are piecewise continuous on Γ, i.e., have one-sided limits at points $\tau \in F$; moreover, $\alpha(\tau \pm 0) \in D \cup F$ and $\alpha'(\tau \pm 0) \neq 0$ for $\tau \in F$. In addition, the curve $\alpha(\Gamma\backslash F)$ is not tangent to Γ.

The Bitsadze–Samarskii problem. Find an analytic function $\phi \in C\left(\bar{D}\backslash F\right)$ in D satisfying the boundary condition

$$\operatorname{Re}\left(a\phi + a^0\phi \circ \alpha\right)\big|_{\Gamma} = f, \tag{1}$$

where the coefficients a and a^0 are piecewise continuous on Γ. Furthermore, in the case of an unbounded domain D, we assume that the function ϕ is bounded at infinity.

One can also consider the case in which the coefficient a^0 and the shift α are defined on some part Γ' of the curve Γ and the boundary condition splits accordingly as

$$\operatorname{Re}(a\phi)|_{\Gamma\backslash\Gamma'} = f_0, \qquad \operatorname{Re}\left(a\phi + a^0\phi \circ \alpha\right)\big|_{\Gamma'} = f_1. \tag{1'}$$

By continuing α to the entire curve $\Gamma\backslash F$ and by completing the definition of a^0 by zero, one can always reduce this problem to the form (1). If $a^0 \equiv 0$, then we obtain the classical Riemann–Hilbert problem.

Obviously, the Bitsadze–Samarskii problem [1] for the Laplace equation [1] can be reduced to problem (1). The Bitsadze–Samarskii problem was comprehensively studied [2–15] for general elliptic equations. In the present paper, we consider problem (1) for Douglis analytic functions, that is, solutions $\phi = (\phi_1, \dots, \phi_l)$ of the first-order canonical system

$$\frac{\partial\phi}{\partial x_2} - J\frac{\partial\phi}{\partial x_1} = 0 \tag{2}$$

with matrix $J \in \mathbb{C}^{l\times l}$ whose eigenvalues $\nu \in \sigma(J)$ lie in the upper half-plane $\operatorname{Im}\nu > 0$. Therefore, the coefficients a and a^0 in (1) are piecewise continuous $l \times l$ matrix functions. If $J = i$, then system (2) becomes the Cauchy–Riemann system and problem (1), (2) corresponds to the problem for analytic vector functions.

The interest in the statement of problem (1), (2) is due to the fact that the Bitsadze–Samarskii problem for elliptic equations and systems with constant (and only leading) coefficients [16, 17] can be reduced to it.

We note the special case of problem (1′), (2) in which Γ' is a smooth arc and the image $\alpha(\Gamma')$ splits D into two subdomains. As indicated in [18], in this case, the problem can be reduced to the generalized Riemann-Hilbert problem, to which general results in [17] can be applied. In this connection, we note the papers [19, 20] for the linearized Stokes system, the paper [21] for the Lamé system of 2D elasticity, and the paper [22] for analytic functions.

Let us return to problem (1), (2) in the above-represented general statement. Following [17], we reduce this problem to an equivalent system of singular integral equations, to which we apply the general results in [23, 24].

All considerations will be performed in the weighted Hölder class $C^{\mu}_{\lambda}\left(\bar{D};F\right)$, $0<\mu<1$, where the weight order λ is a family λ_τ, $\tau\in F$, of real numbers. Recall [17] that if $\lambda_\tau=\mu$, $\tau\in F$, then this space coincides with the Hölder subspace $C^{\mu}\left(\bar{D}\right)$ of functions vanishing at points $\tau\in F$. In the general case, it consists of functions of the form $\varphi(x)=\prod_F|x-\tau|^{\lambda_\tau-\mu}\varphi_*(x)$, $\varphi_*\in C^{\mu}_{\mu}$, with the norm $|\varphi|=|\varphi_*|_{C^{\mu}}$.

For the weight order λ, we assume that

$$\lambda_\tau=\lambda_{\tau'},\qquad \tau=\alpha\left(\tau'\pm 0\right)\in F,\qquad \lambda_{\tau'}<0,\qquad \alpha\left(\tau'\pm 0\right)\in D. \tag{3}$$

This condition ensures that the operator $\phi\to\phi\circ\alpha$ occurring in (1) is a bounded operator in the spaces $C^{\mu}_{\lambda}\left(\bar{D};F\right)\to C^{\mu}_{\lambda}(\Gamma;F)$.

Let us describe the corresponding smoothness conditions for Γ, α, a, and a^0. Let $C^{n,\mu+0}$ stand for the union of classes $C^{n,\mu+\varepsilon}$ over all $\varepsilon>0$. Then for each smooth arc $\Gamma_0\subseteq\Gamma$ that does not contain $\tau\in F$ as interior points, we require that $\Gamma_0\in C^{1,\mu+0}$, $\alpha\in C^{1,\mu+0}\left(\Gamma_0\right)$, and $a,a^0\in C^{\mu+0}\left(\Gamma_0\right)$.

2. THE REDUCTION OF THE PROBLEM TO A SYSTEM OF SINGULAR INTEGRAL EQUATIONS

By [16], all main facts of the theory of analytic functions based on the Cauchy integral can be generalized to the case of Douglis analytic functions. The role of a Cauchy type integral for these functions is played by the integral

$$(I\varphi)(x)=\frac{1}{\pi i}\int\limits_{\Gamma}[dy][y-x]^{-1}\varphi(y),\qquad x\in D, \tag{4}$$

with matrix kernel $[y-x]^{-1}$, where the integration contour Γ is oriented in such a way that the domain D lies to the left.

Here and throughout the following, we use the notation $[x]=x_1\cdot 1+x_2J$, $x\in\mathbb{R}^2$, and $[dy]=(dy_1)\cdot 1+(dy_2)\,J$, where 1 is the identity matrix. The similar notation $[x]_\nu=x_1+x_1\nu\in\mathbb{C}$ is used in the scalar case $\nu\in\mathbb{C}$. By [17], I is a bounded operator in the spaces $C^{\mu}_{\lambda}(\Gamma;F)\to C^{\mu}_{\lambda}\left(\bar{D};F\right)$, $-1<\lambda<0$, and we have the Sokhotskii–Plemelj formula

$$(I\varphi)^{+}(x)=\varphi(x)+(K\varphi)(x),\qquad x\in\Gamma, \tag{5}$$

where $(K\varphi)(x)$ corresponds to the singular integral (4) for $x\in\Gamma\backslash F$.

If the domain D is unbounded (in this case, we write $\infty\in D$), then the function $(I\varphi)(x)$ vanishes at infinity, and in a neighborhood of infinity it can be expanded in the absolutely and uniformly convergent series

$$(I\varphi)(x)=\sum_{k=0}^{\infty}[x]^{-k-1}c_k,\qquad c_k=-\frac{1}{\pi i}\int\limits_{\Gamma}[y]^k[dy]\varphi(y).$$

A similar expansion (in powers of $[x]^{-k}$, $k\geq 0$) is valid for each Douglis analytic function $\phi(x)$ bounded at infinity. Therefore, the class $C^{\mu}_{\lambda}\left(\bar{D};F\right)$ for such functions can be defined with respect to a bounded domain $\bar{D}\cap\{|x|\leq R\}$, where R is sufficiently large.

The reduction of problem (1), (2) to a system of singular integral equations is based on the following analog of the Vekua–Muskhelishvili theorem [17].

