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On limit behavior of a solution to boundary value problem in a plane sector

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We study a certain boundary value problem in Sobolev-Slobodetskii spaces with integral condition in a plane excluding a ray from origin. Using auxiliary problem in outside of a convex cone and the wave factorization concept, we construct a general solution and consider transfer to limit boundary value problem. It was shown that limit boundary value problem can be solvable only if the boundary function satisfies to a certain singular functional equation.

KEYWORDS
elliptic pseudo-differential equation, limit boundary value problem, wave factorization, general solution

MSC CLASSIFICATION
35S15; 47G30

1 | INTRODUCTION

Theory of pseudo-differential operators and equations on manifolds with a smooth boundary was very intensive which was developed in the second half quarter of the last century,[1-3] and now, there are a lot of applications of the theory.[4-6] Many papers and books are related to a theory of elliptic pseudo-differential operators and equations on nonsmooth manifolds or on manifolds with nonsmooth boundaries.[7-14] According to the local principle to obtain Fredholm property for a general pseudo-differential operator, we need to study invertibility properties for model operators in so called canonical domains. One of such domains is a cone.

The authors develop a special approach to studying elliptic pseudo-differential equations on manifolds with a non-smooth boundary. Key point of the approach is studying a unique solvability for a model equation in a canonical domains. Such canonical domains can be a whole space $\mathbb{R}^m$, a half-space $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$ or a certain cone in $\mathbb{R}^m$. Some results in this direction are included in the books[15,16] and papers,[17-19] but now, some new results were obtained,[15,20-24] and it permits developing the approach more explicitly. Moreover, some results[25,26] can help to describe more complicated situations than ordinary m-dimensional cone in $\mathbb{R}^m$, namely, we would like to consider here the situation when starting cone degenerates into a cone of a lower dimension. We will start from two-dimensional case.

Outline of the paper is the following. We introduce boundary value problem with additional integral condition for a model elliptic pseudo-differential operator in a plane sector. For solving this problem, we describe functional spaces, operators, and a special factorization for an elliptic symbol. Further, we find a solution for the boundary value problem, and study conditions under which the solution exists for limit value of a cone.

2 | STATEMENT OF THE PROBLEM AND AUXILIARIES

Let D be a plane domain of the following type $D = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\}$, and A be an elliptic pseudo-differential operator with the symbol $A(\xi)$[3] satisfying the condition:

$$a_1(1+|\xi|)^\alpha \le |A(\xi)| \le a_2(1+|\xi|)^\alpha, \tag{*}$$

with positive constants a_1, a_2.

We will study the following boundary value problem

$$(Au)(x) = v(x), \quad x \in D, \tag{1}$$

$$\int_{-\infty}^{+\infty} u(x_1, x_2)dx_2 = g(x_1). \tag{2}$$

Let us note that the condition (2) is the so-called nonlocal or integral condition. It appears in studies not often, but nevertheless, it is used in studying some problems.[27-29]

Our strategy is the following. Let $C_+^a = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\}$ be plane sector with the size 2α (so that $a = \cot\alpha$), and we will study Equation (1) with the condition (2) in the domain $\mathbb{R}^2 \setminus C_+^a$. So our starting equation will be the following:

$$(Au)(x) = v(x), \quad x \in \mathbb{R}^2 \setminus C_+^a. \tag{3}$$

Further, if we can find the solution of the problem (3), (2) then we will try to obtain limit expression for the solution under $a \to \infty$ ($\alpha \to 0$). We will see that under our assumptions the problem (1),(2) can be solvable only if the function g satisfies a certain equation.

