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On the Global-in-Time Existence of a Generalized Solution to a Free-Boundary Problem

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Abstract—A problem with free (unknown) boundary for a one-dimensional diffusion-convection equation is considered. The unknown boundary is found from an additional condition on the free boundary. By the extension of the variables, the problem in an unknown domain is reduced to an initial boundary-value problem for a strictly parabolic equation with unknown coefficients in a known domain. These coefficients are found from an additional boundary condition that enables the construction of a nonlinear operator whose fixed points determine a solution of the original problem.

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1. INTRODUCTION

Problems with free (unknown) boundary for differential equations belong to the most complicated problems in the theory of partial differential equations. In these problems, it is necessary to not only solve a differential equation but also determine a domain in which the solution is sought. As a rule, this domain (the boundary) is determined from an additional boundary condition on the free boundary. In the theory of free boundary problems, the following two problems are well known: the Stefan problem [1], [2] and Hele–Show problem [3] for the heat equation and the Laplace equations, respectively. These problems are fairly easy to formulate, but, up to now, the existence of a classical solution has only been proved for small times (except in certain simple cases). As to systems of differential equations, we note the papers of Solonnikov and his colleagues on free-boundary problems for the system of Navier–Stokes equations [4]–[6], as well as those of Friedman [7]. But, just as in the case of scalar equations, only the existence of a classical solution for small times has been proved.

There is also a separate large class of free-boundary problems for equations of gas dynamics and the hydrodynamics of an ideal incompressible liquid. These problems are well studied and have a long history [8]–[10]. A typical problem of this class is the Cauchy–Poisson problem for waves on the surface of an ideal incompressible liquid. In these problems, either approximate solutions (in "shallow water") or approximately exact solutions are studied.

We consider the mathematical model formulated in [3] and describing the underground leaching of uranium by an acid solution. In the corresponding initial free boundary-value problem, the dynamics of the carrier liquid in an unknown domain $\Omega_f(t)$ obeys the linear Stokes equations

$$\alpha_\mu \Delta \boldsymbol{v} - \nabla p = \varrho_f \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{v} = 0. \tag{1.1}$$

Here $\boldsymbol{v}$ is the velocity of the liquid, p is the pressure in the liquid, ϱ_f is the dimensionless density of the liquid relateive to that of water in natural conditions, $\boldsymbol{f}$ is the dimensionless vector of given mass forces, $\alpha_\mu = \mu/(TLg\rho^0) = \mu_1\varepsilon^2$ is the dimensionless viscosity of the liquid, μ is the viscosity of the liquid, $\mu_1 = \text{const}$, $\mu_1 > 0$, T is the characteristic time of the physical process, L is the characteristic

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size of the physical domain, g is the acceleration of gravity, and ρ^0 is the density of water in natural conditions.

The diffusion and convection of the acid concentration c in $\Omega_f(t)$ is described by the diffusion-convection equations

$$\frac{\partial c}{\partial t} + \boldsymbol{v} \cdot \nabla c = \alpha_c \Delta c, \tag{1.2}$$

where $\alpha_c = DT/L^2$ is the dimensionless diffusion coefficient of the acid and D is the diffusion coefficient of the acid.

Equations (1.1) and (1.2) are conventional, as well as the mass conservation law

$$v_n = -d_n \sigma, \qquad \sigma = \frac{(\varrho_s - \varrho_f)}{\varrho_f}, \tag{1.3}$$

on the free boundary $\Gamma(t)$ [8].

Here v_n is the normal velocity of the liquid in the direction of the outer unit normal $\boldsymbol{n}$ to the domain $\Omega_f(t)$, d_n is the normal velocity of the free boundary, and ϱ_s is the dimensionless density of the solid skeleton.

The condition of the equality to zero of the tangential component of the velocity on the free boundary is also quite natural.

In addition, on the free boundary,

$$(d_n - v_n)c + \alpha \frac{\partial c}{\partial n} = -\beta c, \tag{1.4}$$

$$d_n = \beta \gamma c. \tag{1.5}$$

Problem (1.1)–(1.5) is closed by boundary conditions on the given boundary of the domain $\Omega_f(t)$ and by initial conditions.

The resulting mathematical model is strongly nonlinear; in our view, its well-posedness can be proved (if at all) only either for small times or in the case of one spatial variable. The latter case will be the subject of our consideration.

The scheme of proof is standard; it is based on the Schauder fixed-point theorem [11]. First, we fix a set $\mathfrak{M}$ of functions $X(t)$ defining the free boundary and then solve the linear problem of finding a solution $c(x,t)$ of problem (1.1)–(1.4) with corresponding initial conditions and boundary conditions on a given boundary. Further, from the boundary condition (1.5), we recover a function

$$Y(t) = X_0 + \beta\gamma \int_0^t c(X(\tau), \tau)\, d\tau, \qquad Y = \Phi(X).$$

We show that the operator Φ is completely continuous and takes the set $\mathfrak{M}$ to itself. The application of the Schauder fixed-point theorem concludes the proof of the theorem.