Theorem 1. *Let J be a triangular matrix, and let $-1/2<\lambda<0$. Then each Douglis analytic function $\phi\in C^{\mu}_{\lambda}\left(\bar{D};F\right)$ can be represented in the form*

$$\phi=I\varphi+\xi, \tag{6}$$

where $\varphi \in C^\mu(\Gamma; F)$ *is a real vector function and* $i\xi \in \mathbb{R}^l$ *(respectively,* $\xi \in \mathbb{C}^l$*) if* $\infty \notin D$ *(respectively,* $\infty \in D$*). If* $\phi = 0$ *in this representation, then* $\xi = 0$ *and the function* φ *is constant on the connected components of the contour* Γ *and, for* $\infty \notin D$*, vanishes on its "exterior" component surrounding the other components.*

By Γ^j, $j = 1, \ldots, s_D$, we denote all connected components of Γ, with Γ^1 for $\infty \notin D$ being treated as the exterior contour. Then the function φ occurring in the representation (6) can be uniquely determined by the condition

$$\int_{\Gamma^j} \varphi(y) ds_y = 0, \tag{7}$$

where $j = 2, \ldots, s_D$ for $\infty \notin D$ and $j = 1, \ldots, s_D$ for $\infty \in D$.

The requirement that J is a triangular matrix is not very restrictive in the theorem. Indeed, let B be an invertible matrix such that $J_0 = B^{-1} J B$ is a triangular matrix (for example, has a Jordan form). Then the linear substitution $\phi = B\phi_0$ reduces system (2) to the case of the matrix J_0. In this connection, throughout the following, we assume that J is a triangular matrix.

First, we perform the reduction of the problem in the weight class C^μ_λ, $-1/2 < \lambda < 0$. Then, by (5) and (6), problem (1) is equivalent to the system

$$\operatorname{Re} \left\{ a(\varphi + K\varphi) + a^0 (I\varphi) \circ \alpha + \left(a + a^0\right) \xi \right\} = f \tag{8}$$

for a real l vector function $\varphi \in C^\mu_\lambda(\Gamma; F)$ and a constant l vector ξ. Under the additional condition (7), this system is equivalent to problem (1), (2).

We rewrite the left-hand side of system (8) in the form $N\varphi +$ Re $(a + a^0)\,\xi$ with an $\mathbb{R}$-linear operator N. It has the form $N\varphi = \operatorname{Re} M\varphi$, where M is a linear operator over $\mathbb{C}$. The operator $\bar{M}\varphi = \overline{M\bar{\varphi}}$ has the same property; here the bar stands for complex conjugation. In the case of real functions $\varphi = \bar{\varphi}$, we have $2N\varphi = M\varphi + \bar{M}\varphi$. Therefore,

$$2N = a(1 + K) + \bar{a}\left(1 + \bar{K}\right) + a^0 K^0 + \bar{a}^0 \bar{K}^0, \tag{9}$$

where, for brevity, we have set $K^0\varphi = (K\varphi) \circ \alpha$. In the explicit form, the operators $\bar{K}$ and $\bar{K}^0$ are given by the relations

$$\left(\bar{K}\varphi\right)(x) = -\frac{1}{\pi i} \int_\Gamma \overline{[dy]}\; \overline{[y - x]}^{-1} \varphi(y),$$

$$\left(\bar{K}^0\varphi\right)(x) = -\frac{1}{\pi i} \int_\Gamma \overline{[dy]}\; \overline{[y - \alpha(x)]}^{-1} \varphi(y), \qquad x \in \Gamma.$$

So far, we have assumed that $-1/2 < \lambda < 0$ in accordance with Theorem 1. This condition can readily be eliminated with the use of a weight substitution. We introduce the J-analytic weight matrix function

$$\varrho(x) = \prod_{\tau \in F} [x - \tau]^{\delta_\tau}, \tag{10}$$

where, in addition to (8), the weight order δ is subjected to the condition $-1/2 < \lambda - \delta < 0$. Here the factors are treated as the values $[x - \tau]_u^{\delta_\tau}\big|_{u=J}$ of functions of matrices, where $[x - \tau]_u^{\delta_\tau}$ stands for a branch of the power-law function continuous with respect to $x \in D$ and $\operatorname{Im} u > 0$. In the case of a multiply connected domain D, this definition should be modified; namely, the matrix $[x - \tau]$ should be replaced by $[x - \tau]\,[x - \tau_*]^{-1}$ with a given point $\tau_* \notin \bar{D}$.

Let ϱ_Γ be a smooth positive weight function of the same order δ on $\Gamma \backslash F$. Then

$$a_* = a\left(\varrho \varrho_\Gamma^{-1}\right), \qquad a^0_* = a^0 (\varrho \circ \alpha) \varrho_\Gamma^{-1} \tag{11}$$

are piecewise continuous matrix functions on Γ and belong to the same class as a and a^0. Let N_* be the singular operator obtained by the replacement of a and a^0 by these coefficients; we consider it in the class $C^\mu_{\lambda-\mu}$.

By applying Theorem 1 to $\varrho^{-1}\phi \in C^{\mu}_{\lambda-\delta}$, we obtain the representation $\varrho^{-1}\phi = I\left(\varrho^{-1}\varphi\right) + \xi$. By using this representation, by analogy with (7), (8), one can reduce problem (1), (2) in the class C^{μ}_{λ} to the equivalent system

$$\left(\varrho_\Gamma^{-1} N_* \varrho_\Gamma\right) + \varrho_\Gamma \operatorname{Re}\left(a_* + a_*^0\right)\xi = f, \qquad \int\limits_{\Gamma^j} \left(\varrho_\Gamma^{-1}\varphi\right) ds_y = 0, \tag{12}$$

where j ranges over $2,\ldots,s_D$ for $\infty \notin D$ and $j = 1,\ldots,s_D$ otherwise. Therefore, in this system, the operator $N = \varrho_\Gamma^{-1} N_* \varrho_\Gamma$ plays the role of (9).

3. MAIN RESULTS

In a small neighborhood of F, the domain D splits into pairwise disjoint curvilinear sectors D_τ with vertex $\tau \in F$. The boundary ∂D_τ consists of two smooth arcs with common endpoint τ (the lateral sides of a sector) and a circle arc. A vector $q \in \mathbb{R}^2$ is said to be associated with the sector D_τ if it is either tangent to one of its lateral sides at the point τ or directed inwards, i.e., satisfies $\tau + qt \in D_\tau$ for all $t > 0$. Let F consist of m points. We number the lateral sides of the sectors D_τ, $\tau \in F$, in a common sequence $\Gamma_{F,j}$, $j = 1,\ldots,2m$, and introduce their smooth parametrizations $\gamma_j : [0,1] \to \Gamma_{F,j}$ of the class $C^{1,\mu+0}$; we assume that the points $\gamma_j(0)$ belong to F. We set

$$\begin{aligned} &P_\tau = \{j \mid \gamma_j(0) = \tau\}, \qquad P_\tau^0 = \{s \mid \left(\alpha\circ\gamma_s\right)(0) = \tau\}, \\ &q_j = \gamma_j'(0), \qquad q_s^0 = \left(\alpha\circ\gamma_s\right)'(0). \end{aligned} \tag{13}$$

Obviously, each set P_τ consists of exactly two elements, and some of the sets P_τ^0 can be empty. Obviously, the vectors q_s^0, $s \in P_\tau^0$, and q_s, $s \in P_\tau$, are associated with the sector D_τ. It follows from the conditions imposed on the shift α that vectors of one pair P_τ, as well as vectors q_j, $j \in P_\tau$, and q_s^0, $s \in P_\tau^0$, cannot lie on one ray.