2.1 | Spaces and operators

We study Equation (3) in Sobolev-Slobodetskii space $H^s(\mathbb{R}^2 \setminus C_+^a)$. By definition, this space consists of functions u from $H^s(\mathbb{R}^m)$ which supports belong to $\overline{\mathbb{R}^2 \setminus C_+^a}$. A norm in the space $H^s(\mathbb{R}^2 \setminus C_+^a)$ is induced by the H^s-norm

$$||u||_s = \left(\int_{\mathbb{R}^3} \tilde{u}(\xi)(1+|\xi|)^{2s} d\xi \right)^{1/2},$$

where the sign $\sim$ over u denotes its Fourier transform

$$\tilde{u}(\xi) = \int_{\mathbb{R}^3} u(x(e^{ix\cdot\xi} dx.$$

The right hand side v in Equation (3) is taken from the space $H_0^{s-\alpha}(\mathbb{R}^2 \setminus C_+^a)$ of functions defined in $\mathbb{R}^2 \setminus C_+^a$ which admit a continuation ℓv into whole $H^{s-\alpha}(\mathbb{R}^m)$. The norm in such a space is defined as

$$||v||_s^+ = \inf ||\ell v||_s,$$

where inf is taken over all continuations ℓv.

Let us remind[3] that a pseudo-differential operator A is defined by its symbol $A(\xi)$ in the following way:

$$(Au)(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} \tilde{u}(\xi) d\xi.$$

Generally speaking usually, they consider more general symbols $A(x,\xi)$ depending on a spatial variable x, but here, we will consider the simplest variant.

The operator A with the symbol $A(\xi)$ satisfying the condition $(*)$ is a linear bounded operator $H^s(\mathbb{R}^2 \setminus C_+^a) \to H^{s-\alpha}(\mathbb{R}^2 \setminus C_+^a)$.[3]

2.2 | Wave factorization

Our study is based on a concept of the wave factorization.[15,16] Before introducing the concept, we will remind some definitions from multidimensional analysis.[30]

If C is a convex cone in $\mathbb{R}^2$, then the conjugate cone $\overset{*}{C}$ is defined as follows:

$$\overset{*}{C} = \{x \in \mathbb{R}^2 : x \cdot y = x_1 y_1 + x_2 y_2 > 0, \quad \forall y \in C\}.$$

Obviously, $\overset{*}{C^a_+} = \{x \in \mathbb{R}^2 : ax_2 > |x_1|\}$. Let us denote $\overset{*}{C^a_-} = -\overset{*}{C^a_+}$.

A radial tube domain over the cone C is called a subset of two-dimensional complex space $\mathbb{C}^2$ of the following type:

$$T(C) = \{z \in \mathbb{C}^2 : z = x + iy, x \in \mathbb{R}^2, y \in C\}.$$

Definition 1. The wave factorization of an elliptic symbol $A(\xi)$ with respect to the cone C^a_+ is called its representation in the form

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where factors $A_{\neq}(\xi), A_{=}(\xi)$ must satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^2$ may be except $\{\xi \in \mathbb{R}^2 : |\xi_1|^2 = a^2 \xi_2^2\}$;
2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytic continuation into radial tube domains $T(\overset{*}{C^a_-}), T(\overset{*}{C^a_+})$ respectively with estimates

$$|A^{\pm 1}_{\neq}(\xi + i\tau)| \leq c_1 (1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A^{\pm 1}_{=}(\xi - i\tau)| \leq c_2 (1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \quad \forall \tau \in \overset{*}{C^a_-}.$$

The number $æ \in \mathbb{R}$ is called an index of the wave factorization.

Remark 1. Let us note that we replace in the definition $\overset{*}{C^a_+}, \overset{*}{C^a_-}$ in a comparison with standard definition of the wave factorization.[16]

2.3 | A special integral operator G_2

Let us define this operator in the following way[16] first for functions u from Schwartz space $S(\mathbb{R}^2)$

$$(G_2 \tilde{u})(\xi_!, \xi_2) = \lim_{\tau \to 0+} \int_{\mathbb{R}^2} \frac{2a\tilde{u}(\eta_1, \eta_2) d\eta}{(\xi_1 - \eta_1)^2 - a^2 (\xi_2 - \eta_2 + i\tau)^2}.$$

This operator plays an important role for constructing a solution of Equation (3). It is linear bounded operator $H^s(\mathbb{R}^2 \to H^s(\mathbb{R}^2)$ for $|s| < 1/2$.[16]

If we denote by $\widetilde{H}^s(C^a_+), \widetilde{H}^s(\mathbb{R}^2 \setminus C^a_+)$ the Fourier images of spaces $H^s(C^a_+), H^s(\mathbb{R}^2 \setminus C^a_+)$, respectively, then an arbitrary function $\tilde{f} \in \widetilde{H}^s(\mathbb{R}^2$ can be uniquely represented in the form

$$\tilde{f} = \tilde{f}_+ + \tilde{f}_-,$$

where $\tilde{f}_+ \in \widetilde{H}^s(C^a_+), \tilde{f}_- \in \widetilde{H}^s(\mathbb{R}^2 \setminus C^a_+)$ and

$$\tilde{f}_= = G_1 \tilde{f} + (I - G_2) \tilde{f}$$

for $|s| < 1/2$.