2. MAIN RESULTS

In the dimensionless variables

$$x \to \frac{x}{L}, \qquad t \to \frac{t}{T}, \qquad v \to \frac{T}{L} v, \qquad p \to p^* p,$$

the behavior of an incompressible liquid in the domain

$$Q_X(t) = \bigcup_{\tau=0}^{t} \Omega_X(\tau), \qquad \Omega_X(t) = \{x : 0 < x < X(t) < 1\},$$

is described by the system of differential equations

$$\frac{\partial p}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\alpha\frac{\partial c}{\partial x} - vc\right) \tag{2.3}$$

for the pressure p, the velocity v of the liquid, and the concentration c of the acid. The differential equations are supplemented with the initial and boundary conditions

$$c = c^0(t), \qquad x = 0, \quad 0 < t < 1, \tag{2.4}$$

$$\alpha\frac{\partial c}{\partial x} + \left(\frac{dX}{dt} - v\right)c = -\beta c, \qquad x = X(t), \quad 0 < t < 1, \tag{2.5}$$

$$\frac{dX}{dt} = \beta\gamma c, \qquad x = X(t), \quad 0 < t < 1, \tag{2.6}$$

$$v(t) = -\sigma\frac{dX}{dt}(t), \qquad 0 < t < 1, \tag{2.7}$$

$$c(x,0) = c_0(x), \qquad 0 < x < X_0, \quad X(0) = X_0. \tag{2.8}$$

In (2.1)–(2.8),

$$\alpha = \frac{DT}{L^2}, \qquad \sigma = \frac{(\rho_s - \rho_f)}{\rho_f},$$

where L is the characteristic size of the domain, T is the characteristic time of the process, ρ_f is the density of the liquid, ρ_s is of the density of the rocks, D is the diffusion coefficient of the acid, and β and γ are given positive constants.

For simplicity, we assume that $c^0(t) \equiv 0$.

To determine a generalized solution of problem (2.1)–(2.8), it is necessary to rewrite the original problem in the form of differential equations in the equivalent form of integral identities supplemented with the corresponding boundary and initial conditions.

First, using equalities (2.6) and (2.7), we change the boundary condition (2.5):

$$\alpha\frac{\partial c}{\partial x} + \left(\frac{dX}{dt} - v\right)c = -\beta c = -\frac{1}{\gamma}\frac{dX}{dt} = \frac{1}{\gamma\sigma}v.$$

By their physical meaning, the concentration $c(x,t)$ of the acid and the velocity $(dX/dt)(t)$ of the variation of the boundary $x = X(t)$ are nonnegative. Unfortunately, this does not follow directly from the auxiliary linear problem (i.e., from the original problem without the boundary condition (2.6)), even if we assume that $(dX/dt)(t) \ge 0$.

To prove the nonnegativity of the acid concentration, instead of the last condition, we consider the modified condition

$$\alpha\frac{\partial c}{\partial x} - v\left(c + \frac{\operatorname{sgn} c}{\gamma\sigma}\right) = -\frac{dX}{dt}c, \qquad x = X(t), \quad 0 < t < 1. \tag{2.9}$$

The boundary condition (2.9) will coincide with the boundary condition (2.5) if we show that the solution $c(x,t)$ of the linear problem (2.4), (2.7)–(2.9) in the domain $Q_X(1)$ with a given boundary $x = X(t)$ such that $(dX/dt)(t) \ge 0$ is nonnegative.

To do this, let us rewrite Eq. (2.3) in the equivalent form

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\alpha\frac{\partial c}{\partial x} - v\left(c + \frac{\operatorname{sgn} c}{\gamma\sigma}\right)\right), \qquad 0 < x < X(t), \quad 0 < t < 1. \tag{2.10}$$

We now multiply the diffusion-convection equation (2.10) by an arbitrary smooth function ξ equal to zero at $x = 0$ and then integrate the resulting equality by parts over the domain $Q_X(t)$, using the boundary condition (2.9):

$$\int_{\Omega_X(t)} c(x,t)\xi(x,t)\,dx + \int_0^t \int_{\Omega_X(\tau)} \biggl(-c\frac{\partial \xi}{\partial \tau} + \biggl(\alpha\,\frac{\partial c}{\partial x} - v\biggl(c + \frac{\operatorname{sgn} c}{\gamma\,\sigma}\biggr)\biggr)\frac{\partial \xi}{\partial x}\biggr)\,dx\,d\tau$$
$$= \int_{\Omega_X(0)} c_0(x)\,\xi(x,0)\,dx. \tag{2.11}$$

Identity (2.11) with sufficiently smooth functions contains the differential equation (2.10) and the boundary condition (2.9).