By [17, 23], the singular operator K belongs to the algebra $\mathscr{K}\left(C^{\mu}_{\lambda}, -1 < \lambda < 0\right)$. Moreover, $K + \bar{K} \in \mathscr{K}_0$, and the $2m \times 2m$ matrix $\left(\hat{K}_{sj}\right)$ of the terminal symbol is given by the relation

$$\pm(i\sin\pi\zeta)\hat{K}_{sj}(\zeta) = \begin{cases} \cos\pi\zeta & \text{if} \quad s = j \\ \left\{-\left[q_s\right]\left[q_j\right]^{-1}\right\}^{\zeta} & \text{if} \quad \gamma_s(0) = \gamma_j(0),\ s \neq j \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where the upper (respectively, lower) sign is chosen if the parametrization γ_j preserves (respectively, changes) the orientation of Γ. Furthermore, the analytic function u^{ζ}, whose value on the matrix in curly braces occurs on the right-hand side in (14), is determined by the choice of the argument $|\arg u| < \pi$.

A similar result is valid for the operator $K^0\varphi = (I\varphi)\circ\alpha$.

Lemma 1. *The integral operator K^0 belongs to $\mathscr{K}_0\left(C^{\mu}_{\lambda}, -1 < \lambda < 0\right)$, and its terminal symbol is given by the relation*

$$\pm(i\sin\pi\zeta)\hat{K}^0_{sj}(\zeta) = \begin{cases} \left\{-\left[q_s^0\right]\left[q_j\right]^{-1}\right\}^{\zeta} & \textit{if} \quad \left(\alpha\circ\gamma_s\right)(0) = \gamma_j(0) \\ 0 & \textit{otherwise} \end{cases} \tag{15}$$

Proof. Let a smooth function $\chi(y)$ on $\Gamma\backslash F$ be identically equal to unity in a neighborhood of F and vanish outside the arcs $\Gamma_{F,j}$. Since the operator $\chi K^0 - K^0\chi$ is compact, it suffices to prove the assertion of the lemma for $\chi K^0\chi$. Without loss of generality, one can assume that all functions $\chi\circ\gamma_j$ coincide with some smooth function $\chi_0(t)$, $t \geq 0$, identically equal to unity in a neighborhood of $t = 0$ and vanishing for $t \geq 1$. Then

$$\left(\left(\chi K^0\chi\varphi\right)\gamma_s\right)(t_0) = \sum_{j=1}^{2m} \frac{\pm 1}{\pi i} \int\limits_0^1 \chi_0\left(t_0\right)\chi_0(t)\left[\gamma_j(t) - \left(\alpha\circ\gamma_s\right)\left(t_0\right)\right]\left[\gamma_j'(t)\right]\left(\varphi\circ\gamma_j\right)(t)dt, \quad t_0 > 0.$$

If $\gamma_j(0) \neq (\alpha \circ \gamma_s)(0)$, then the integral occurring in the sum is compact in the spaces $C^{\mu}_{\lambda'}([0,1];0) \to C^{\mu}_{\lambda''}([0,1];0)$ for $-1 < \lambda', \lambda'' < 0$.

Let $\gamma_j(0) = (\alpha \circ \gamma_s)(0)$. In this case, the kernel of the integral can be represented in the form $k(t_0,t) \times [q_j]\,[q_j t - q^0_s t_0]^{-1}$ with some matrix function $k(t_0,t)$ such that

$$k(t_0,t) - 1 \in C^{\mu+\varepsilon}_{\varepsilon}([0,1] \times [0,1];0)$$

with a small $\varepsilon > 0$. Moreover, $k \equiv 0$ in a neighborhood of the sides $t = 1$ and $t_0 = 1$ of the square $[0,1] \times [0,1]$. Therefore, neglecting a compact term, the expressions $\gamma_j(t) - (\alpha \circ \gamma_s)(t_0)$ and $\gamma'_j(t)$ in the integrand can be replaced by $q_j t - q^0_s t_0$ and q_j, respectively. Just as in the proof of the corresponding result for K, we obtain the desired assertion.

We treat the weight order λ as a function $\lambda : k \to \lambda(\gamma_k(0))$ constant on pairs P_τ of the partition $P = (P_\tau)$ of the set $\{1, \ldots, 2m\}$. By (14), the matrix $\hat{K}$ has a block diagonal form with respect to this partition, and its P_τth diagonal block $\hat{K}(\zeta, P_\tau)$ is considered on the axis $\operatorname{Re}\zeta = \lambda_\tau$.

Consider a partition (F_j) of the set F such that $\tau \in F_j$, $\alpha(\tau \pm 0) \in F$ (for some sign) implies that $\alpha(\tau \pm 0) \in F_j$. Then the first condition in (3) can be replaced by the requirement that the weight order λ is constant on the elements of this partition. Let E_j consist of pairs P_τ, $\tau \in F_j$. Then each set P^0_τ entirely lies in some E_j. This, together with (15), implies that the terminal symbol $\hat{K}^0$ is a $2m \times 2m$ matrix that has a block diagonal form in the partition (E_j) and its diagonal block $\hat{K}^0(\zeta_j, E_j)$ is defined on the line $\operatorname{Re}\zeta = \lambda$; here and throughout the following, λ_j stands for the restriction to λ on E_j.

Consider the singular operator $N = \varrho_\Gamma N_* \varrho_\Gamma^{-1}$ occurring in (12). We subject the weight function $\varrho_\Gamma(t)$ of order δ to the additional condition

$$(\varrho_\Gamma \circ \gamma_s)(t) = t^{\delta_j}, \quad s \in E_j. \tag{16}$$

This, together with the results of [23], implies that this operator belongs to the algebra $\mathscr{K}(C^{\mu}_{\lambda}; E)$. In this case, its terminal symbol $\hat{N}$ has a block diagonal form in the partition E and is given by the relation

$$\begin{gathered} \hat{N}(\zeta, E_j) = \hat{N}_*(\zeta - \delta_j, E_j), \qquad \operatorname{Re}\zeta = \lambda_j, \\ 2\hat{N}_* = \hat{a}_*(1+K)^\wedge + \hat{\bar{a}}_*\left(1+\bar{K}\right)^\wedge + \hat{a}^0_*\hat{K}^0 + \widehat{\overline{a^0}}_*\widehat{\overline{K^0}}. \end{gathered} \tag{17}$$

Note that $\hat{a}_*$ and $\hat{a}^0_*$ are diagonal $2m \times 2m$ matrices with constant entries $(a_* \circ \gamma_j)(0)$ and $(a^0_* \circ \gamma_j)(0)$, respectively, along the diagonal. To compute the terminal symbols of the operators $\bar{K}$ and $\bar{K}_0$, one should use the relation $\hat{\bar{M}}(\zeta) = \bar{\hat{M}}(\zeta)$, where the bar on the right-hand side stands for the conjugation involution in the class of functions $x(\zeta)$: $\bar{x}(\zeta) = \overline{x(\bar{\zeta})}$ with the complex conjugation on the right-hand side. In the case of the operators K and K^0, this involution corresponds to the replacement of $i\sin\pi\zeta$ in (14) and (15) and the matrix J in the definition of $[x]$ by $-i\sin\pi\zeta$ and $\bar{J}$, respectively.

Obviously, the Fredholm properties of system (12) and the operator N are equivalent, and their indices are related by $\varkappa = \operatorname{ind} N + (2 - s_D)\,l$. Therefore, by applying the corresponding results in [24] to N, we obtain the following criterion for the Fredholm property.