2.4 | Transmutation operator

Our further considerations are based on a special transmutation operator which is related to the Fourier transform.

We introduce $T_a : \mathbb{R}^2 \to \mathbb{R}^2$ of the following type:

$$\begin{cases} t_1 = x_1, \\ t_2 = x_2 - a|x_1|. \end{cases}$$

This operator transforms ∂C_+^a into hyperplane $x_2 = 0$.

We are interested in the operator FT_aF^{-1}; therefore, first, we need to study FT_1 and to find its explicit form. We have

$$(FT_a u)(\xi) = \int_{-\infty}^{+\infty} e^{ia|y_1|\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1$$

$$= \int_{-\infty}^{+\infty} \chi_+(y_1) e^{iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1 + \int_{-\infty}^{+\infty} \chi_-(y_1) e^{-iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2) dy_1$$

$$= \int_{-\infty}^{+\infty} \chi_+(y_1) e^{iy_1(a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2) dy_1 + \int_{-\infty}^{+\infty} \chi_-(y_1) e^{iy_1(-a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2) dy_1,$$

where $\chi_\pm$ is an indicator of the half-axis $\mathbb{R}_\pm$.

The last two summands are the Fourier transforms of functions

$$\chi_+(y_1) e^{iy_1(a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2), \qquad \chi_-(y_1) e^{iy_1(-a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2),$$

on the first variable y_1, respectively, so we can use Plemelj-Sokhotskii formulas[31-33] (see also Eskin[3]), and we write them as follows:

$$\int_{-\infty}^{+\infty} \chi_+(x) e^{ix\xi} u(x) dx = \frac{1}{2}\tilde{u}(\xi) + v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta) d\eta}{\xi - \eta},$$

$$\int_{-\infty}^{+\infty} \chi_-(x) e^{ix\xi} u(x) dx = \frac{1}{2}\tilde{u}(\xi) - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta) d\eta}{\xi - \eta},$$

where $v.p.$ denotes principal value of the integral in Cauchy sense.[31]

Thus, we obtain

$$(FT_a u)(\xi) = \frac{\tilde{u}(\xi_1 + a\xi_2, \xi_2) + \tilde{u}(\xi_1 - a\xi_2, \xi_2)}{2}$$

$$+ v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{\xi_1 + a\xi_2 - \eta} - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{\xi_1 - a\xi_2 - \eta}.$$

Let us denote

$$P_1 = \frac{1}{2}(I + S_1), \qquad Q_1 = \frac{1}{2}(I - S_1),$$

where

$$(S_1\tilde{u})(\xi_1, \xi_2) = \frac{i}{\pi} v.p.\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta, \xi_2) d\eta}{\xi_1 - \eta},$$

then we can write

$$(FT_a u)(\xi_1, \xi_2) = (P_1\tilde{u})(\xi_1 + a\xi_2, \xi_2) + (Q_1\tilde{u})(\xi_1 - a\xi_2, \xi_2).$$

Corollary 1. *If*

$$u(x_1, x_2) = \sum_{k=0}^{n-1} c_k(x_1) \delta^{(k)}(x_2),$$

then

$$(FT_a u)(\xi_1, \xi_2) = \sum_{k=0}^{n-1} \xi_2^k \left((P_1\tilde{c}_k)(\xi_1 + a\xi_2) + (Q_1\tilde{c}_k)(\xi_1 - a\xi_2)\right).$$

3 | A GENERAL SOLUTION

If the symbol $A(\xi)$ admits the wave factorization[16] under the condition $1/2 < æ - s < 3/2$, where $æ$ is the index of wave factorization, then one can show[34] that a general solution of the homogeneous Equation (3) in Sobolev-Slobodetskii space $H^s(C_+^a)$ in Fourier image has the following form:

$$\tilde{u}(\xi) = \frac{\tilde{c}_0(\xi_1 + a\xi_2) + \tilde{c}_0(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1, \xi_2)} +$$
$$A_{\neq}^{-1}(\xi_1, \xi_2)\left(v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta)d\eta}{\xi_1 - a\xi_2 - \eta} \right),$$

where c_0 is an arbitrary function from $H^{s-æ+1/2}(\mathbb{R})$.