To verify identities (2.11), it suffices to return to the original differential equation (2.10), using the Stokes theorem in the form

$$\iint_{Q_X(t)} \xi\biggl(\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial x}\biggr)\,dx\,d\tau$$
$$= \int_{\Omega_X(\tau)} \xi\, A|_{\tau=0}^{\tau=t}\,dx - \iint_{Q_X(t)} \biggl(A\frac{\partial \xi}{\partial \tau} + B\frac{\partial \xi}{\partial x}\biggr)\,dx\,d\tau + \int_0^t \widetilde{\xi}\biggl(\widetilde{B} - \widetilde{A}\,\frac{dX}{d\tau}\biggr)\,d\tau$$
$$= \int_{\Omega_X(t)} \xi(x,t)\,A(x,t)\,dx - \int_{\Omega_X(0)} \xi(x,0)\,A(x,0)\,dx - \iint_{Q_X(t)} \biggl(A\frac{\partial \xi}{\partial \tau} + B\frac{\partial \xi}{\partial x}\biggr)\,dx\,d\tau \tag{2.12}$$

and the corresponding boundary condition on the free boundary $x = X(t)$.

In (2.12), we take

$$A = -c, \qquad B = \alpha\frac{\partial c}{\partial x} - v\biggl(c + \frac{\operatorname{sgn} c}{\gamma\sigma}\biggr), \qquad \widetilde{\varphi}(t) = \varphi(X(t),t).$$

Definition 1. The functions $\{c(x,t), X(t)\}$ are called a *generalized solution of problem* (2.1)–(2.7) if

$$c \in \mathbb{W}_2^{1,0}(Q_X(1)), \qquad X \in \mathbb{W}_\infty^1(0,1), \qquad \frac{dX}{dt}(t) \ge 0 \quad \text{a.e. for} \quad 0 < t < 1$$

and the integral identity (2.11), Eqs. (2.6) and (2.7), and the boundary condition (2.4) hold.

Definition 2. A function $c(x,t) \in \mathbb{W}_2^{1,0}(Q_X(1))$ is called a *generalized solution of problem* (2.4)*,* (2.7)–(2.10) for a given function

$$X \in \mathbb{W}_\infty^{1,0}(0,1), \qquad \frac{dX}{dt}(t) \ge 0$$

if the integral identity (2.11), Eq. (2.7), the boundary condition (2.4), and the initial condition (2.8) hold.

In the present paper, we use the notation of [1] for function spaces and norms on these spaces.

Lemma 1. *Let $c_0 \in \mathbb{L}_\infty(\Omega_X(0))$, and let $0 \le c_0(x) \le M_1$ almost everywhere in $\Omega_X(0)$. Then, for any function $X \in \mathbb{W}_\infty^1(0,1)$ such that $dX/dt \ge 0$, there exists a unique generalized solution of problem* (2.4)*,* (2.7)–(2.10).

Lemma 2. *Under the assumptions of Lemma* 1*, the following estimate holds*:

$$\operatorname{vrai} \min_{(x,t)\in Q_X(1)} c(x,t) \ge 0. \tag{2.13}$$

Lemma 3. *Under the assumptions of Lemma* 1*, the following estimate holds*:

$$\operatorname{vrai} \max_{(x,t)\in Q_X(1)} c(x,t) \le \operatorname{vrai} \max_{x\in\Omega_X(0)} c_0(x) = M_1. \tag{2.14}$$

Theorem 1. *Under the assumptions of Lemma* 1*, let $c_0 \in \mathbb{W}_2^1(\Omega_X(0))$. Then problem* (2.1)–(2.8) *has at least one generalized solution.*

3. PROOF OF LEMMA 1

The proof of this lemma is standard, because the problem possesses the following a priori estimate:

$$\sup_{0<t<1}\int_{\Omega_X(t)} c^2(x,t)\,dx \le M_0^2 = \int_{\Omega_X(0)} c_0^2(x)\,dx \tag{3.1}$$

with constant M_0 depending only on the data of the problem; this estimate follows from the equality

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_X(t)} c^2(x,t)\,dx + \alpha\int_{\Omega_X(t)}\left|\frac{\partial c}{\partial x}(x,t)\right|^2 dx + \frac{dX}{dt}(t)\left(\left(\frac{1}{2}+\sigma\right)\widetilde{c}^{\,2}(t) + \frac{1}{\gamma}|\widetilde{c}(t)|\right) = 0. \tag{3.2}$$

It will be shown in Sec. 5 how to obtain equalities similar to equality (3.2).