Theorem 2. *Problem* (1), (2) *has the Fredholm property in the class* $C^{\mu}_{\lambda}\left(\bar{D}; F\right)$ *if and only if* $\det a(t) \neq 0$ *everywhere on* Γ, *including the limit values at the point* $\tau \in F$, *and*

$$\det \hat{N}(\zeta, E_j) \neq 0, \qquad \operatorname{Re}\zeta = \lambda_j, \qquad j = 1, \ldots \tag{18}$$

In this case, the index $\varkappa$ *of the problem is given by the formula*

$$\varkappa = -\frac{1}{\pi}\arg\left(\det a_*\right)\big|_\Gamma + l\,(2 - s_D) - \sum_j \frac{1}{2\pi i}\ln\det\hat{N}\,(\zeta, E_j)\big|^{\lambda_j + i\infty}_{\zeta = \lambda_j - i\infty}. \tag{19}$$

By [24], the solvability of the equation $N\varphi = f$ can be described in terms of conditions of orthogonality of the right-hand side f to the solutions $\psi \in C^{\mu}_{-\lambda-1+0}$ of the homogeneous adjoint equation $N'\psi = 0$. The orthogonality and adjointness are treated in the sense of the form

$$(\varphi, \psi) = \int_{\Gamma} \varphi(y)\psi(y) ds_y,$$

where $\varphi(y)\psi(y)$ is the inner product in $\mathbb{R}^l$.

System (12) is a "finite-dimensional" perturbation of the above-mentioned equation, and one can readily write out the homogeneous adjoint system. To be definite, we assume that D is a finite domain, i.e., $\infty \notin D$. Then the adjoint system of (12) is given by the homogeneous system

$$\begin{aligned} &(N'\psi)(x) + \varrho_{\Gamma}^{-1}(x)\eta_i = 0, \qquad x \in \Gamma^i, \qquad i = 1, 2, \ldots, s_D, \\ &\int_{\Gamma} \operatorname{Im}\left(a_* + a_*^0\right)(y)\varrho_{\Gamma}(y)\psi(y) ds_y = 0 \end{aligned} \tag{12'}$$

for the pair $\psi \in C^{\mu}_{-\lambda-1}$ and $\eta = (\eta_i) \in \mathbb{R}^{l(s-1)}$, where $s = s_D$ and $\eta_1 = 0$.

Therefore, the solvability of problem (1), (2) is determined by the orthogonality $(f, \psi) = 0$ for all solutions (ψ, η) of the homogeneous system (12′).

By analogy with [17], we supplement Theorem 2 with results on the asymptotics and smoothness of a solution of problem (1). We assume that the problem is of normal type, i.e., the assumption of Theorem 2 for $\det a$ is valid. However, condition (18) can fail for some block $\hat{N}(\zeta, E_j)$, say, for $\hat{N}(\zeta, E_1)$. The matrix function $\hat{N}(\zeta, E_1)$ is analytic in some open strip $\lambda_1 - \varepsilon < \operatorname{Re}\zeta < \lambda_1 + \varepsilon$ and has the form $x(\zeta) + \bar{x}(\zeta)$. Therefore, if the function $\det \hat{N}(\zeta + u)$ vanishes for $u = 0$, then the same is true for $\det \hat{N}\left(\bar{\zeta} + u\right)$ as well. Moreover, the orders of the poles r_ζ and $r_{\bar{\zeta}}$ of the matrix functions $\hat{N}^{-1}(\zeta + u)$ and $\hat{N}^{-1}\left(\bar{\zeta} + u\right)$ coincide. If $\det \hat{N}(\zeta; E_i) \neq 0$, then, by the definition, we set $r_\zeta = 0$.

The theorem on the asymptotics deals with the behavior of an arbitrary solution $\phi \in C^{\mu}_{\lambda-0}\left(\bar{D}; F\right)$ in the sectors D_τ, $\tau \in F_1$, under the corresponding conditions for the behavior of the functions $(f \circ \gamma_j)(t)$, $j \in E_1$, as $t \to 0$. Here and throughout the following $C^{\mu}_{\lambda-0}$, is treated as the union of $C^{\mu}_{\lambda-\varepsilon}$ over $\varepsilon > 0$. Likewise, $C^{\mu}_{\lambda+0} = \bigcap C^{\mu}_{\lambda+\varepsilon}$.

Theorem 3. *Let*

$$(f \circ \gamma_j)(t) - \operatorname{Re} \sum_{i=0}^{k} \left(\ln^i t\right) t^\zeta c_{ij} \in C^{\mu}_{\lambda+0}([0,1]; 0), \qquad j \in E_1,$$

with some $c_{ij} \in \mathbb{R}^l$ for a given ζ, $\operatorname{Re}\zeta = \lambda_1$. Then for each solution $\phi \in C^{\mu}_{\lambda-0}(D; F)$ of problem (1), (2), *there exist $c^{\pm}_{\tau j} \in \mathbb{C}^l$, $\tau \in F_1$, $j = 1, \ldots, k + r_s$, such that*

$$\phi(x) - \sum_{j=0}^{k+r_s} \left([x-\tau]^\zeta c^{+}_{\tau j} + [x-\tau]^{\bar{\zeta}} c^{-}_{\tau j}\right)(\ln[x-\tau])^j \in C^{\mu}_{\lambda+0}\left(\bar{D}_\tau; \tau\right), \qquad \tau \in F_1.$$

Proof. We rewrite the first equation in (12) in the form $N\varphi = \tilde{f}$, where the right-hand side $\tilde{f}$ satisfies the same condition as f. By the theorem on asymptotics [24], the solution of this equation satisfies the inclusion

$$(\varphi \circ \gamma_j)(t) - \operatorname{Re} \sum_{i=0}^{k+r_\zeta} \left(\ln^i t\right) t^\zeta d_{ij} \in C^{\mu}_{\lambda+0}([0,1]; 0), \qquad j \in E_1,$$

with some $d_{ij} \in \mathbb{R}^l$. This, together with the corresponding property [17] of the Cauchy type integral (4), implies that the corresponding property is valid for $\phi(x)$ as well.

We state the smoothness theorem for one of smooth arcs Γ_0 composing Γ. More precisely, let Γ_0 be the closure of one of connected components of $\Gamma\backslash F$. The set F_0 of endpoints of this arc can consist of two points or a single point. In the latter case, Γ_0 is treated as a "closed" arc.

Theorem 4. *In addition to the assumptions of Theorem* 2, *suppose that* $\Gamma_0 \in C^{n,\mu+0}$, $\alpha \in C^{n,\mu+0}(\Gamma_0)$, *and* $a, b \in C^{n,\mu+0}(\Gamma_0)$. *If* $f \in C^{n,\mu}_{\lambda}(\Gamma_0; F_0)$, *then each solution* $\phi \in C^{\mu}_{\lambda}\left(\bar{D}; F\right)$ *of problem* (1), (2) *has the similar property* $\phi \in C^{n,\mu}_{\lambda}(\Gamma_0; F_0)$.