Here, we will consider Equation (3) for the case $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$ for the cone $\mathbb{R}^2 \setminus C_+^a$.

Theorem 1. *Let the symbol $A(\xi)$ satisfies the condition $(*)$ and admits the wave factorization with respect to the cone C_+^a with the index $æ$, $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. A general solution of Equation (3) in Fourier image is given by the formula*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)(I - G_2)Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{\ell v}(\xi) + A_{\neq}^{-1}(\xi)FT_{-a}F^{-1}\left(\sum_{k=0}^{n-1}\tilde{c}_k(\xi_1)\xi_2^k\right), \tag{4}$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ are arbitrary functions, $s_k = s - æ + k + 1/2, k = 0, 1, 2, \ldots, n-1$, ℓv is an arbitrary continuation of f on $H^{s-\alpha}(\mathbb{R}^m)$, $Q_n(\xi)$ is an arbitrary polynomial satisfying the condition $(*)$ for $\alpha = n$.

Proof. After wave factorization for the symbol with preliminary Fourier transform, we write

$$A_{\neq}(\xi)\tilde{u}(\xi) + A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = A_{=}^{-1}(\xi)\tilde{\ell v}(\xi),$$

where $u_-(x) = \ell v(x) - u(x)$, ℓv is an arbitrary continuation of v on the whole $\mathbb{R}^2$.

One can see that $A_{=}^{-1}(\xi)\tilde{\ell v}(\xi)$ belongs to the space $\tilde{H}^{s-æ}(\mathbb{R}^2)$, and if we choose the polynomial $Q_n(\xi)$, satisfying the condition

$$|Q_n(\xi)| \sim (1 + |\xi|)^n,$$

then $Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{\ell v}(\xi)$ will belong to the space $\tilde{H}^{-\delta}(\mathbb{R}^2)$.

Further, according to the theory of multi-dimensional Riemann problem,[16] we can decompose the last function on two summands (jump problem):

$$Q_n^{-1}A_{=}^{-1}\tilde{\ell v} = f_+ + f_-,$$

where $f_+ \in \tilde{H}(C_+^a)$, $f_- \in \tilde{H}(\mathbb{R}^2 \setminus C_+^a)$, and

$$f_+ = (I - G_2)(Q_n^{-1}A_{=}^{-1}\tilde{\ell v}), \qquad f_- = G_2(Q_n^{-1}A_{=}^{-1}\tilde{\ell v}).$$

Therefore, we obtain

$$Q_n^{-1}A_{\neq}\tilde{u} + Q_n^{-1}A_{=}^{-1}\tilde{u}_- = f_+ + f_-,$$

or

$$Q_n^{-1}A_{\neq}\tilde{u} - f_+ = f_- - Q_n^{-1}A_{=}^{-1}\tilde{u}_-.$$

Rewriting we have

$$A_{\neq}\tilde{u} - Q_n f_+ = Q_n f_- - A_{=}^{-1}\tilde{u}_-.$$

The left-hand side of the equality belongs to the space $\tilde{H}^{-n-\delta}(\mathbb{R}^2 \setminus C_+^a)$, and the right-hand side belongs to $\tilde{H}^{-n-\delta}(C_+^a)$. Hence,

$$F^{-1}(A_{\neq}\tilde{u} - Qf_+) = F^{-1}(Qf_- - A_{=}^{-1}\tilde{u}_-), \tag{5}$$

where the left-hand side belongs to $H^{-n-\delta}(\mathbb{R}^2 \setminus C_+^a)$, and right-hand side belongs to $H^{-1-\delta}(C_+^a)$; therefore, we conclude immediately that this is a distribution supported on ∂C_+^a.