4. PROOF OF LEMMA 2

To prove (2.13), we set $w(x,t) = \sup\{0, -c(x,t)\}$,

$$w(0,t) = 0, \quad 0<t<1, \qquad w(x,0) = 0, \quad 0<x<1, \tag{4.1}$$

$$w = \frac{\partial w}{\partial x} = 0, \quad \text{if} \quad c \ge 0, \qquad w = -c, \quad \frac{\partial w}{\partial x} = -\frac{\partial c}{\partial x}, \quad \text{if} \quad c<0. \tag{4.2}$$

Besides,

$$wc = w^2, \qquad w\frac{\partial c}{\partial x} = w\frac{\partial w}{\partial x}, \qquad c\frac{\partial w}{\partial x} = w\frac{\partial w}{\partial x} \tag{4.3}$$

almost everywhere in Q_X.

Further, we consider the Steklov averagings

$$\xi_{(h)}(x,t) = \frac{1}{h}\int_t^{t+h}\xi(x,s)\,ds, \qquad \xi_{(\overline{h})}(x,t) = \frac{1}{h}\int_{t-h}^{t}\xi(x,s)\,ds$$

of the function ξ over the variable t.

We assume that all the functions under consideration are identically zero for $t<0$. Then

$$\int_0^t \xi_{(h)}(x,t)c(x,t)\,dt = \int_0^t \xi(x,t)c_{(\overline{h})}(x,t)\,dt, \qquad \frac{\partial \xi_{(h)}}{\partial t} = \left(\frac{\partial \xi}{\partial t}\right)_{(h)}$$

and, in view of the properties of averagings [1, Lemma 4.7, Sec. 4, Chap. 2], the integrated (over time) identity (2.11) for the test function $\xi = \zeta_{(\overline{h})}(x,t)$ with nonpositive functions ξ and ζ will take the form

$$\begin{aligned}\int_0^1\int_{\Omega_X(t)} & c_{(h)}(x,t)\zeta(x,t)\,dx\,dt \\ &+ \int_0^1\int_0^t\int_{\Omega_X(\tau)}\left(-c_{(h)}\frac{\partial \zeta}{\partial \tau} + \left(\alpha\frac{\partial c}{\partial x} - v\left(c + \frac{\operatorname{sgn} c}{\gamma\sigma}\right)\right)_{(h)}\frac{\partial \zeta}{\partial x}\right)dx\,d\tau\,dt \\ =& \int_{\Omega_X(0)} c_0(x)\xi(x,0)\,dx \le 0. \end{aligned}\tag{4.4}$$

Now, for the test function $\zeta(x,t)$ in (4.4) we will choose the function

$$\zeta = w_h^{(0)}(x,t) = (-c_{(h)})^{(0)}(x,t) = \max\{-c_{(h)}, 0\},$$

where

$$u^{(k)}(x,t) = \max\{u(x,t), k\}, \qquad w_{(h)}^{(0)}(x,0) = 0, \quad w_{(h)}^{(0)}(0,t) = 0.$$

By analogy with (4.3), we have

$$c_{(h)}(x,t)\frac{\partial w_{(h)}^{(0)}}{\partial t}(x,t) = w_h^{(0)}(x,t)\frac{\partial w_{(h)}^{(0)}}{\partial t}(x,t)$$

and, therefore,

$$\int_0^t d\tau \int_{\Omega_X(\tau)} c_h(x,\tau)\frac{\partial w^{(0)}_{(h)}}{\partial \tau}(x,\tau)\,dx = \int_0^t d\tau \int_{\Omega_X(\tau)} w^{(0)}_h(x,\tau)\frac{\partial w^{(0)}_h}{\partial \tau}(x,\tau)\,dx$$
$$= \frac{1}{2}\int_{\Omega_X(t)} (w^{(0)}_h(x,t))^2\,dx - \frac{1}{2}\int_0^t (\widetilde{w}^{(0)}_h(\tau))^2\frac{dX}{dt}(\tau)\,d\tau.$$

Thus, (4.4) can be rewritten as

$$\int_0^1\int_{\Omega_X(t)} c(x,t)(w^{(0)}_h)_{(\overline{h})}(x,t)\,dx\,dt$$
$$-\frac{1}{2}\int_0^1\int_{\Omega_X(t)}(w^{(0)}_h(x,t))^2\,dx\,dt + \frac{1}{2}\int_0^1\int_0^t(\widetilde{w}^{(0)}_h(\tau))^2\frac{dX}{dt}(\tau)\,d\tau\,dt$$
$$+\int_0^1\int_0^t\int_{\Omega_X(\tau)}\biggl(\biggl(\alpha\frac{\partial c}{\partial x} - v\biggl(c+\frac{\operatorname{sgn} c}{\gamma\sigma}\biggr)\biggr)\frac{\partial (w^{(0)}_h)_{(h)}}{\partial x}\biggr)\,dx\,d\tau\,dt = 0. \tag{4.5}$$