Proof. By analogy with Lemma 1 from [25], one can readily show that $K_0\varphi = (I\varphi)\circ\alpha$ belonging to $C^{n,\mu}_{\lambda}(\Gamma_0, F_0)$ for $\varphi \in C^{\mu}_{\lambda}(\Gamma; F)$. Under the additional requirement $\varphi|_{\Gamma_0} = 0$, a similar property is valid for $K\varphi$. Therefore, the first equation in (12) can be represented in the form

$$a_0\left(1 + K_0\right)\varphi_0 + \bar{a}_0\left(1 + \bar{K}_0\right)\varphi_0 = 2f_0 \in C^{n,\mu}_{\lambda}(\Gamma_0; F),$$

where $a_0(\varphi_0)$ is the restriction of $a_*(\varphi)$ to Γ_0 and the Cauchy operator K_0 is defined with respect to $\Gamma = \Gamma_0$. An application of the smoothness theorem to this equation implies that $\varphi_0 \in C^{n,\mu}_{\lambda}(\Gamma_0, F_0)$. Consequently, it holds for

$$\phi(x) = \varphi_0(x) + \left(K_0\varphi_0\right)(x) + \frac{1}{\pi i}\int\limits_{\Gamma\backslash\Gamma_0}[x-y]^{-1}dy\,\varphi(y).$$

4. THE TERMINAL SYMBOL OF THE PROBLEM

The coefficients a_* and a^0_* and the operator $N = \varrho_\Gamma N_* \varrho_\Gamma^{-1}$ itself depend on the choice of the weight order δ in (10). Therefore, it is desirable to replace the terminal symbol $\hat{N}$ of this operator by some matrix function free of this dependence. For the Riemann–Hilbert problem (and for more general problems of this type), the corresponding constructions were given in [17]. The case of problem (1) with $a^0 \neq 0$ needs additional considerations.

We order each pair $P_\tau = \{k, r\}$ (and write $P_\tau = \overline{k, r}$) by assuming that the motion through the point τ on Γ in the positive direction (for which the domain D lies on the left) is performed from $\Gamma_{F,r}$ towards $\Gamma_{F,k}$. Vectors $q \in \mathbb{R}^2$ associated with D_τ form a sector (a cone), which is denoted by S_τ. The opening angle, which is obtained from S_τ under the transformation

$$q = (q_1, q_2) \to [q]_\nu = q_1 + \nu q_2,$$

where $\operatorname{Im}\nu > 0$, is denoted by $\theta_{\tau,\nu}$. We fix the branch of $\arg[q]_\nu$ that continuously depends on $q \in S_\tau$ and $\operatorname{Im}\nu > 0$. It defines a power-law function $[q]^\zeta_\nu$ and the matrix $[q]^\zeta = [q]^\zeta_J$. Obviously, it, as well as J, is a triangular matrix with diagonal entries $[q]^\zeta_\nu$, $\nu \in \sigma(J)$. The number l_ν of their occurrences on the diagonal is equal to the multiplicity of the eigenvalue ν in the characteristic polynomial of the matrix J. In a similar way, one can define the matrix $\overline{[q]}^\zeta$ with respect to $\arg[q]_{\bar{\nu}} = -\arg[q]_\nu$. In the following, these matrices are used for the tangent vectors q_k and q_r of the sector D_τ and for its "interior" vectors q^0_s, $s \in P^0_\tau$. By the definition of a continuous branch of $\arg[q]_\nu$, we have

$$\arg\left[q_k\right]_\nu - \arg\left[q_r\right]_\nu = \theta_{\tau,\nu}, \qquad \overline{k, r} = P_\tau. \tag{20}$$

In what follows, we often deal with $2m \times 2m$ matrices $x(\zeta)$ that have a block-diagonal form in the partition P. We write out their diagonal blocks $x(\zeta, P_\tau)$ in the form of a 2×2 table with respect to the order $\overline{k, r}$ of elements of the pair P_τ.

Consider a matrix $W_*(\zeta)$ of this type analytic in the strip $-1/2 < \operatorname{Re}\zeta < 0$ and given by the relation

$$W_*\left(\zeta, P_\tau\right) = \frac{1}{2i\sin\pi\zeta}\begin{pmatrix} -e^{-\pi i\zeta}\overline{[q_k]}^{-\zeta} & e^{\pi i\zeta}\overline{[q_r]}^{-\zeta} \\ e^{\pi i\zeta}\left[q_k\right]^{-\zeta} & -e^{\pi i\zeta}\left[q_r\right]^{-\zeta} \end{pmatrix}, \qquad P_\tau = \overline{k, r}. \tag{21}$$

If the entries $a_{ij} \in \mathbb{R}^{l\times l}$ of the block matrix $(a_{ij})_1^2$ are invertible and triangular, then the determinant of this matrix coincides with the determinant of the matrix $d = a_{11}a_{22} - a_{12}a_{21}$ and is equal to the product of its diagonal entries d_{ii}, $i = 1, \dots, l$. By applying this remark to the matrix (21) and by taking account of (20), we obtain

$$\begin{aligned} \det W_*(\zeta, P_\tau) &= \prod_{\nu\in\sigma(J)} \left\{ \frac{e^{-2\pi i\zeta}}{(2i\sin\pi\zeta)^2} [q_k]_{\bar\nu}^{-\zeta} [q_r]_\nu^{\zeta} - \frac{e^{2\pi i\zeta}}{(2i\sin\pi\zeta)^2} [q_k]_\nu^{-\zeta} [q_r]_\nu^{-\zeta} \right\}^{l_\nu} \\ &= \prod_{\nu\in\sigma(J)} \left\{ \frac{|[q_k]_\nu [q_r]_\nu|^{-\zeta}}{(-2i\sin\pi\zeta)^2} \times 2i\sin\left(-2\pi + \theta_{\tau,\nu}\right)\zeta \right\}^{l_\nu}. \end{aligned} \tag{22}$$

Using W_*, we introduce the $2m \times 2m$ matrix W with diagonal blocks

$$W(\zeta, P_\tau) = \operatorname{diag}\left(\overline{\varrho(\tau,\tau)}, \varrho(\tau,\tau)\right) W_*(\zeta - \delta_\tau, P_\tau), \qquad \varrho(x,\tau) = [x-\tau]^{-\delta_\tau}\varrho(x), \tag{23}$$

defined on the line $\operatorname{Re}\zeta = \lambda_\tau - \delta_\tau$.

The following assertion represents the terminal symbol $\hat N$ in the form of the product of W by a matrix that is naturally referred to as the terminal symbol of the problem. In view of applications to more general problems, it is convenient to replace the operators a and a^0 occurring in (1) by arbitrary functional operators A and A^0 belonging to the algebra $\mathscr{A}$ introduced in [17, 24]. Moreover, by virtue of the condition imposed on λ, the terminal symbol of these operators has the same structure as $\hat N$, i.e., it has a block-diagonal form in the partition E. The operators A_* and A_*^0 are defined by analogy with (11) on the basis of A and A^0.

We introduce the $2m\times 2m$ matrices $X(\zeta)$ and $X^0(\zeta)$ whose columns with numbers in the ordered pair $P_\tau = \overline{k,r}$ are given by the relation

$$X_{sr}(\zeta) = \sum_{j\in P_\tau} \hat A_{sj} [q_j]^\zeta = \bar X_{sk}(\zeta), \qquad X^0_{sr}(\zeta) = \sum_{j\in P^0_\tau} \hat A^0_{sj} \left[q^0_j\right]^\zeta = \bar X^0_{sk}(\zeta), \tag{24}$$

where s ranges over $1, \dots, 2m$.

These matrices, together with $\hat A$ and $\hat A^0$, have a block diagonal form with respect to the partition (E_j). Indeed, if $s \in E_i$ and $P_\tau \subseteq E_{i'}$, $i \ne i'$, then $\hat A_{sj} = 0$, $j \in P_\tau$, and therefore, $s \in E_i$, $j \in P^0_\tau$. Likewise, $\hat A^0_{sj} = 0$ for $s \in E$, $j \in P^0_\tau$, since, in addition to P_τ, P^0_τ also belongs to $E_{i'}$.