Taking into account a general form for a distribution from $S'(\mathbb{R}^2)$ supported on the straight line $x_2 = 0$[3,30]

$$c(x_1, x_2) = \sum_{k=0}^{m} c_k(x_1)\delta^{(k)}(x_2), \tag{6}$$

we need to apply the transform T_{-a} to the formula (6)to obtain the distribution supported on ∂C_+^a.

The formula (6) in the fourier image looks as follows:

$$\tilde{c}(\xi_1, \xi_2) = \sum_{k=0}^{m} \tilde{c}_k(\xi 1)\xi_2^k.$$

Because such distribution should be belonging to $\tilde{H}^{-n-\delta}(\mathbb{R}^2)$, we need to estimate the integrals

$$\int_{\mathbb{R}^2} |\tilde{c}_k(\xi_1)|^2 |\xi_2|^{2k} (1 + |\xi|)^{2(-n-\delta)} d\xi = \int_{\mathbb{R}^2} |\tilde{c}_k(\xi_1)|^2 |\xi_2|^{2k} (1 + |\xi|)^{2(s-æ)} d\xi \leq$$

$$const \int_{\mathbb{R}^2} |\tilde{c}_k(\xi_1)|^2 (1 + |\xi|)^{2(k+s-æ)} d\xi = const \int_{-\infty}^{+\infty} |\tilde{c}_k(\xi_1)|^2 \left(\int_{-\infty}^{+\infty} (1 + |\xi_1| + |\xi_2|)^{2(k+s-æ)} d\xi_2 \right) d\xi_1.$$

The latter inner integral converges only if

$$2(k + s - æ) < -1. \tag{7}$$

If the condition (7) is valid, then by integrating on ξ_2, we obtain

$$\int_{\mathbb{R}^2} |\tilde{c}_k(\xi_1)|^2 |\xi_2|^{2k} (1 + |\xi|)^{2(-n-\delta)} d\xi \leq const \int_{-\infty}^{+\infty} |\tilde{c}_k(\xi_1)|^2 (1 + |\xi_1|)^{2(k+s-æ+1/2)} d\xi_1,$$

so that $c_k \in H^{k+s-æ+1/2}(\mathbb{R})$. Because $s - æ = -n - \delta$ we see that the condition (7) can be fulfil only for $k = 0, 1, \ldots, n-1$.

Thus, we have exactly n summands in the formula (6), that is, $m = n - 1$.

Now, in equality (5), we will write as follows:

$$F^{-1}(A_{\neq}\tilde{u} - Qf_+) = T_{-a}c.$$

Further, applying the Fourier transform F to both left and hand side of the latter formula, we obtain the formula (4). □

Remark 2. According to Corollary 1, it is obvious that

$$(FT_{-a}c)(\xi_1, \xi_2) = \sum_{k=0}^{n-1} \xi_2^k \left((Q_1\tilde{c}_k)(\xi_1 + a\xi_2) + (P_1\tilde{c}_k)(\xi_1 - a\xi_2) \right).$$

Corollary 2. *If $a \to \infty$ then a general solution of the equation of Equation (1) depends on unique function $c_0(x_1)$.*

Proof. According to Corollary 1, we have

$$(FT_a u)(\xi_1, \xi_2) = \sum_{k=0}^{n-1} \xi_2^k \left((P_1\tilde{c}_k)(\xi_1 + a\xi_2) + (Q_1\tilde{c}_k)(\xi_1 - a\xi_2) \right).$$

Let us make the change of variables

$$\begin{cases} t_1 = \xi_1 + a\xi_2, \\ t_2 = \xi_1 - a\xi_2, \end{cases}$$

Then, we obtain

$$(FT_a u)\left(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a}\right) = P_1\tilde{c}_0)(t_1) + (Q_1\tilde{c}_0)(t_2) + \\ +\sum_{k=1}^{n-1}\left(\frac{t_1-t_2}{2a}\right)^k ((P_1\tilde{c}_k)(t_1) + (Q_1\tilde{c}_k)(t_2)),$$

so we see that under $a \to \infty$ the limit exists for arbitrary fixed collection $\{\tilde{c}_k\}_{k=1}^{n-1}$. □

Therefore, we conclude that for studying the limit boundary value problem under $a \to \infty$ we need to determine only one arbitrary function c_0.