Passing to the limit as $h\to 0$ in (4.5), we obtain the equalities

$$0 = \frac{1}{2}\int_0^1\int_{\Omega_X(t)}(w^{(0)}(x,t))^2\,dx\,dt + \frac{1}{2}\int_0^1\int_0^t(\widetilde{w}^{(0)}(\tau))^2\frac{dX}{dt}(\tau)\,d\tau\,dt$$
$$+\int_0^1\int_0^t\int_{\Omega_X(\tau)}\biggl(\biggl(\alpha\frac{\partial c}{\partial x} - v\biggl(c+\frac{\operatorname{sgn} c}{\gamma\sigma}\biggr)\biggr)\frac{\partial w^{(0)}}{\partial x}\biggr)\,dx\,d\tau\,dt$$
$$= \frac{1}{2}\int_0^1\int_{\Omega_X(t)}(w^{(0)}(x,t))^2\,dx\,dt + \alpha\int_0^1\int_0^t\int_{\Omega_X(\tau)}\biggl(\frac{\partial w^{(0)}}{\partial x}\biggr)^2\,dx\,d\tau\,dt$$
$$+\int_0^1\int_0^t\frac{dX}{dt}(\tau)\biggl(\frac{1}{2}+\sigma(\widetilde{w}^{(0)}(\tau)\biggr)^2+\frac{1}{\gamma}|\widetilde{w}^{(0)}(\tau)|\biggr)\,d\tau\,dt = 0. \tag{4.6}$$

Therefore,

$$w^{(0)}(x,t) = \max\{0, -c(x,t)\} = 0 \qquad \text{a.e. in} \quad Q_X(1),$$
$$c(x,t)\ge 0 \qquad \text{a.e. in} \quad Q_X(1). \tag{4.7}$$

5. PROOF OF LEMMA 3

The proof of Lemma 3 repeats, except for small changes, the proof of Lemma 2.

Here for the test function ξ in identity (2.11) integrated over time we must take the function

$$\xi(x,t) = w^{(k)}_h(x,t) = \max\{c^{(k)}_{(h)}(x,t)-k, 0\}, \qquad \text{where} \quad w^{(k)}_h(x,0)=0, \quad w^{(k)}_h(0,t)=0,$$

for

$$k > M_1 = \operatorname{vrai}\max_{x\in\Omega_f(0)} c_0(x). \quad \square$$

6. PROOF OF THEOREM 1

6.1. Definition of the Domain of the Operator Φ

The estimates (2.13) and (2.14) show that, for any monotonically increasing function $X(t)$,

$$0\le c(x,t)\le M_1 \qquad \text{a.e. in the domain} \quad Q_X(1). \tag{6.1}$$

In other words, for any function $X(t)$ from the set

$$\mathfrak{M} = \biggl\{X\in \mathbb{W}^1_\infty(0,1) : 0\le \frac{dX}{dt}(t)\le M_1 \text{ a.e. for } 0<t<1\biggr\}, \tag{6.2}$$

the following inequalities always hold:

$$0 \le \frac{dY}{dt}(t) = \widetilde{c}(t) \le M_1 \qquad \text{a.e. for} \quad 0 < t < 1, \quad Y = \Phi(X). \tag{6.3}$$

Thus, the operator Φ takes the set $\mathfrak{M}$ to itself. The fixed points of this operator define a solution of the original problem (2.1)–(2.8).

6.2. Continuity of the Operator Φ

The continuity of the operator Φ follows from the continuous dependence of the solutions of the parabolic equation (2.3) on the coefficient $v = -\sigma\, dX/dt$ of this equation and on the position of the boundary $x = X(t)$ of the domain $\Omega_X(t)$. This obvious fact is well known, but it is impossible to give any specific reference, perhaps, because of the simplicity of the result.

Our case is a little more complicated, because the function $X(t)$ not only appears in the coefficients of the equation and the boundary condition but also defines the domain in which the solution is sought.

To prove this assertion, we consider $X^{(1)}, X^{(2)} \in \mathfrak{M}$, and let $c^{(1)}$ and $c^{(2)}$ be generalized solutions of problem (2.1)–(2.5), (2.7), (2.8) in the domains $Q_{X^{(1)}}(1)$ and $Q_{X^{(2)}}(1)$ corresponding to $X^{(1)}$ and $X^{(2)}$. To estimate the difference $c = c^{(1)} - c^{(2)}$, we must first consider these problems in one of the domains, for example, in $Q = Q_{X^{(1)}}(1)$.