Lemma 2. *One has*

$$\hat N = \left(X + X^0\right) W. \tag{25}$$

Proof. First, we justify the desired assertion under the assumption that $\delta = 0$. Then $-1/2 < \lambda < 0$, and the asterisk in the notation A_*, A^0_*, N_*, and N^0_* can be omitted. By (14) and the definition of $[q]^\zeta$, the diagonal block $\hat K(\zeta, P_\tau)$ has the form

$$\hat K(\zeta, P_\tau) = \frac{1}{i\sin\pi\zeta} \begin{pmatrix} \cos\pi\zeta & -e^{-\pi i\zeta} [q_k]^\zeta [q_r]^{-\zeta} \\ e^{\pi i\zeta} [q_k]^{-\zeta} [q_r]^\zeta & -\cos\pi\zeta \end{pmatrix};$$

here and throughout the following, $P_\tau = \overline{k,r}$. Hence it follows that

$$\left(1 + \hat K\right)(\zeta, P_\tau) = \frac{1}{i\sin\pi\zeta} \begin{pmatrix} e^{\pi i\zeta} & -e^{-\pi i\zeta} [q_k]^\zeta [q_r]^{-\zeta} \\ e^{\pi i\zeta} [q_k]^{-\zeta} [q_r]^\zeta & -e^{-\pi i\zeta} \end{pmatrix},$$

$$\left(1 + \overline{\hat K}\right)(\zeta, P_\tau) = \frac{1}{i\sin\pi\zeta} \begin{pmatrix} -e^{-\pi i\zeta} & e^{\pi i\zeta} \overline{[q_k]}^\zeta \overline{[q_r]}^{-\zeta} \\ -e^{-\pi i\zeta} \overline{[q_k]}^{-\zeta} \overline{[q_r]}^\zeta & e^{\pi i\zeta} \end{pmatrix}.$$

This, together with (21), implies that

$$\begin{aligned} \frac{1}{2}\left(1+\hat{K}\right)(\zeta, P_\tau) &= \begin{pmatrix} 0 & [q_k]^\zeta \\ 0 & [q_r]^\zeta \end{pmatrix} W(\zeta, P_\tau), \\ \frac{1}{2}\left(1+\overline{\hat{K}}\right)(\zeta, P_\tau) &= \begin{pmatrix} \overline{[q_k]}^\zeta & 0 \\ \overline{[q_r]}^\zeta & 0 \end{pmatrix} W(\zeta, P_\tau). \end{aligned} \tag{26}$$

Consider the terminal symbol $\hat{K}^0$ of the operator K^0. By (15), its matrix row $\hat{K}^0(\zeta, s\times P_\tau)$ consisting of the entries $\hat{K}^0_{sk}$ and $\left(\hat{K}_0\right)_{sr}$ is nonzero only for $s\in P^0_\tau$ and has the form

$$(i\sin\pi\zeta)\hat{K}^0(\zeta, s\times P_\tau) = \left(e^{\pi i\zeta}\left[q^0_s\right]^\zeta [q_k]^{-\zeta}, -e^{-\pi i\zeta}\left[q^0_s\right]^\zeta [q_\tau]^{-\zeta}\right).$$

Hence it follows that

$$(i\sin\pi\zeta)\overline{\hat{K}^0}(\zeta, s\times P_\tau) = \left(-e^{-\pi i\zeta}\overline{[q^0_s]}^\zeta\,\overline{[q_k]}^{-\zeta}, e^{\pi i\zeta}\overline{[q^0_s]}^\zeta\,\overline{[q_\tau]}^{-\zeta}\right).$$

In notation (21), this implies that

$$\hat{K}_0(\zeta, \sigma\times P_\tau) = 2\left(0, \left[q^0_s\right]^\zeta\right) W(\zeta, P_\tau), \qquad \overline{\hat{K}_0}(\zeta, \sigma\times P_\tau) = 2\left(\overline{[q^0_s]}^\zeta, 0\right) W(\zeta, P_\tau). \tag{27}$$

We introduce auxiliary $2m\times 2m$ matrices $U_\pm$ and $U^0_\pm$ by setting

$$\begin{aligned} &U_+(\zeta, s\times P_\tau) = \left(0, [q_s]^\zeta\right), && U_-(\zeta, s\times P_\tau) = \left(\overline{[q_s]}^\zeta, 0\right), \quad s\in P_\tau, \\ &U^0_+(\zeta, s\times P_\tau) = \left(0, \left[q^0_s\right]^\zeta\right), && U^0_-(\zeta, s\times P_\tau) = \left(\overline{[q^0_s]}^\zeta, 0\right), \quad s\in P^0_\tau, \\ &U_\pm(\zeta, s\times P_\tau) = 0, \quad s\notin P_\tau; && U^0_\pm(\zeta, s\times P_\tau) = 0, \quad s\notin P^0_\tau. \end{aligned} \tag{28}$$

Then one can rewrite relations (26), (27) in the form

$$1+\hat{K} = 2U_+W, \qquad 1+\overline{\hat{K}} = 2U_-W, \qquad \hat{K}^0 = 2U^0_+W, \qquad \overline{\hat{K}^0} = 2U^0_-W. \tag{29}$$

Consequently, for the terminal symbol $\hat{N}$, we have relation (25) with the matrices

$$X = \hat{A}U^+ + \overline{\hat{A}}U^-, \qquad X^0 = \hat{A}^0U^0_+ + \overline{\hat{A}^0}U^0_-,$$

which, by virtue of (28), coincide with (24).

Let us proceed to the case in which $\delta\neq 0$ and λ is an arbitrary weight order. In this case, we have $A_* = Ac$ and $A^0_* = A^0c^0$ with the multiplication operators $c = \varrho\varrho_\Gamma^{-1}$ and $c^0 = (\varrho\circ\alpha)\varrho_\Gamma^{-1}$. By (10), (16), and the definition of the function $\varrho(t,x) = \varrho(x)[x-\tau]^{-\delta_\tau}$, we have

$$\begin{aligned} (c\circ\gamma_j)(t) &= \left[t^{-1}(\gamma_j(t)-\gamma_j(0))\right]^{\delta_\tau}\varrho(\tau, \gamma_j(t)), \qquad \tau = \gamma_j(0), \\ \left(c^0\circ\gamma_s\right)(t) &= \left[t^{-1}((\alpha\circ\gamma_s)(t) - (\alpha\circ\gamma_s)(0))\right]^{\delta_\tau}\varrho(\tau, \gamma_s(\tau)), \quad \tau = (\alpha\circ\gamma_s)(0)\in F, \end{aligned}$$

where we have used the relation $\lambda\left(P^0_\tau\right) = \lambda(P_\tau) = \lambda_\tau$. In notation (13), this implies that

$$(c\circ\gamma_s)(0) = [q_s]^{\delta_\tau}\varrho(\tau,\tau), \quad s\in P_\tau; \qquad \left(c^0\circ\gamma_s\right)(0) = \left[q^0_s\right]^{\delta_\tau}\varrho(\tau,\tau), \quad s\in P^0_\tau. \tag{30}$$

Finally, by the definition of δ in (10), the relation $\delta_\tau < 0$ should be valid for $\tau = \gamma_s(0)\in D$; therefore, $\left(c^0\circ\gamma_s\right)(0) = 0$.