4 | BOUNDARY VALUE PROBLEMS

Let us denote $\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)(I - G_2)Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{\ell}v(\xi) \equiv \tilde{f}$. Then according to Theorem 1 and Remark 2, we have the following formula for a general solution of Equation (3):

$$\tilde{u}(\xi) = \tilde{f}(\xi) + A_{\neq}^{-1}(\xi)\sum_{k=0}^{n-1}\xi_2^k ((Q_1\tilde{c}_k)(\xi_1 + a\xi_2) + (P_1\tilde{c}_k)(\xi_1 - a\xi_2)). \tag{8}$$

Taking into account that the condition (2) in Fourier image takes the form,

$$\tilde{u}(\xi_2, 0) = \tilde{g}(\xi_1),$$

and substituting it into the formula (8), we obtain

$$\tilde{u}(\xi_1, 0) = \tilde{g}(\xi_1) = \tilde{f}(\xi_1, 0) + A_{\neq}^{-1}(\xi_1, 0)\tilde{c}_0(\xi_1).$$

Therefore, we can find $\tilde{c}_0$

$$\tilde{c}_0(\xi_1) = (\tilde{g}(\xi_1) - \tilde{f}(\xi_1, 0))A_{\neq}(\xi_1, 0).$$

4.1 | The case $v \equiv 0$

For this case, the formula (8) reduces to the following:

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\sum_{k=0}^{n-1}\xi_2^k ((Q_1\tilde{c}_k)(\xi_1 + a\xi_2) + (P_1\tilde{c}_k)(\xi_1 - a\xi_2)), \tag{9}$$

and formula for $\tilde{c}_0$ looks as follows:

$$\tilde{c}_0(\xi_1) = \tilde{g}(\xi_1)A_{\neq}(\xi_1, 0).$$

We make change of variables in the formula (9) like the proof of Corollary 2 and obtain

$$\tilde{u}\left(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a}\right) = A_{\neq}^{-1}\left(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a}\right)\sum_{k=0}^{n-1}\left(\frac{t_1-t_2}{2a}\right)^k ((Q_1\tilde{c}_k)(t_1) + (P_1\tilde{c}_k)(t_2)).$$

Then we see that under $a \to \infty$, the following equality,

$$u\left(\frac{t_1+t_2}{2}, 0\right) = A_{\neq}^{-1}\left(\frac{t_1+t_2}{2}, 0\right)((Q_1\tilde{c}_0)(t_1) + (P_1\tilde{c}_0)(t_2)), \tag{10}$$

appears.

5 | SOLVABILITY CONDITION

Now, we are able to make a certain conclusion on solvability of starting boundary value problem (1),(2).

Let us denote

$$\tilde{b}(t) = A_{\neq}(t, 0).$$

Taking into account our additional condition (2), we can write

$$(\tilde{b}\tilde{g})\left(\frac{t_1 + t_2}{2}\right) = (Q_1(\tilde{b}\tilde{g}))(t_1) + (P_1(\tilde{b}\tilde{g}))(t_2). \tag{11}$$

Theorem 2. *Let elliptic symbol $A(\xi)$ admits wave factorization with respect to C_+^a with index æ such that $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$ for all enough large a, and $v \equiv 0, g \in H^{s+1/2}(\mathbb{R})$. Then the limit problem (1),(2) can be solvable in the space $H^s(D)$ if and only if the function g satisfies Equation (11) for all t_1, t_2.*

Proof. We act like previous steps. First, we find a general solution (8) using wave factorization method. Second, we verify that limit of a general solution under $a \to \infty$ includes only one arbitrary function, which can be found using the condition (2). Third, changing variables and passing to limit we obtain Equation (11). □

6 | CONCLUSION

As it was shown for limit boundary value problem, the value of index of wave factorization does not play such important role in comparison with standard case of a cone. Although there are a lot of solutions to preliminary boundary value problem, we have only one limit solution, and the solvability condition for considered limit boundary value problem is the same like the case $n = 1$. Maybe if we will consider other types of additional conditions to determine arbitrary functions in a general solution, we will not find such a phenomenon.

CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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