To this end, we change the variables:

$$\begin{aligned} x &= y, \quad t = t \quad \text{in the domains } Q_{X^{(1)}}(1), \\ y &= zx, \quad t = t \quad \text{in the domains } Q_{X^{(2)}}(1), \end{aligned} \qquad z = \frac{X^1(t)}{X^2(t)}.$$

In the domain Q, the functions $u^{(1)}(y,t) = c^{(1)}(x,t)$, $u^{(2)}(y,t) = c^{(2)}(x,t)$ satisfy the differential equations

$$\begin{aligned} \frac{\partial u^{(1)}}{\partial t} &= \frac{\partial}{\partial y}\left(\alpha \frac{\partial u^{(1)}}{\partial y} + \sigma \frac{dX^{(1)}}{dt} u^{(1)}\right), \\ \frac{\partial u^{(2)}}{\partial t} &= \frac{\partial}{\partial y}\left(\alpha \frac{\partial u^{(2)}}{\partial y} + z\sigma \frac{dX^{(2)}}{dt} u^{(2)} + (1 - z^2)\alpha \frac{\partial u^{(2)}}{\partial y}\right) - \frac{y}{z}\frac{dz}{dt}\frac{\partial u^{(2)}}{\partial y} \end{aligned}$$

and the boundary conditions

$$\begin{aligned} \alpha \frac{\partial u^{(1)}}{\partial y} &= -(1+\sigma)\frac{dX^{(1)}}{dt}\widetilde{u}^{(1)} - \frac{1}{\gamma}\frac{dX^{(1)}}{dt}, \\ \alpha \frac{\partial u^{(2)}}{\partial y} &= -\left(\frac{1+\sigma}{z}\right)\frac{dX^{(2)}}{dt}\widetilde{u}^{(2)} - \frac{1}{\gamma z}\frac{dX^{(2)}}{dt} \end{aligned}$$

on the boundary $y = X^{(1)}$.

In addition,

$$u^{(1)}(0,t) = u^{(2)}(0,t) = 0, \quad 0 < t < 1, \qquad u^{(1)}(y,0) = u^{(2)}(y,0), \quad 0 < y < X_0.$$

Thus, for the difference $\{u = u^{(1)} - u^{(2)},\ X = X^{(1)} - X^{(2)}\}$, we obtain the initial boundary-value problem consisting of the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial y}\left(\alpha \frac{\partial u}{\partial y} + +\sigma \frac{dX^{(1)}}{dt} u\right) = \frac{\partial F}{\partial y} + F_0 \tag{6.4}$$

in the domain Q, the boundary condition

$$\alpha \frac{\partial u}{\partial y} + \sigma \frac{dX^{(1)}}{dt}\widetilde{u} = -\frac{dX^{(1)}}{dt}\widetilde{u} + f \tag{6.5}$$

on the boundary $y = X^{(1)}$, and the boundary and initial conditions

$$u(0,t) = 0, \quad 0 < t < 1, \qquad u(y,0) = 0, \quad 0 < y < X_0, \tag{6.6}$$

respectively, on the boundary $y = 0$ and at the initial instant of time $t = 0$.

In (6.4) and (6.5),

$$F = \sigma u^{(2)} \frac{dX}{dt} + \alpha(z^2 - 1)\frac{\partial u^{(2)}}{\partial y}, \qquad F_0 = \left(\sigma(z-1)\frac{dX^{(2)}}{dt} + \frac{y}{z}\frac{dz}{dt}\right)\frac{\partial u^{(2)}}{\partial y},$$
$$f = -\left(\frac{1}{\gamma} + (1+\sigma)\widetilde{u}^{(2)}\right)\frac{dX}{dt} + \left(\frac{1}{z} - 1\right)\frac{dX^{(2)}}{dt}\left(\frac{1}{\gamma} + (1+\sigma)\widetilde{u}^{(2)}\right).$$

If we now multiply Eq. (6.4) by $u(y,t)$ and integrate by parts over the domain Q, taking into account the boundary and initial conditions (6.5) and (6.6), then, after a few manipulations, we obtain the following chain of inequalities:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_{X^{(1)}}(t)} u^2(y,t)\,dy + \int_{\Omega_{X^{(1)}}(t)} \alpha\left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy + \frac{1}{2}(\sigma+1)\frac{dX^{(1)}}{dt}\,\widetilde{u}^2(t)\\
&\quad\le \widetilde{u}(t)\widetilde{f}(t) + \int_{\Omega_{X^{(1)}}(t)}\left(-F(y,t)\frac{\partial u}{\partial y}(y,t) + F_0(y,t)u(y,t)\right)dy\\
&\quad\le \frac{1}{2}\widetilde{u}^2(t) + \frac{1}{2}\int_{\Omega_{X^{(1)}}(t)} u^2(y,t)\,dy + \varepsilon\int_{\Omega_{X^{(1)}}(t)}\left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\\
&\qquad + \frac{1}{2}\widetilde{f}^2(t) + \int_{\Omega_{X^{(1)}}(t)}\left(\frac{1}{2}|F_0(y,t)|^2 + \frac{1}{4\varepsilon}|F(y,t)|^2\right)dy\\
&\quad\le \left(\frac{1}{2} + \frac{1}{\varepsilon_0}\right)\int_{\Omega_{X^{(1)}}(t)} u^2(y,t)\,dy + (\varepsilon+\varepsilon_0)\int_{\Omega_{X^{(1)}}(t)}\left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\\
&\qquad + C_2(\varepsilon)\left(|X(t)|^2 + \left|\frac{dX}{dt}(t)\right|^2\right).
\end{aligned}$$