In the case under consideration, the expression (17) for the terminal symbol of the operator N acquires the form

$$\begin{aligned} 2\hat{N}(\zeta, E_j) &= \hat{A}\hat{c}(\zeta, E_j)\left(1+\hat{K}\right)(\zeta-\delta_j, E_j) + \overline{\hat{A}\hat{c}}(\zeta, E_j)\left(1+\overline{\hat{K}^0}\right)(\zeta-\delta_j, E_j) \\ &+ \hat{A}^0\hat{c}^0(\zeta, E_j)\,\hat{K}^0(\zeta-\delta_j, E_j) + \overline{\hat{A}^0\hat{c}^0}(\zeta, E_j)\,\overline{\hat{K}^0}(\zeta-\delta_j, E_j). \end{aligned} \tag{31}$$

The expressions (29), where W should be replaced by W_*, can be substituted into the right-hand side of this relation. On the other hand, by (22), (28), and (30), we have

$$\begin{aligned} &(\hat{c}U_+)(\zeta-\delta_\tau, s\times P_\tau) = U_+(\zeta, s\times P_\tau)\,\varrho(\tau,\tau), \\ &\left(\bar{\hat{c}}U_-\right)(\zeta-\delta_\tau, s\times P_\tau) = U_-(\zeta, s\times P_\tau)\,\overline{\varrho(\tau,\tau)}, \qquad s\in P_\tau, \end{aligned}$$

and similarly,

$$\begin{aligned} &\left(\hat{c}^0U_+^0\right)(\zeta-\delta_\tau, s\times P_\tau) = U_+^0(\zeta, s\times P_\tau)\,\varrho(\tau,\tau), \\ &\left(\overline{\hat{c}^0}U_-^0\right)(\zeta-\delta_\tau, s\times P_\tau) = U_-^0(\zeta, s\times P_\tau)\,\overline{\varrho(\tau,\tau)}, \qquad s\in P_\tau^0. \end{aligned}$$

Moreover,

$$\begin{aligned} \left(\hat{c}^0U_+^0\right)(\zeta-\delta_\tau, s\times P_\tau) &= \left(\overline{\hat{c}^0}U_+^0\right)(\zeta-\delta_\tau, s\times P_\tau) = U_+^0(\zeta-\delta_\tau, s\times P_\tau) \\ &= U_-^0(\zeta-\delta_\tau, s\times P_\tau) = 0, \qquad (\alpha\circ\gamma_s)(0)\in D. \end{aligned}$$

By substituting all these relations into (31), by using (23), and by arguing as in the case $\delta=0$, we obtain relation (25) with the matrices $X = \hat{A}U_+ + \overline{\hat{A}}U_-$ and $X^0 = \hat{A}^0U_+^0 + \overline{\hat{A}^0}U_-^0$, which completes the proof of the lemma.

In the case of the multiplication operators $a=A$ and $a^0=A^0$ in problem (1), the expressions (24) are simplified, since the terminal symbols of these operators are the diagonal matrices $\hat{a}=(\hat{a}_i\delta_{ij})$, $\hat{a}_i = (a\circ\gamma_j)(0)$, and similarly for $\hat{a}^0$. Therefore, relation (24) can be represented in the form

$$\begin{aligned} X(\zeta, s\times P_\tau) &= \begin{cases} \left(\overline{\hat{a}_s[q_s]}^{\zeta}, \hat{a}_s[q_s]^{\zeta}\right) & \text{for} \quad s\in P_\tau \\ 0 & \text{for} \quad s\notin P_\tau, \end{cases} \\ X^0(\zeta, s\times P_\tau) &= \begin{cases} \left(\overline{\hat{a}_s^0[q_s^0]}^{\zeta}, \hat{a}_s^0[q_s]^0\right) & \text{for} \quad s\in P_\tau^0 \\ 0 & \text{for} \quad s\notin P_\tau^0. \end{cases} \end{aligned} \tag{32}$$

It readily follows from Lemma 2 that the terminal symbol $\hat{N}$ can be replaced by $X+X^0$ in the Fredholm property criterion (18) in Theorem 2 as well as in Theorem 3. For the index of the problem, we have the following assertion.

Theorem 5. (a) *The indices $\varkappa(\lambda)$ and $\varkappa\left(\tilde{\lambda}\right)$ of problem* (1), (2) *in the classes C_λ^μ and $C_{\tilde{\lambda}}^\mu$, respectively, are related by the formula*

$$\varkappa(\lambda) - \varkappa\left(\tilde{\lambda}\right) = \sum_j \varkappa_j\left(\lambda, \tilde{\lambda}\right), \tag{33}$$

where $\varkappa_j\left(\lambda,\tilde{\lambda}\right)$ is the number of zeros (*with regard to multiplicities*) *of the function* $\det\left(X+X^0\right)(\zeta; E_j)$ *in the strip lying between the lines* $\operatorname{Re}\zeta = \lambda_j$ *and* $\operatorname{Re}\zeta = \tilde{\lambda}_j$ *with the* $+$ *sign for $\lambda_j < \tilde{\lambda}_j$ and the* $-$ *sign otherwise*;

(b) *the formula*

$$\varkappa(\lambda) = -\frac{1}{\pi}\arg\det a|_\Gamma + l\,(2-s_D) - \frac{1}{2\pi i}\sum_j \ln\left\{\frac{\det\left(X+X^0\right)(\zeta, E_j)}{\det Y(\zeta, E_j)}\right\}\Bigg|_{\operatorname{Re}\zeta=\lambda_j-i\infty}^{\lambda_j+i\infty}, \tag{34}$$
$$-1/2<\lambda<0,$$

is valid for $-1/2<\lambda<0$, where Y is the matrix defined similarly to X in (32) *for $a=1$.*

Proof. (a) It suffices to prove the assertion under the assumption that λ is sufficiently close to $\tilde{\lambda}$. In this case, $\varkappa_j\left(\lambda, \tilde{\lambda}\right)$ coincides with the similar quantity computed for $\det \hat{N}\left(\zeta, E_j\right)$, as follows from (25). Therefore, relation (33) is a corollary to the index formula (19) and the Rouche theorem for analytic functions.

(b) If $-1/2 < \lambda < 0$, then in definition (23) of the matrix W, one can set $\delta = 0$, whence $W = W_*$. Likewise, $a = a_*$. Therefore, by taking account of the relation

$$\det \hat{N} = \det\left(X + X^0\right)\det W = \frac{\det\left(X + X^0\right)}{\det Y}\det(YW)$$

and the index formula (19), one can reduce the proof of formula (34) to the proof of the relation

$$\ln(\det YW)|_{\operatorname{Re}\zeta=\lambda} = 0. \tag{35}$$

By analogy with (22), for the determinant of the matrix

$$Y\left(\zeta, P_\tau\right) = \begin{pmatrix} \overline{[q_k]}^\zeta & [q_k]^\zeta \\ \overline{[q_r]}^\zeta & [q_r]^\zeta \end{pmatrix},$$

we have

$$\det Y\left(\zeta, P_\tau\right) = \prod_{\nu\in\sigma(J)}\left\{\left|[q_k]_\nu\,[q_r]_\nu\right|^\zeta\left(-2i\sin\theta_{\tau,\nu}\zeta\right)\right\}^{l_\nu}.$$

Therefore, in turn, relation (35) can be reduced to the form

$$\ln\left.\left(\frac{\sin\theta\zeta\,\sin(2\pi-\theta)\zeta}{\sin^2\pi\zeta}\right)\right|_{\operatorname{Re}\zeta=\lambda} = 0,$$

where $-1/2 < \lambda < 0$ and $0 < \theta < 2\pi$. The left-hand side of this relation is continuous with respect to θ and λ and remains integer. Therefore, it is constant, and one should set $\theta = \pi$.