Here we have used the simplest interpolation inequality [1, Theorem 2.2, Sec. 2, Chap. 2]

$$\widetilde{u}^2(t) = 2\int_{\Omega_{X^{(1)}}} u(y,t)\frac{\partial u}{\partial y}(y,t)\,dy \le \frac{1}{\varepsilon_0}\int_{\Omega_{X^{(1)}}} u^2(y,t)\,dy + \varepsilon_0\int_{\Omega_{X^{(1)}}}\left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy$$

and the properties of the functions F_0, F, and f:

$$\int_0^1 \widetilde{f}^2(t)\,dt + \int_0^1 dt\int_{\Omega_{X^{(1)}}}(|F_0(y,t)|^2 + |F(y,t)|^2)\,dy \le C\int_0^1\left(|X(t)|^2 + \left|\frac{dX}{dt}(t)\right|^2\right)dt.$$

Choosing $\varepsilon_0 + \varepsilon = \alpha/2$ and using Gronwall's inequality [1, Lemma 5.5, Sec. 2, Chap. 2], we obtain the necessary estimate

$$\int_{\Omega_{X^{(1)}}} u^2(y,t)\,dy + \int_0^t\int_{\Omega_{X^{(1)}(\tau)}}\left|\frac{\partial u}{\partial y}(y,\tau)\right|^2 dy\,d\tau \le C\left(|X(t)|^2 + \left|\frac{dX}{dt}(t)\right|^2\right), \qquad (6.7)$$

which yields the required statement.

6.3. Complete Continuity of the Operator Φ

To use Schauder's theorem [11] and prove the existence of at least one fixed point of the operator Φ, we must show that this operator is compact. The simplest way to do this is to make the change of variables

$$t = t, \quad y = \frac{x}{X(t)} \qquad \text{in the domain} \quad Q_X(1),$$

which reduces problem (2.4), (2.7)–(2.10) in the domain $Q_X(1)$ to the integral identity

$$\begin{aligned}\int_0^1 u(y,t)\eta(y,t)\,dy &+ \int_0^t\int_0^1\Big(-u\frac{\partial\eta}{\partial\tau}+\Big(\frac{\alpha}{X^2}\frac{\partial u}{\partial y}+\frac{1}{X}\frac{dX}{dt}(y+\sigma)u+\frac{1}{\gamma}\Big)\Big)\frac{\partial\eta}{\partial y}\Big)\,dy\,d\tau\\ &= \int_0^1 u(y,t)\eta(y,t)\,dy+\int_0^t\int_0^1\Big(-u\frac{\partial\eta}{\partial\tau}+\Big(a_{11}\frac{\partial u}{\partial y}+a_1u+f_1\Big)\Big)\frac{\partial\eta}{\partial y}\Big)\,dy\,d\tau\\ &= \int_0^1 u_0(y)\xi(y,0)\,dy \end{aligned}\tag{6.8}$$

for the function $u(y,t)=X(t)c(x,t)$ in the square $Q_{0,1}$, where

$$Q_{a,b}=\Omega\times(a,b)=\{(y,t):0<y<1,\ a<t<b\},\qquad \Omega=\{y:0<y<1\};$$

this integral identity will be valid for any smooth function $\eta(y,t)$ vanishing at $y=1$.

In (6.8), we set

$$a_{11}=\frac{\alpha}{X^2(t)},\qquad a_1=\frac{1}{X}\frac{dX}{dt}(t)(y+\sigma),\qquad f_1=\frac{1}{\gamma}.$$

By construction,

$$u(0,t)=0,\qquad 0<t<1.\tag{6.9}$$

Obviously, the complete continuity of the operator Φ will follow from the Hölder continuity of the function $u(y,t)$. After the corresponding integral identity is obtained, the proof of this fact repeats that of the estimate of the Hölder constant in [1, Sec. 2, Chap. 3]. Without going into details, we repeat the scheme of proof.

In identity (6.8), we consider the test function $\eta=\widehat{\eta}_{(\overline{h})}$, where $\widehat{\eta}$ vanishes for $t\le 0$ and $1-h\le t\le 1$ and is equal to a function $\eta(y,t)$ from $\mathbb{W}_2^{1,1}(Q_{-h,1})$ that vanishes at $y=0$. We have

$$\begin{aligned}0&=\int_0^{1-h}\int_0^1\Big(-u_{(h)}\frac{\partial\eta}{\partial t}+\Big(a_{11}\frac{\partial u}{\partial y}+a_1u+f_1\Big)\Big)_{(h)}\frac{\partial\eta}{\partial y}\Big)\,dy\,dt\\ &=\int_0^{1-h}\int_0^1\Big(\frac{\partial u_{(h)}}{\partial t}\eta+\Big(a_{11}\frac{\partial u}{\partial y}+a_1u+f_1\Big)\Big)_{(h)}\frac{\partial\eta}{\partial y}\Big)\,dy\,dt=0.\end{aligned}\tag{6.10}$$

Let $\chi_k(\tau)$ denote the following continuous piecewise linear functions:

$$\chi_k(\tau)=\begin{cases}0 & \text{for } \tau\le t_0-\dfrac{1}{k},\\ k(\tau-t_0)+1 & \text{for } t_0-\dfrac{1}{k}\le\tau\le t_0,\\ 1 & \text{for } t_0\le\tau\le t,\\ k(t-\tau)+1 & \text{for } t\le\tau\le t+\dfrac{1}{k},\\ 0 & \text{for } \tau\ge t+\dfrac{1}{k},\end{cases}\qquad 0\le t_0<t\le 1-h.$$

For $\eta=\chi_k(\tau)\zeta(y,\tau)$, in identity (6.10), we shall pass to the limit as $k\to\infty$, obtaining

$$\int_{t_0}^t\int_\Omega\Big(\frac{\partial u_{(h)}}{\partial\tau}\zeta+\Big(a_{11}\frac{\partial u}{\partial y}+a_1u+f_1\Big)\Big)_{(h)}\frac{\partial\zeta}{\partial y}\Big)\,dy\,d\tau=0.\tag{6.11}$$

Further, we take an arbitrary cylinder

$$Q(\varrho,\delta)=K_\varrho\times(t_0,t_0+\delta),\qquad K_\varrho=\{y:1-\varrho<y\le 1\}$$

and, in (6.11), we set

$$\zeta(y,\tau)=\xi^2(y,\tau)u^{(k)}_{(h)}(y,\tau)=\xi^2(y,\tau)\max\{u_{(h)}(y,\tau)-k,0\},$$

where $\xi(y,\tau)$ is a nonnegative continuous piecewise smooth function not exceeding 1 and equal to zero on the lateral surface $Q(\varrho,\delta)$ and outside the cylinder.

After standard manipulations and the passage to the limit as $h\to 0$, we obtain the equality

$$\frac{1}{2}\|u^{(k)}(y,t)\xi(y,t)(y,t)\|^2_{K_\varrho}-\frac{1}{2}\|u^{(k)}(y,t_0)\xi(y,t_0)\|^2_{K_\varrho}-\int_{t_0}^{t}\int_{A_{k,\varrho}(t)}(u^{(k)})^2\xi\frac{\partial\xi}{\partial\tau}\,dy\,d\tau$$
$$+\int_{t_0}^{t}\int_{A_{k,\varrho}(t)}\biggl(\biggl(a_{11}\frac{\partial u^{(k)}}{\partial y}+a_1u+f_1\biggr)\biggl(\xi^2\frac{\partial u^{(k)}}{\partial y}+2u^{(k)}\xi\frac{\partial\xi}{\partial y}\biggr)\biggr)\,dy\,d\tau=0, \tag{6.12}$$

where

$$u^{(k)}(y,\tau)=\max\{u(y,\tau)-k,0\},\qquad t_0<\tau<t<\tau<t_0+\delta,$$
$$A_{k,\varrho}(t)=\{y\in K_\varrho: u(y,t)>k\}.$$

Since the functions $u(y,t)$, a_1, and f_1 are bounded, it follows that

$$\frac{1}{2}\|u^{(k)}(y,t)\xi(y,t)\|^2_{K_\varrho}+\frac{\alpha}{X_0^2}\int_{t_0}^{t}\int_{K_\varrho}\biggl(\frac{\partial u^{(k)}}{\partial y}\biggr)^2\xi^2\,dy\,d\tau$$
$$\le\frac{1}{2}\|u^{(k)}(y,t_0)\xi(y,t_0)\|^2_{K_\varrho}+C\int_{t_0}^{t}\int_{K_\varrho}\biggl(\biggl(\frac{\partial\xi}{\partial y}\biggr)^2+\xi\biggl|\frac{\partial\xi}{\partial\tau}\biggr|\biggr)(u^{(k)})^2\,dy\,d\tau$$
$$+C(M^2+1)\int_{t_0}^{t}\int_{K_\varrho}\xi^2\,dy\,d\tau. \tag{6.13}$$

The last inequality guarantees the boundedness of the Hölder norm of the function $u(y,t)$ with respect to time on the boundary $y=1$ and hence the complete continuity of the operator Φ. Thus, the operator Φ has at least one fixed point. As already noted, all the fixed points of the operator Φ determine a solution of the original problem (2.1)–(2.8), which concludes the proof of Theorem 1.

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