Note that the multiplication of the matrix X in (24) on the right by the diagonal matrix x with diagonal entries $x_r = [q_r]^{-\zeta}$, $x_k = \bar{x}_r$, where $\overline{k, r} = P_\tau$, $\tau \in F$, reduces it to the matrix $\tilde{X}$ for which $\tilde{X}_{sr} = \hat{A}_{sr} + \hat{A}_{sk}v_\tau = \overline{\tilde{X}}_{sk}$ and $P_\tau = \overline{k, r}$; here $v_\tau(\zeta) = [q_k]^\zeta\,[q_r]^{-\zeta}$. This matrix $\tilde{X}$ exactly coincides with the terminal symbol [17] of the generalized Riemann–Hilbert problem corresponding to $a = A$ and $a^0 = 0$ in (1).

5. THE SPECIAL BLOCK STRUCTURE OF THE TERMINAL SYMBOL OF THE PROBLEM

We denote points $\tau \in F$ by the symbols τ_i, $i = 1, \ldots, m$, and represent the sets P_i and P_i^0 in (13) in the corresponding form. The expressions (24) imply that the change of order in the pair P_i results only in a rearrangement of the corresponding columns of the matrices X and X^0. Therefore, this order can be chosen arbitrarily. We denote the first and second elements of the ordered pair P_i by $P_i(p)$, $p = 1, 2$.

Let $X_{(ij)}$ stand for the 2×2 matrix consisting of the entries of the matrix X in the rows and columns with indices in P_i and P_j, respectively; $X^0_{(ij)}$ has a similar meaning. By (32), $\left(X_{(ij)}\right)$ is a diagonal matrix (i.e., $X_{(ij)} = 0$, $i \neq j$) with diagonal entries

$$X_{(ii)p1} = \bar{X}_{(ii)p2} = \hat{a}_k\,[q_k]^\zeta, \qquad k = P_i(p).$$

Likewise,

$$X^0_{(ij)p1} = \overline{X^0}_{(ii)p2} = \begin{cases} \hat{a}^0_k\,[q^0_k]^\zeta & \text{if} \quad k = P_i(p) \in P^0_j \\ 0 & \text{otherwise.} \end{cases}$$

If the sets F_k of points $\tau = (i)$ are treated as subsets of $\{1, \ldots, m\}$, then the $m \times m$ matrix $X^0 = \left(X^0_{(ij)}\right)$ has a block-diagonal form with respect to the partition (F_k). In other words, the block matrix $X^0\,(\zeta, F_k)$ consists of the entries $X^0_{(ij)}$, $i, j \in F_k$.

In the description (32) of the matrix X^0, the sets $P^0_{(j)}$ can be diminished by supplementing their definition (13) with the condition $(a^0 \circ \gamma_k)\,(0) \neq 0$. This requirement is especially convenient for problem (1′) in which the shift α is defined on the part Γ' of the arc Γ. By way of illustration, we consider a special case of this problem in which Γ' is an arc with endpoints τ_1 and τ_2 and with no interior point $\tau \in F$. If $\alpha\,(\Gamma') \subseteq D$, then, obviously, X^0 is the zero matrix. Therefore, one can assume that $\alpha\,(\tau_1) \in F$ and $\alpha\,(\tau_2) \in D \cup F$. The numbering of the arcs $\Gamma_{F,j}$ is chosen as to ensure that Γ' finishes by arcs $\Gamma_{F,1}$ and $\Gamma_{F,2}$ with endpoints τ_1 and τ_2, respectively. We subject the numbering of $P_{(1)}$ and $P_{(2)}$ to the condition $(i)1 = i$, $i = 1, 2$.

Then, by (32), the second row of each matrix $X^0_{(ij)}$ is zero. More precisely, the $m \times m$ matrix $\left(X^0_{(ij)}\right)$ contains only two nonzero entries

$$x^0_k = \begin{pmatrix} a^0\,(\tau_k)\,[q^0_k]^\zeta & \overline{a^0\,(\tau_k)\,[q^0_k]}^{\,\zeta} \\ 0 & 0 \end{pmatrix}, \qquad k = 1, 2.$$

To be definite, we assume that $\tau_1 \in F_1$ and write out the diagonal block

$$X^0\,(\zeta, F_1) = \left(X^0_{(ij)},\ \tau_i, \tau_j \in F_1\right)$$

of the matrix X^0 that depends on the mutual arrangement of the points τ_k and $\alpha\,(\tau_k)$, $k = 1, 2$. We separately consider all possible cases of this arrangement.

(1) Let $\alpha\,(\tau_1) = \tau_1$ and $\alpha\,(\tau_2) \neq \tau_1$. Then $F_1 = \{\tau_1\}$, $P^0_1 = \{1\}$, and $X^0_{(11)} = x^0_1$.

(2) Let $\alpha\,(\tau_1) = \tau_2$ and $\alpha\,(\tau_2) = \tau_1$. Then $F_1 = \{\tau_1, \tau_2\}$, $P^0_1 = \{2\}$, $P^0_2 = \{1\}$, $X^0_{(11)} = X^0_{(22)} = 0$, $X^0_{(12)} = x^0_1$, and $X^0_{(21)} = x^0_2$.

(3) Let $\alpha\,(\tau_1) = \tau_1$ and $\alpha\,(\tau_2) = \tau_1$. Then $F_1 = \{\tau_1, \tau_2\}$, $P^0_1 = \{1, 2\}$, $X^0_{(12)} = X^0_{(22)} = 0$, and $X^0_{(i1)} = x^0_i$, $i = 1, 2$.

(4) Let $\alpha\,(\tau_1) = \tau_2$ and $\tau_3 = \alpha\,(\tau_2) \neq \tau_1, \tau_2$. Then $F_1 = \{\tau_1, \tau_2, \tau_3\}$, $P^0_2 = \{1\}$, $P^0_3 = \{2\}$, $X^0_{(12)} = x^0_1$, $X^0_{(23)} = x^0_2$, $X^0_{(ij)} = 0$ in the remaining cases.

(5) Let $\tau_4 = \alpha\,(\tau_1) \neq \tau_1, \tau_2$ and $\alpha\,(\tau_2) = \tau_4$. Then $F_1 = \{\tau_1, \tau_2, \tau_4\}$, $P^0_4 = \{1, 2\}$, and $X^0_{(i4)} = x^0_i$, $i = 1, 2$, $X^0_{(ij)} = 0$ in the remaining cases.

(6) Let $\tau_4 = \alpha\,(\tau_1) \neq \tau_1, \tau_2$ and $\alpha\,(\tau_2) \neq \tau_1, \tau_2, \tau_4$. Then $F_1 = \{\tau_1, \tau_4\}$, $P^0_1 = \{1\}$, and $X^0_{(14)} = x^0_1$, $X^0_{(11)} = X^0_{(44)} = X^0_{(41)} = 0$.

Note that, in cases (4)–(6), the matrices $(X + X^0)(\zeta, F_j)$ are block triangular for all F_j. Therefore, their determinant coincides with $\det X(\zeta, F_j)$, and the characteristic of the poles $(X + X^0)^{-1}\,(\zeta, F_j)$ coincides with the corresponding characteristic of the matrix function $X^{-1}\,(\zeta, F_j)$. This, in view of the geometric interpretation of these cases, implies the following assertion.

Theorem 6. *Let the curve* Γ' *occurring in problem* (1′), (2) *be an arc that does not contain the set* F *in its interior, and let the intersection* $\Gamma' \cap \alpha\,(\Gamma')$ *be either empty or consist of a unique point* $\tau \in F$, $\alpha(\tau) \neq \tau$. *Then one can set* $X^0 = 0$ *in Theorems* 4 *and* 5.

Note that the statement of problem (1′) suggested in [1] satisfies the assumption $\Gamma' \cap \alpha\,(\Gamma') = \varnothing$ of the theorem.

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